Computational Continuum Mechanics
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Lecture – 25 – 26
Linearization of internal and external virtual work

So, today we are going to start the Linearization of internal virtual work. So, remember we have to linearize both the internal virtual work and the external virtual work ok. So, we first start with linearization of the internal virtual work ok.

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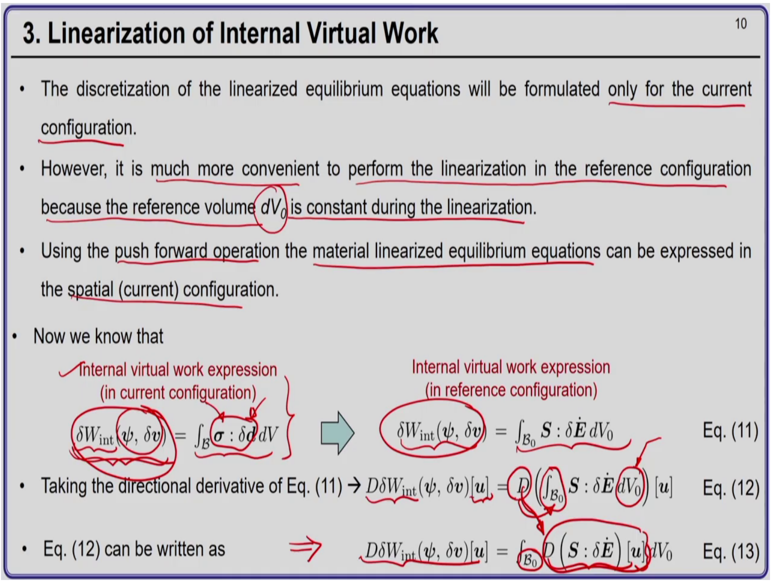

So, you have to remember that the equilibrium equations are formulated in the spatial configuration. Therefore, the final discretization of the equilibrium equation which is obtained after linearization has to be formulated only in the current configuration or the spatial configuration ok. But, the problem is in the spatial configuration the volume element dV is not known is unknown.

So, therefore, it is much more convenient to perform the linearization in the reference configuration that is the undeformed configuration because, here the reference volume which is dV 0 is constant during the linearization. So, when you have to take the linearization it means you have to take some derivatives and if you are taking the linearization with respect to the reference configuration therefore, the integral over the volume so, that is basically constant dV 0 is constant. So, you can take those derivatives inside the integral sign ok.

And, then we can obtain the spatial linearize internal virtual work expression from the push forward operation of the material linearize equilibrium equation ok. So, we first do the linearization of the equilibrium equations in the material or the reference configuration or the undeformed configuration and then finally, we use the push forward operations of various kinematical and kinetical quantities to get the linearize equilibrium equation in the spatial or the current configuration that is our idea.

So, now we know that the internal virtual work expression in the current configuration is given by del W and int for internal which is a function of deformation mapping psi and the virtual velocity del v is equal to integral over the current configuration of the double contraction of the Cauchy stress tensor with the virtual rate of deformation tensor. So, that we already know.

And, we also know that this spatial virtual work expression can be transformed to the material configuration and this we already did in our kinetics and when we were discussing the objective stress measures and there we saw that the internal virtual work expression written over the reference configuration reference.

So, v 0 and dV 0 is nothing, but the integral of the double contraction of the second Piola–Kirchhoff stress tensor with the virtual material rate of the Green–Lagrange strain tensor. So, if you linearize this you see the left hand side of both are same. So, if you linearize either of these expressions you are going to get the same quantity.

So, now, if you take the directional derivative of equation 11 which means, you have to take the directional derivative of this particular integral. So, the directional derivative of the internal virtual work in the direction of a displacement u or a change u in psi is nothing, but the directional derivative of the integral of the double contraction of the second Piola–Kirchhoff stress tensor with the virtual material rate of Green–Lagrange strain tensor in the direction u ok.

So, now as we said in our second point that now we know that we have to take the directional derivative of this integral ok. Now, in the reference configuration your volume dV 0 is constant. Therefore, this directional derivative which will involve a derivative can be taken inside. So, that is what we do because our volume is constant I can take this directional derivative inside the integral sign.

Then linearize internal virtual work in the direction u is nothing, but integral over the reference configuration of the directional derivative of second Piola–Kirchhoff stress tensor double contracted with the virtual material rate of Green–Lagrange strain tensor in the direction u. So, this now is my integrand which is nothing, but the directional derivative.

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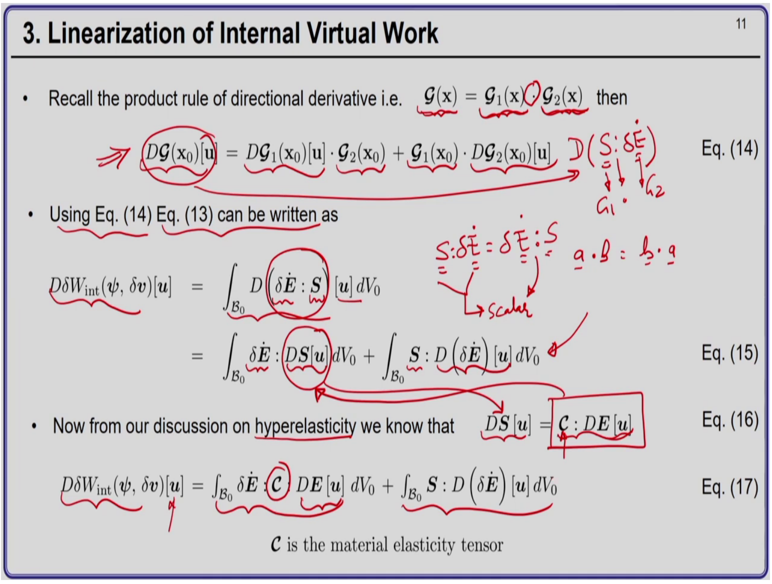

So, the next thing is we recall the product rule of directional derivative ok. So, if you have a set of non-linear equations and these equations can be written as a product of two set of non-linear equations G 1 and G 2 where this dot denotes any kind of product. It can be simple dot product, it can be double contraction ok.

So, G is a product of 2 functions G 1 and G 2 ok. Then we know and this we had already discussed in our mathematical basics that the directional derivative of G at point x 0 in the direction u is nothing, but the directional derivative of G 1 evaluated at x 0 in the direction u dotted with or the product with function G 2 evaluated at x 0 plus function G 1 evaluated at x 0 dotted with the directional derivative of G 2 evaluated at x 0 in the direction u ok.

So, why we are discussing this? Because we have to take the directional derivative of the second Piola–Kirchhoff stress tensor with the virtual material weight of Green–Lagrange strain tensor. So, here if you see this can be our G 1 and this can be our G 2 and this double contraction is nothing, but the product ok.

