Computational Continuum Mechanics

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Lecture – 25-26
Linearization of internal and external virtual work

So, welcome to this module on Linearization, ok. So, in this we are going to discuss the linearization of the equilibrium equations, both the internal and the external virtual work, ok. So, we have to go we have linearize, the internal virtual work which comes from the developed stresses inside the body and the external virtual work which comes because of the externally applied tractions and the body forces, ok.

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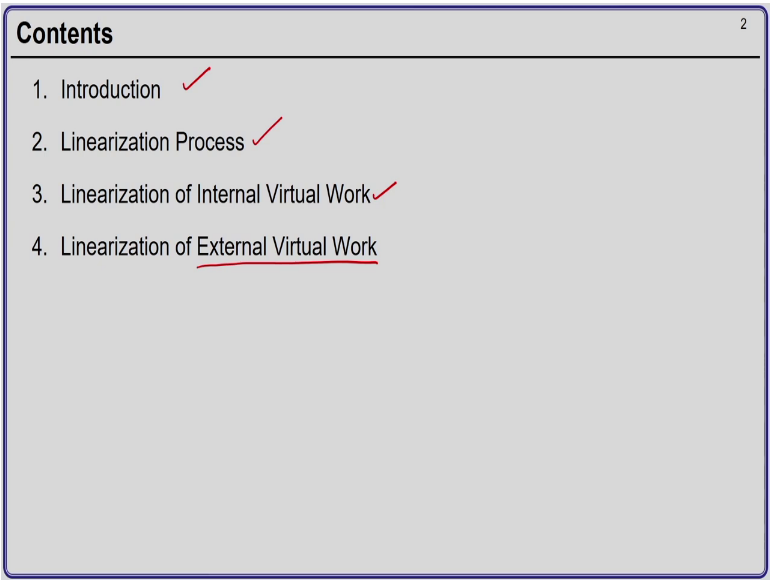

So, following are the contents of this module. First, we have to see some of the underlined basis form for which we require the linearization, ok. And, we also revise the virtual work expression that we derived when we were discussing kinetics, ok.

Then, we discussed what is meant by the linearization process, ok, what exactly linearization of the virtual work expression actually means, ok. Then, we discussed the linearization of the internal virtual work expression, which means the linearization of the term containing the Cauchy stress and the rate of deformation tensor.

That is the virtual work associated with the internal stresses that are generated inside the body. And then, we discuss the linearization of the external virtual work expression. External virtual work which means the virtual work because of the body forces and the externally applied surface traction over the surface of the body, ok.

So, in this course, we consider that the surface forces or tractions that we are specified are not deformation dependent, ok. So, they are not follower loads, which means they will be acting like a dead-loads in that way our linearization of the external virtual work will be very simple, ok.

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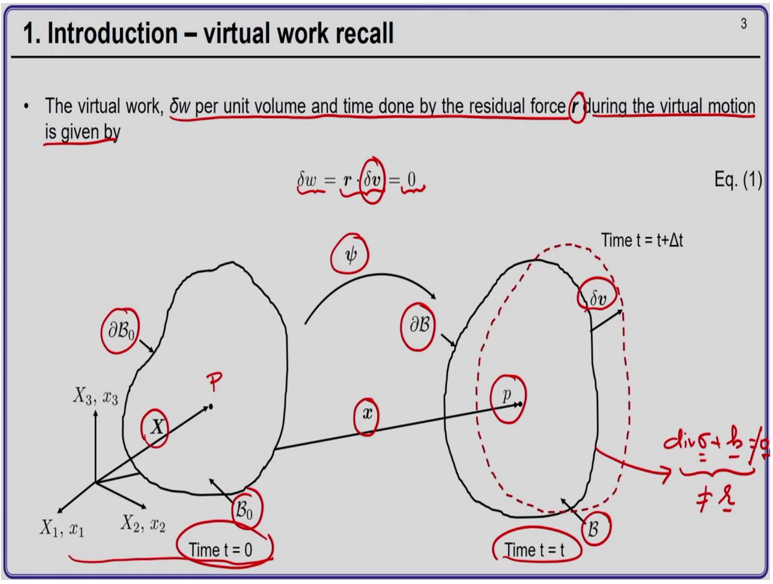

So, we first recall our derivation of virtual work expression, ok. So, in the figure as you see here you have a body in the initial configuration at time 0, ok. So, the configuration is B 0 and the surface of the body is del B 0.

So, we have a material particle X whose initial position, so consider this particle P whose initial position is X, capital A. And after deformation, so after this deformation the body occupies at time t, a configuration B bounded by surface del B, and the material particle has moved to a spatial location p with the spatial coordinate x, ok.

Now, we apply a small virtual velocity, ok. So, here you will have your equilibrium equation, divergence of sigma plus B equal to 0, ok. Now, because of the Newton-Raphson iterative procedure that we will employ this equilibrium equation will not in general be satisfied. So, what will happen is this will not be equal to 0, but rather will be equal to some residual force r, ok. So, the virtual work del w per unit volume and time done by this residual force r during the virtual motion will be given by del w equal to r dot del v, ok. And this virtual work should be equal to 0 for all values of virtual velocities del v. So, this virtual velocity del v are arbitrary, ok.