So, now, we have to take the directional derivative. This is what we have to take. So, therefore, I can use now the product rule of directional derivative to further simplify my internal directional derivative of internal virtual work ok.

So, using this equation number 16, I can write equation number 13 as the directional derivative of the internal virtual work in the direction u is nothing, but the directional derivative of del E dot dot double contracted with S in the direction u. Now, remember S double contracted with del E dot is same as del E dot double contracted with S because the eventual outcome of a double contraction is a scalar ok.

So, it does not matter you will get a same scalar. So, you can interchange both of them it is just like a dot b is same as b dot a ok. Therefore, I can interchange. So, that is what we have done here ok. I have written the directional derivative of the virtual material rate of Green–Lagrange strain tensor double contracted with the second Piola–Kirchhoff stress tensor in the direction u.

Is nothing, but the virtual material rate of Green–Lagrange strain tensor double contracted with the directional derivative of second Piola–Kirchhoff stress tensor in the direction u plus the second Piola–Kirchhoff stress tensor double contracted with the directional derivative of the variation of material rate of the Green–Lagrange strain tensor in the direction u ok. So, that is what we get after applying equation number 14 ok.

Now, from our discussion on hyper elasticity that we had in our module on hyper elasticity we had derived that the directional derivative of the second Piola–Kirchhoff stress tensor in the direction u is nothing, but the material elasticity tensor c double contracted with the directional derivative of the Green–Lagrange strain tensor in the direction u ok.

So, we have to find out the directional derivative of the Piola second Piola–Kirchhoff stress tensor in direction u and this is nothing, but given by equation 16. So, I can now substitute equation 16 here and then I can obtain the directional derivative of the virtual internal work in the direction u as del E dot double contracted with the material elasticity tensor C double contracted with the directional derivative of the Green–Lagrange strain tensor in the direction u plus our second term which remains as it is.

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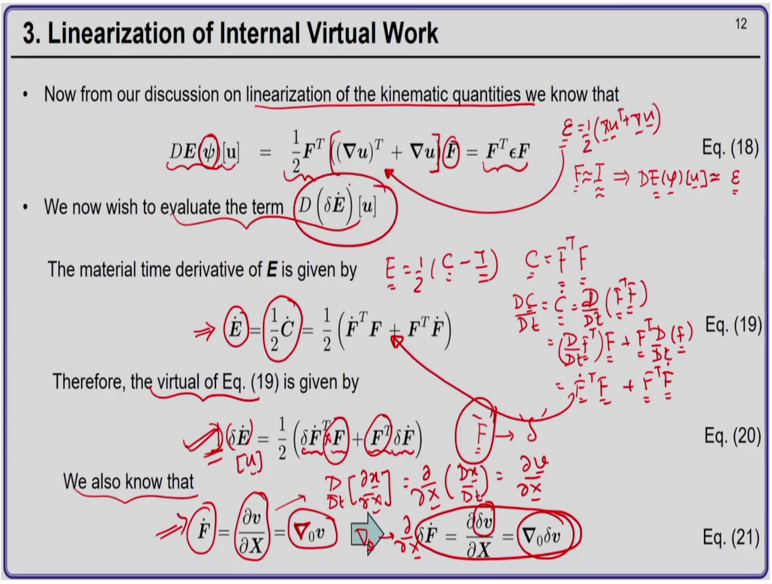

Now, let us see the different terms in the directional derivative of the internal virtual work. So, from our discussion on the linearization of the kinematical quantities we know that the directional derivative of the Green–Lagrange strain tensor which is a function of the deformation mapping psi.

In the direction of a displacement say u is nothing, but 1 by 2 F transpose del u transpose plus del u into F ok. And, now we know that the small strain tensor is given by del u transpose plus del u ok. So, if I substitute this here I can write the directional derivative of the Green–Lagrange strain tensor in the direction u as F transpose epsilon F ok.

Now, we also discuss the case where we had very small deformation in case of say elasticity in linear elasticity when the deformations are very small in that case F was nearly equal to identity and we had discussed that the directional derivative of the Green–Lagrange strain tensor in the direction u is nothing, but small strain tensor epsilon.

So, now, remember equation 18 ok. Now, we have to evaluate this term ok. So, if you see equation number 17 we have evaluated this term over here. Now, we have to evaluate this term over here ok. Now, our objective is to evaluate the directional derivative of the virtual material rate of Green–Lagrange strain tensor in the direction u let us see how to do that.

So, first we notice that the Green–Lagrange strain tensor is 1 by 2 right Cauchy–Green tensor minus second order identity tensor I ok. Therefore, if you take the material time derivative of the Green–Lagrange strain tensor which is E dot is nothing, but 1 by 2 the material time derivative of the right Cauchy–Green tensor F dot.

Now, I know that C is nothing, but F transpose F. So, if I take the material time derivative del C by delta T which is nothing, but C dot it will be D by Dt of F transpose F and this will be nothing, but D by Dt of F transpose into F plus F transpose D by Dt of F and this is nothing, but F dot transpose F plus F transpose F dot ok. So, that is what we have substituted and we have got ok.

Now, the virtual variation of the material rate of the Green–Lagrange strain tensor is given by del E dot equal to 1 by 2 the virtual variation of the material rate of the deformation gradient tensor and its transpose plus multiplied by F plus F transpose the virtual variation of the material rate of the deformation gradient tensor F.

Now, we have discussed in our previous lectures that F all the quantities are independent of the virtual variations when you apply virtual velocities. These kinematic quantities like F do not change. Therefore, there will be no change F here.

Now, we also note that the material time derivative of the deformation gradient tensor is nothing, but del v by del X. So, this we had like D by Dt of del x by del capital X. So, I can write this as del by del X of Dx by Dt and Dx by Dt is nothing, but our velocity.

So, the material time derivative of the deformation gradient tensor is del v by del X which in short can be written as the gradient ok. So, del 0 v. So, this del 0 denotes derivative with respect to the material coordinates. So, therefore, the virtual variation of the material time derivative of the deformation gradient tensor is nothing, but the material time derivative of the virtual velocities which in short is written as del 0 del v.

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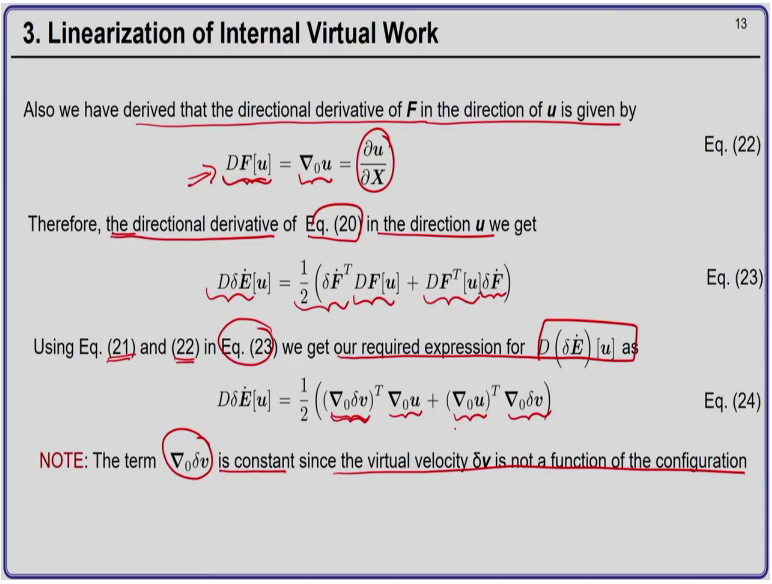

So, we have also derived that the directional derivative of the deformation gradient tensor in the direction u is nothing, but ok. So, this is the directional derivative of the deformation gradient tensor in the direction u is nothing, but the material time derivative of u and then if we now substitute now, if we take the material time derivative or we take the directional derivative of equation 20 in the direction u ok. So, our equation 20 if you see was this.

So, I have to take the direction derivative of this quantity in the direction u because that is what we want to find out. So, this directional derivative of virtual material rate of Green–Lagrange strain tensor in the direction u will be nothing, but 1 by 2 del F dot transpose the directional derivative of the deformation gradient tensor F in the direction u plus the directional derivative of F transpose in the direction u times the virtual material rate of the deformation gradient tensor F.

So, now I can use equation number 21 and equation number 22. In equation number 23 to get our required expression for the directional derivative of the virtual variation of the material rate of Green–Lagrange strain tensor in the direction u as 1 by 2 del 0 ok. So, material time derivative of the virtual velocities transpose times the material time derivative of the displacement of the direction u plus the material time derivative of u transpose times the material time derivative of the virtual velocities v.

Now, we note that this term the material time derivative of the virtual velocities is a constant since the virtual velocities are not a function of configuration, that we discuss in our previous lectures that the virtual velocities are independent of the configuration.

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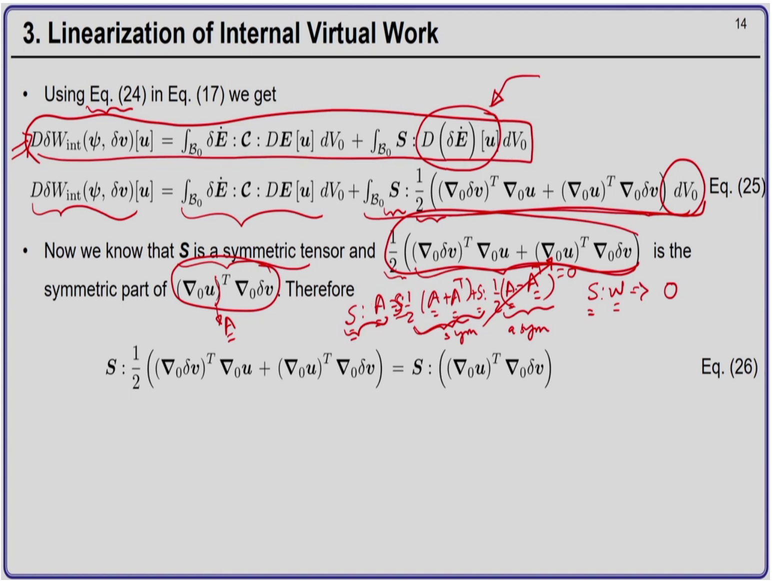

Therefore, we now if we use equation number 24 in equation number 17; so, for simplicity I have reproduced equation 17 here ok. So, equation 24 is the expression for this. Now, if I substitute equation 24 which is an expression for this I will get the directional derivative of the internal virtual work in the direction u as the first term plus the integral of the second Piola–Kirchhoff stress tensor double contracted with 1 by 2 this term in the bracket integrated over the reference configuration.

Now, we can further simplify this second term I can further simplify it and how I can simplify? I notice first that the second Piola–Kirchhoff stress tensor is a symmetric tensor and then this term with which it has been double contracted that is this 1 by 2 and this term in the bracket is nothing, but it is the symmetric part of the tensor second order tensor del 0 u transpose del 0 del v ok.

So, if I take del 0; so, it is a second order tensor let us say if I write it as A then A will be 1 by 2 A plus A transpose plus 1 by 2 A minus A transpose. So, you have this symmetric term and you have this anti-symmetric term ok. So, this is your symmetric term. Now, if I take double contraction of A with S, so, I get S double contraction with this plus. So, S double contracted with the symmetric part plus S double contracted with the anti-symmetric part.

Now, we know that the double contraction of a second order symmetric tensor with an anti symmetric tensor is nothing, but 0 ok, this we remember. If S is a symmetric second order tensor and w is a anti symmetric tensor second order tensor then this value is equal to 0.

So, then what happens? This is equal to 0 and then the double contraction of the second Piola–Kirchhoff stress tensor with the symmetric part of second order tensor A is nothing, but the double contraction with second Piola–Kirchhoff stress tensor with the tensor A ok.

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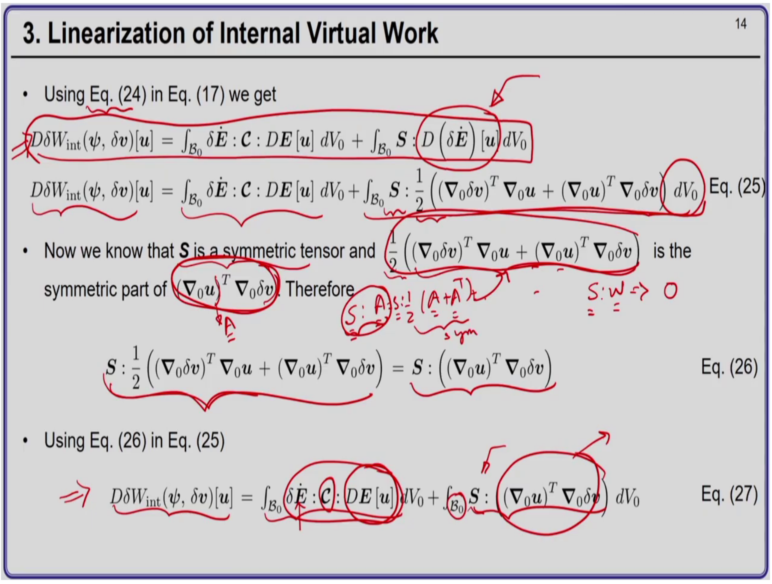

So, that is what we substitute ok. So, this goes away and then you have the double contraction ok. So, that is what we have it here. So, I can substitute this with S double contracted with A, where A is my this expression ok. So, I can substitute this long I can replace this long expression with this expression given by equation number 26.

So, if I do this my directional derivative of the internal virtual work in the direction u becomes the integral over the reference volume of the virtual variation of the material time derivative of the Green–Lagrange strain tensor double contracted with the material elasticity tensor C double contracted with the directional derivative of the Green–Lagrange strain tensor in direction u plus the integral over the reference volume of the double contraction of the second Piola–Kirchhoff stress tensor with the second order tensor given by this expression.

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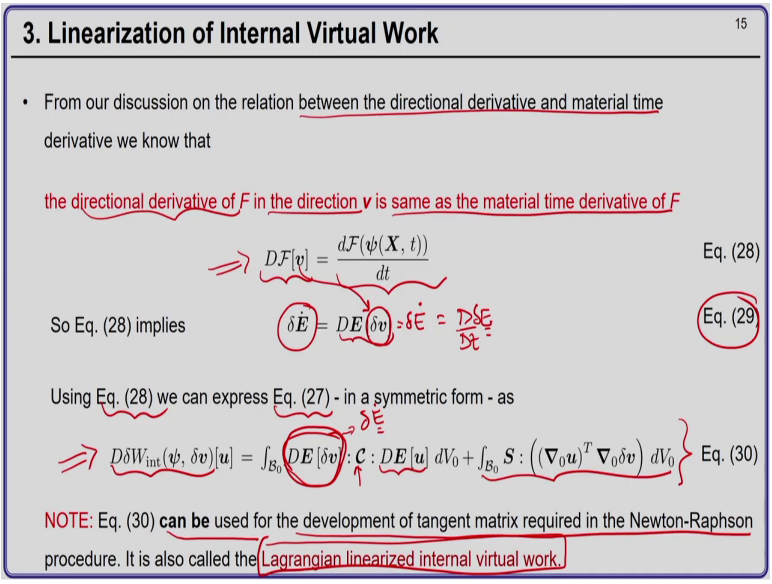

So, now once we have this we know further that the there is a relation between the directional derivative and the material time derivative. So, what is this? To recall the directional derivative of a non-linear function F in the direction v is same as the material time derivative of the function F ok.

Mathematically, if I write the directional derivative of F in the direction of velocity v is nothing, but the material time derivative of F. So, equation 28, now implies the directional derivative of the Green–Lagrange strain tensor in the direction of the virtual velocities. So, it is still the velocity, but it has become now the virtual velocity will be nothing, but the material time derivative of the virtual Green–Lagrange strain tensor which is D by Dt of del E.

So, with this in hand I can replace the del E dot in equation number 27 with our equation number 29 ok. So, our virtual internal virtual work in the direction u will be the directional derivative of the Green–Lagrange strain tensor, remember this was nothing, but del E dot ok.

So, it becomes the directional derivative of the Green–Lagrange strain tensor in the direction of the virtual velocities del v double contract with the material elasticity tensor C double contracted with the directional derivative of the Green–Lagrange strain tensor in the direction u plus as usually our second term ok.

So, this expression given by equation number 30 is also called the Lagrangian linearized internal virtual work expression and in theory equation 30 can be used for the development of tangent matrix required for the Newton–Raphson procedure. However, it will be a very difficult task to do.

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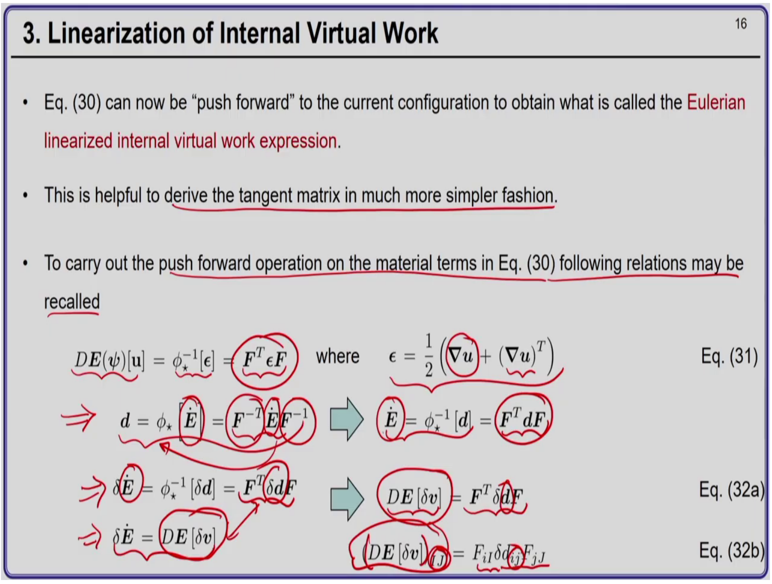

So, what we now need to do is we need to push forward equation number 30 to the spatial configuration and then we will get the Eulerian linearized internal virtual work expression.

So, to push forward there if you noticed there are we have to notice that there are lot of kinematical quantities like you have Green–Lagrange strain tensor and that is the directional derivative. So, you directly cannot take the push forward of the directional derivative that we have to derive you have the kinetic quantity the second Piola–Kirchhoff stress tensor.

And you have the material elasticity tensor C that also we have to push forward and now also we have the kinematic quantity dV 0 this also we need to push forward.

So, first we have to derive the expression for the push forward of each of them ok. So, once we do this push forward operation and we get the Eulerian linearized internal virtual work expression, we can derive the tangent matrix in much more simpler fashion ok. We will get a very elegant expression for tangent matrix.

So, to carry out the push forward operation on the material terms in equation 30, we have to recall the following expression ok. We just saw that the directional derivative of the Green–Lagrange strain tensor in the direction u is nothing, but the pullback of the small strain tensor epsilon and how this pullback is given it is F transpose epsilon F ok, where epsilon is 1 by 2 the spatial derivative of u plus the transpose of the spatial derivative of u that is the displacement.

Now, we also know that the rate of deformation tensor d is the push forward of the material time derivative of the Green–Lagrange strain tensor. And, the way this push forward is carried out is you have to pre-multiply the material time derivative of the Green–Lagrange strain tensor with F inverse transpose and post multiply by F inverse. Therefore, I can get the push forward or the pullback of the material time derivative of the Green–Lagrange strain tensor as F transpose dF ok.

So, what we are trying to do? We are trying to get the expression for various material kinetic quantities in terms of their spatial counterparts that is what we are trying to get ok. Now, if we take the variation of the material time derivative of the Green–Lagrange strain tensor we get F transpose delta dF, where delta d is the virtual variation of the rate of deformation tensor.

Now, we also know that the variation of del E dot that the variation of the material time derivative of the Green–Lagrange strain tensor is nothing, but the directional derivative of the Green–Lagrange strain tensor in the direction of virtual velocities del v ok. If we do this we will get the directional derivative ok. So, if we compare these two we get the directional derivative of the Green–Lagrange strain tensor in the direction of virtual velocities as F transpose the virtual rate of deformation tensor times F ok.

So, in indicial notation this will be helpful later. In indicial notation equation 32 a can be written as the term in the bracket IJ; IJ because the directional derivative of Green–Lagrange strain tensor is a material quantity. So, we use uppercase then F iI del d ij F jJ; remember, d is a spatial quantity therefore, we have to use a small case indices lowercase indices.

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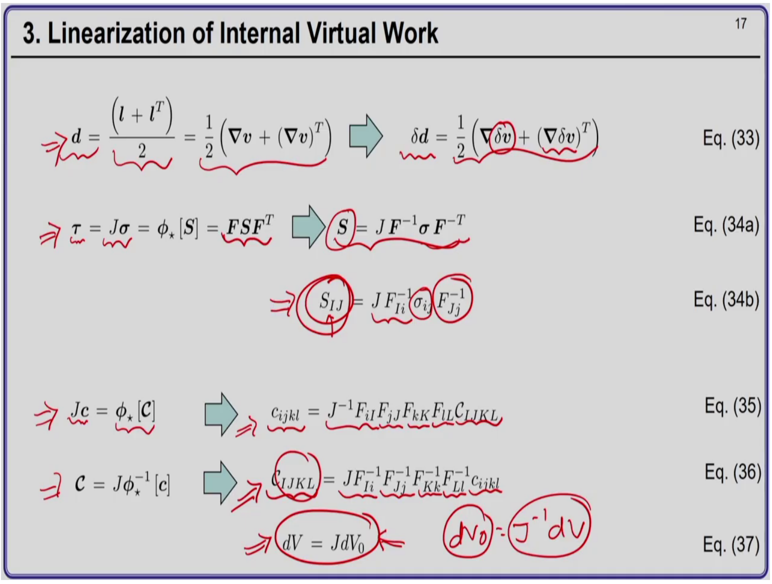

Now, we know that the rate of deformation tensor is nothing, but it is the symmetric part of the velocity gradient tensor l ok. That is l plus l transpose by 2 or it is 1 by 2 del v plus del v transpose. Therefore, the virtual of the rate of deformation tensor given here is nothing, but 1 by 2 the gradient of the virtual velocities plus the gradient of the virtual velocity transpose ok.

Now, from our discussion in the kinetics, we know that the Kirchhoff stress is equal to the Jacobian times the Cauchy stress tensor sigma and the Kirchhoff stress is nothing, but the pullback or the push forward of the second Piola–Kirchhoff stress tensor given by FSF transpose. So, from this we already had derived that the second Piola–Kirchhoff stress tensor is nothing, but j F inverse sigma F inverse transpose.

In indicial notation I can write S IJ; S is a material quantity, so, we use uppercase indices; JF inverse Ii sigma ij, sigma is a spatial quantity. So, we use we use lowercase indices F inverse Jj. Finally, we have also derived in our discussion in hyper elasticity that J times the spatial elasticity tensor c is nothing, but the push forward of the material elasticity tensor C.

So, the spatial tensor c given by in indicial notation was given by c ijkl J inverse F iI F jJ F kK F lL then the material elasticity tensor C C IJKL. Therefore, we I can write the material elasticity tensor C as J times F Ii inverse F Jj inverse F Kk inverse F Ll inverse c ijkl ok. And, we from our discussion in kinetics kinematics we know that the spatial volume dV is related to the material volume dV 0 as shown in equation 37. Therefore, dV 0 will be J inverse dV ok.

So, now remember in our material equation ok. So, directional derivative of the internal virtual work in the material configuration wherever we have dV 0 c ijkl the second Piola–Kirchhoff stress tensor and the other kinematic quantities, we now can directly substitute these expressions and if we simplify we will get our spatial linearize internal virtual work expression ok.

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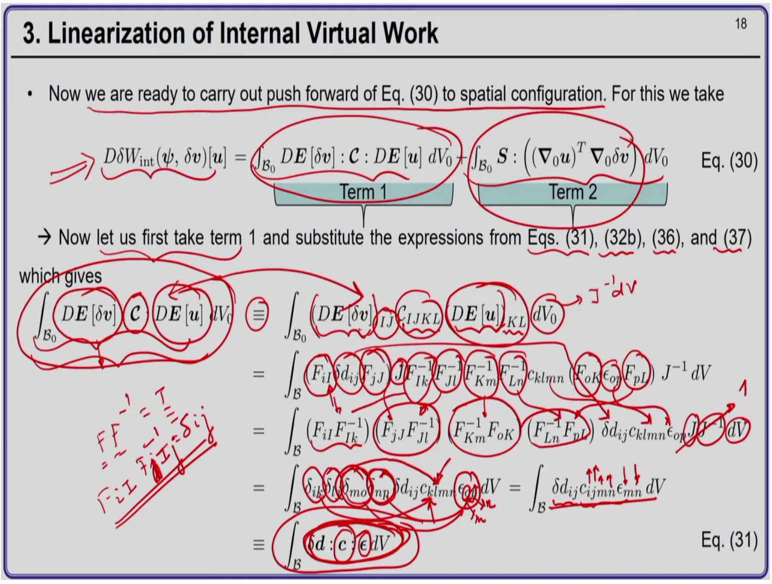

So, now let us do this. So, we are now ready to carry out the push forward of equation thirty to spatial configuration. So, for this we take our material linearized internal virtual work expression which is given by this expression equation 30 we have reproduced again and then we consider the linearization separately as term 1 and term 2 ok.

So, we separately push forward the both the terms term 1 and term 2. Now, let us first take term 1 and we now substitute equations 31, 32b, 36 and 37 in term 1 ok. So, our term 1 is shown here and we now first what we do? We write this direct notation in equivalently into indicial notation ok.

So, this is the directional derivative of the Green–Lagrange strain tensor in the direction of virtual velocities and this is a second order material tensor. So, this is nothing, but IJ ok. So, the term in the bracket IJ and this is nothing, but C IJKL and this directional derivative of

Green–Lagrange strain tensor in the direction of u is nothing, but this quantity in the bracket KL.

And, now I can substitute for example, dV 0 is j inverse dV that is what we have done here, then the directional derivative of E KL is nothing, but F transpose epsilon F in indicial notation is F oK epsilon epsilon op F pL and C IJKL is this expression over here. And, the directional derivative of Green–Lagrange strain tensor in the direction of virtual velocities del v is nothing, but F iI del d d ij F jJ.

Now, what I now do is I can further simplify seems like a very big expression, but if we look closely we can simplify it further ok. So, I see that I have this deformation gradient tensor with an uppercase index I and also have another inverse of deformation gradient tensor with the index uppercase index I. So, I can write these two as one bracket over here then I notice I have another deformation gradient tensor and this one ok.

So, this again I can put inside the bracket then I have F inverse Km F oK ok. So, K is common. So, I can put these two inside one bracket and now I have F inverse Ln Fpl. I have uppercase L common, so, I can write here c lkm is here and our d ij is here, our epsilon op is here and this J we can take on the right hand side. So, it become JJ inverse dV.

So, JJ inverse becomes 1 and now I know that FF inverse is identity which means that F iI F jI is nothing, but delta ij sorry F inverse Ij F inverse Ij is delta ij ok. So, now if I see this and I compare with this expression I get that this expression is nothing, but delta ik; the second expression therefore, also becomes delta lj; the third one becomes delta mo the fourth one becomes delta np, then delta d ij c klmn epsilon op dV.

Now, if I now want to use the substitution property of the Kronecker delta I see that I have delta ik and I have c k. I have delta lj and I have c lj. So, the first Kronecker delta will replace k here; the second Kronecker delta with replace l here and then I see that delta mo epsilon o and I have delta np and there was epsilon p. So, this third Kronecker delta will replace this o with m and the fourth Kronecker delta will replace this p with n ok.

So, if I do this I get delta d ij c ijkl because k was replaced with i, j was l was replaced with j, m and n are already there and o was replaced with m and n was p was replaced with n ok. So, I get this expression which in direct notation I can write the virtual rate of deformation tensor double contacted with the spatial elasticity tensor c double contacted with small strain tensor epsilon integrated over the current volume.

So, now I have carried out the push forward of the first term. So, this was the term in the material configuration and now, equation 31 we get the term in the spatial configuration ok. So, the first term has now been totally pushed forward to the spatial configuration. Now, I take the second term which is over here ok.

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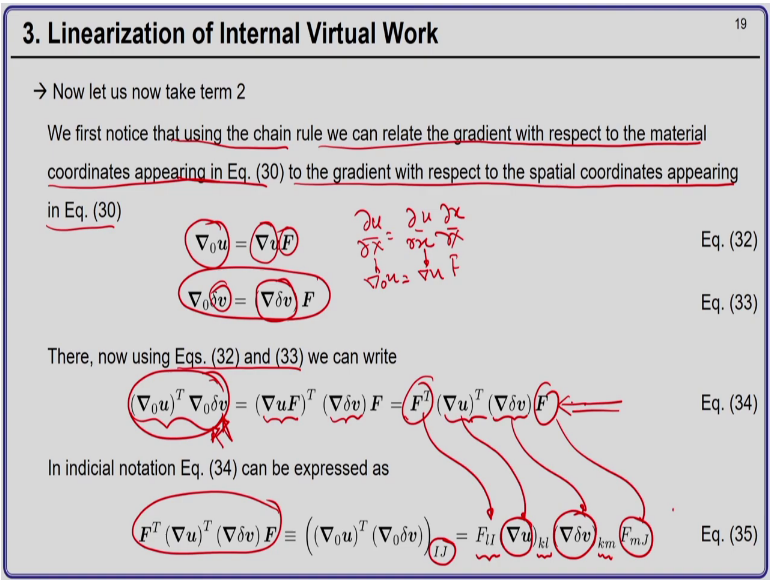

Now, let us take term 2. Now, first we notice that using the chain rule I can relate the gradient with respect to the material coordinates appearing in equation number 30 to the gradient with respect to the spatial coordinates appearing in equation number 30 ok.

So, now the material derivative of u is nothing, but the spatial derivative times F this is because del u by del X is nothing, but del u by del x into del x by del X and this is del 0 u this is del u and this is nothing, but deformation gradient tensor that is what we have in equation number 32. So, the material derivative of the virtual velocities similarly can be written as the spatial derivative of the virtual velocity times the deformation gradient tensor F ok.

Now, if I use equations 32 and 33, I can write this expression remember this is the expression with which the second Piola–Kirchhoff is being double contracted in term 2 ok. So, this expression I can write del u F transpose into del del v F ok. So, if I simplify this become F transpose del u transpose del del v into F.

So, in indicial notation equation number 34 which is this here is nothing, but and remember this is nothing, but the material tensor. So, therefore, we have capital IJ and this F transpose if l F lI and this is the spatial quantity, so, we have two small lowercase indices del u kl and this is del v del del v km and F is nothing, but F mJ ok. So, equation 35 is the indicial expression or the direct notation given by equation number 34 ok.

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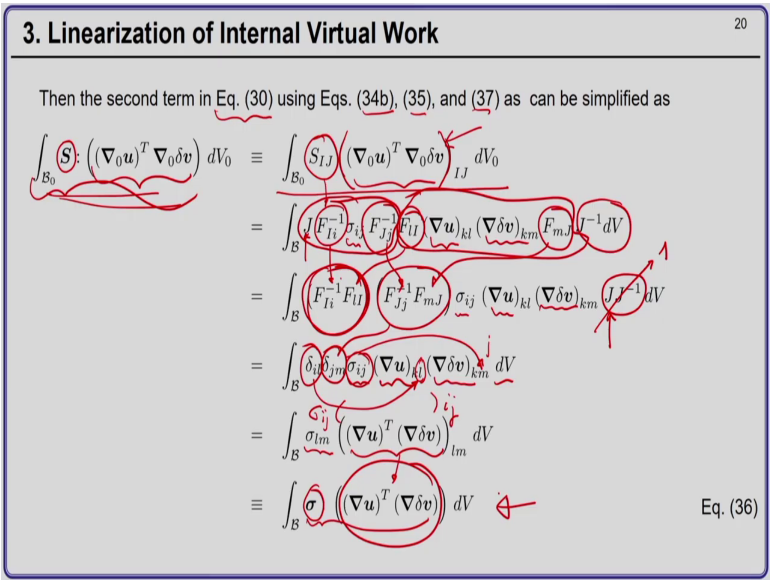

Now, the second term in equation number 30 using equations 34b, 35 and 37 can now be simplified and that is how we do ok. So, this is the second term the double contraction of the second Piola–Kirchhoff stress tensor with the second order material tensor given by this expression.

So, what I do I first express this term in indicial notation ok. So, this is S IJ del 0 u transpose del 0 del v IJ and now, I have already written this expression in indicial notation and S IJ we already know from 34 b that how it can be represented in terms of the Cauchy stress tensor. So, I can replace S IJ in terms of the Cauchy stress tensor. So, I can replace this with the Cauchy stress tensor and this one now becomes this and dV 0 is nothing, but J inverse dV ok.

Now, as we did for the term 2, I can take first of all this Jacobian J on the left hand side and then I have JJ inverse therefore, this becomes 1 and then I have F inverse Ii and I have F lI ok.

So, this I can bring in inside one bracket I have F inverse Jj and I have F mJ. These two I can bring inside one bracket.

And, then sigma ij multiplied by del u kl del del v km del del v k m and from our previous discussion we know that this is nothing, but Kronecker delta del il and this second term is nothing, but the Kronecker delta delta j m and we have sigma I j del u kl del del v km dV.

Now, I can replace my l with I because this Kronecker delta will replace this l with i and this Kronecker delta will replace this m with j or I can replace i and j with lm, it does not matter because all the indices here are dummy indices. So, I have sigma lm or I could have written sigma ij this term in the bracket ij ok. It does not make a difference. So, I have sigma lm and this term in the bracket lm dV.

So, this is nothing, but the double contraction in direct notation the double contraction of the Cauchy stress tensor with the second order spatial tensor del u transpose del del v ok. Now, we have the push forward of term 1 and term 2 ok.

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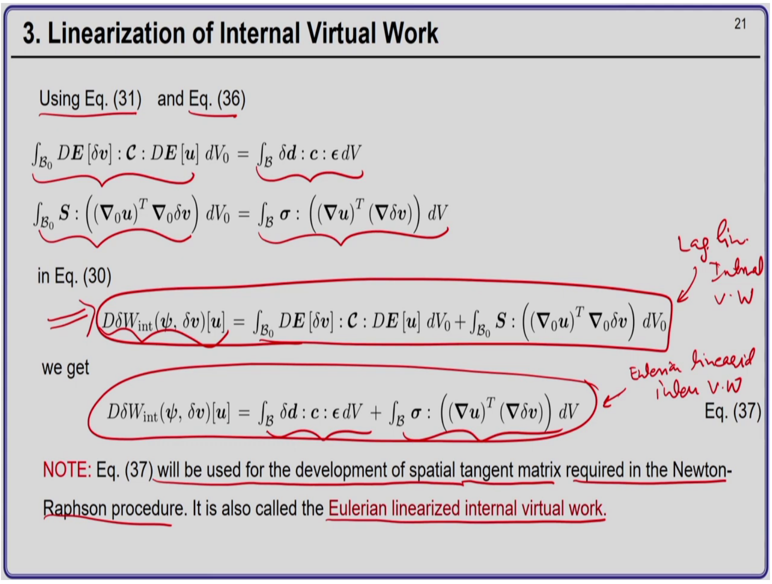

I can write using equation 31 and 36. So, the system 1 and this is the push forward of the term 1 to the spatial configuration, this was our term 2 and this is a push forward of the term 2 in the spatial configuration and using this I can get the Eulerian linearized internal virtual work or the spatial linearized internal virtual work as this ok. So, remember this was the Lagrangian linearized internal virtual work and now we get this expression which is nothing, but the Eulerian linearized internal virtual work.

So, this equation number 37 can now be used for the development of spatial tangent matrix which is required for the Newton–Raphson procedure and this expression as I stated earlier is also called the Eulerian linearized internal virtual work ok.

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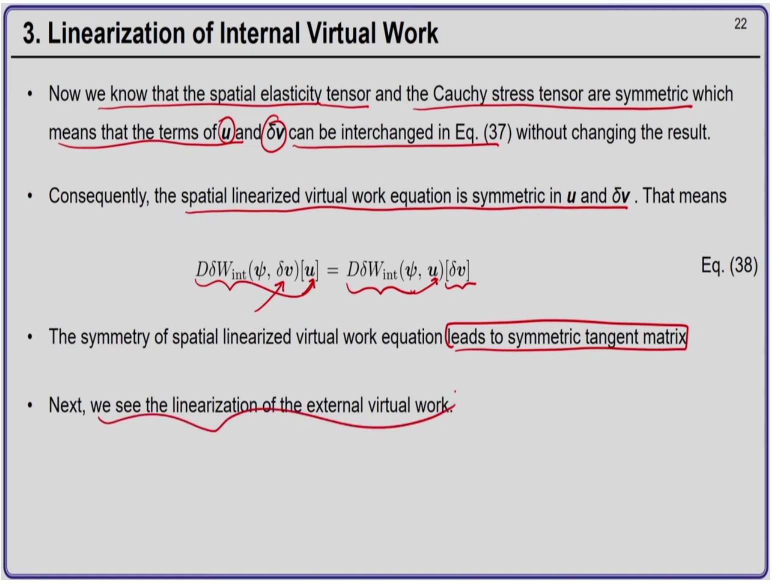

Now, there are some more points to focus on and one of them is we notice that the spatial elasticity tensor and the Cauchy stress tensor are symmetric which means that that the terms with the displacement u and the virtual velocities del v can be interchanged in equation 37 without changing the results.

And, consequently the spatially linearized virtual work equation is symmetric in the displacement u and virtual velocities del v which means that the linearized virtual work expression which is in the direction I mean of function of del v in the direction u is nothing, but the directional derivative of the internal virtual work as a function of u in the direction del v.

What this means is that eventually we are going to get a symmetric tangent matrix when we are going for the discretization ok. So, finally, we see the linearization of the external virtual work expression ok.

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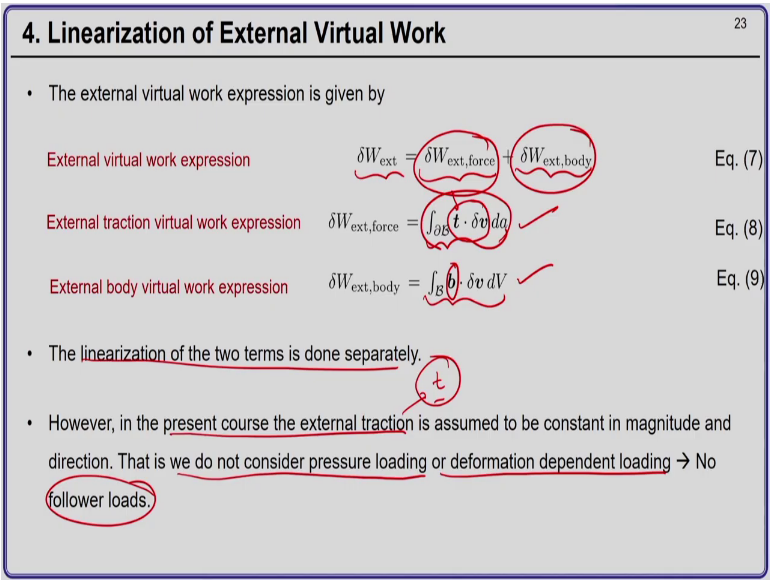

So, the external virtual work expression is given by the external virtual work of the externally applied tractions and the virtual work of the externally applied body forces ok. Now, the external traction virtual work which is this term over here is nothing, but the integral over the current area of the virtual work of the applied surface tractions t.

So, t dot del v is the rate of which virtual work is being done and then integral over the current area gives you the rate of external traction virtual work. So, the external body virtual work is nothing, but the volume integral of the virtual work of the externally applied body forces b ok. Now, we have to linearize these two terms separately ok.

However, in the present course the external traction which is t here we assume that it is constant in magnitude and direction which means that we are not considering any follower loads or pressure loads or any deformation depending loads ok. So, therefore, we do not have any follower loads ok. So, if we had a follower load then it is a little complicated to do the directional derivative of such traction virtual work ok. So, in our course we just treat that t is both constant and magnitude and direction ok.

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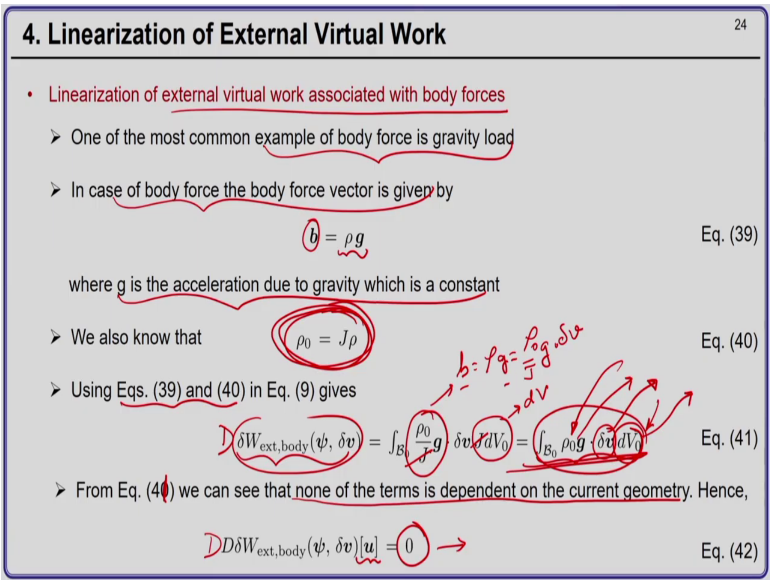

So, first we see the linearization of the external virtual work associated with the body forces and one of the most common example of the body forces is the gravity load ok. There are other kind of body forces like the centrifugal forces, but here we take the some simplest case of the gravity load.

Now, in case the body force is given by the gravity load, then the body force vector b is nothing, but the current density times the acceleration due to gravity g ok, where g is the acceleration due to gravity and this is a constant. Therefore, we also know the relation between the density in the material configuration as J times the density in the spatial configuration rho. So, rho 0 is J rho ok.

Now, writing using equations 39 and 40 in equation number 9 which is the virtual external body forces I can write my b ok. So, this is my b I can write this as rho g and from equation number 40 my rho is rho 0 by J g del v and then dV can be written as J dV 0. So, this J and this J cancel out and I get the virtual work of the external body forces as rho 0 g dot with the virtual velocities integrated over the reference volume.

Now, we notice that none of the terms is dependent on the current geometry in equation number 41 ok. So, equation number 41, none of the terms ok. So, rho 0 is independent of the current geometry g which is acceleration due to gravity is any a constant the virtual velocities are independent of the geometry and dV 0 is also independent of the current geometry.

Therefore, if we take the directional derivative so, directional derivative of the external virtual work because of the body forces in the direction of displacement u, then what we get? Because everything is constant on the right hand side then the directional derivative is equal to 0.

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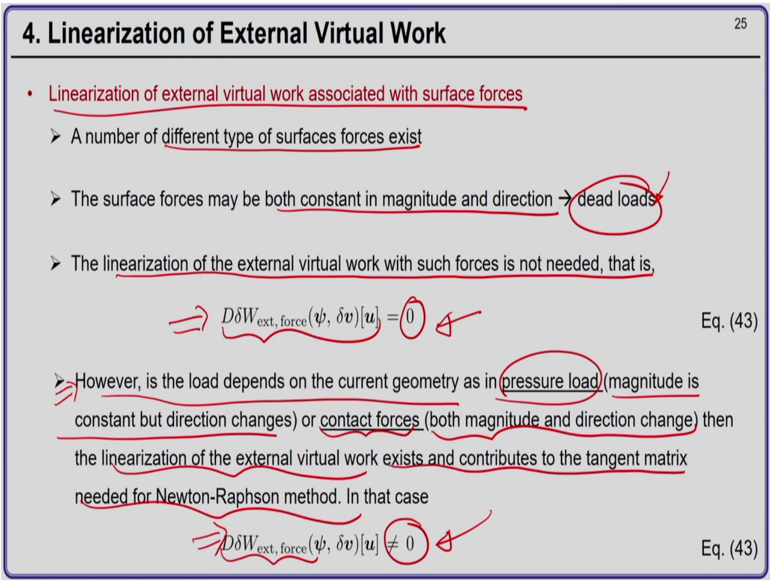

Now, linearization of external virtual work associated with the body forces. So, number of different types of surface forces they exist and the surface forces may both be constant in magnitude and direction which means they are what is called dead loads and linearization of the external virtual work of such forces is not needed ok.

If you have dead loads it means everything is constant then the directional derivative of the external virtual work because of dead loads will be equal to 0, and remember we are not dealing with follower loads in this course therefore, we will not have the directional derivative associated with the follower loads here ok.

However, it is to be noticed that if the load depends on the current geometry as in the pressure load where the magnitude is constant, but the direction changes or in case the external loads are because of the contact forces we change both in the magnitude and in direction, then the linearization of the external virtual work will exist ok. And it will then contribute to the tangent matrix which is needed for the Newton–Raphson method.

In that case, equation number 43 will not be equal to 0 and the derivation for the directional derivative of the external forces of tractions because of follower loads or the pressure loads or contact forces is a very complicated exercise that we are skipping in this course ok.

So, but you remember if you have dead loads as your external tractions, then their directional derivative will be equal to 0. And, if you have pressure loads or the follower loads then your directional derivative will not be equal to 0 and you have to take that and those forces follower loads will contribute to the tangent matrix they are also ok. So, they will have a contribution to the tangent matrix and if you do not compute that correctly then your convergence on your Newton–Raphson method will effect or will suffer.

So, with this we end this module ok.

Thank you.