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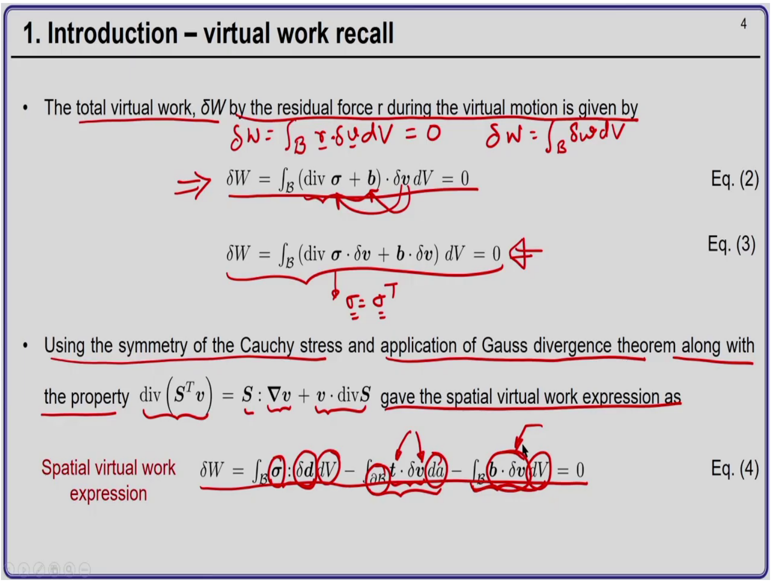

So, the total virtual work, that is del capital W by the residual forces r during the virtual motion will be given by del W equal to integral over the current volume r dot del v dV equal to 0, ok. So, del W, del capital W is nothing, but integral over the volume del w dV, ok, nd that should be equal to 0. So, when we substitute r as divergence of sigma plus p we get the total virtual work by the body forces as given by equation number 2, ok.

So, you can take the virtual velocities inside the bracket, and then we are getting this particular equation, equation number 3 as the expression for the virtual work. And then, if you recall what we did? When we derived this that we use the symmetry of the Cauchy stress tensor, because we know that the Cauchy stress tensor is the symmetric tensor in the absence of external body couples, ok.

So, and this is derived from the principle of conservation of angular momentum and application of Gauss divergence theorem along with the property divergence of a second order tensor S transpose v is equal to the second order tensor S double contracted with the gradient of v velocity plus the velocity vector doted with divergence of the second order tensor S. And this gave us our spatial virtual work expression as this, ok.

So, we got our spatial virtual work expression as del W equal to integral of the Cauchy stress tensor, double contracted with the virtual rate of deformation tensor integrated over the volume, current volume.

Minus, the virtual work of the external traction which is the external traction applied over the surface current surface del v and the dot product of the external traction with the virtual velocities integrated over the current area, or the current surface gives you the external virtual work because of the external forces. And then the body may have body forces, so the virtual work of the body forces b is given by integrating the virtual work of the body force over the current volume, and this should be equal to 0, ok.

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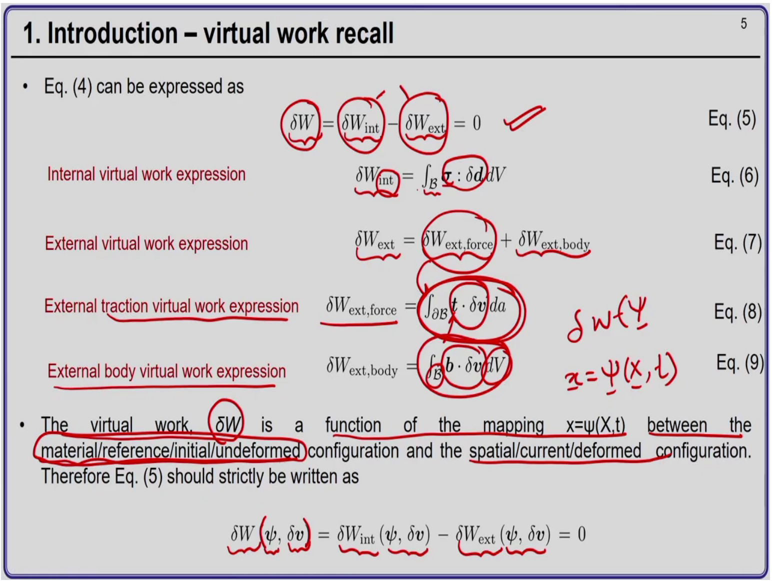

Now, equation 4 can be expressed as the difference of the virtual work of the internal forces and the virtual work of the external forces, ok. So, the total virtual work is nothing, but the difference of the internal virtual work and the external virtual work, and that should be equal to 0. So, in a way the at equilibrium the internal and external virtual work will balance out each other, ok.

So, the expression for the internal virtual work is del W, so subscript int. So, int is the short for internal, ok. So, integration over the volume of the work done by the internal stresses or the stresses generated inside the body sigma, double contracted with virtual rate of deformation tensor.

The external virtual work can be split into two part, ok. So, one because of the external forces or the external tractions and the other is because of the external body forces which might be acting on the body which is like centrifugal forces or the gravity forces, and the external forces may be because of the contract, ok.

Now, the external traction virtual work, which is here is given by this expression, ok. So, the external virtual work because of the external traction is the integral of the externally applied traction, over the current area, ok. Now, the external body virtual work, expression is nothing but the integral over the current volume of the virtual work done by the body forces, ok.

So, now it is worth-while to note that the virtual work given by del capital W is a function of the mapping x, ok. So, this virtual work, so you have this mapping and this virtual work del W is a function of this mapping psi, between the what is called the material or the reference or the initial or the undeformed configuration, ok.

So, all these terms I have written explicitly, so that they are they mean one and the same, ok, that the material configuration or the reference configuration or the initial or the undeformed configuration and the spatial or the current or the deformed configuration, ok. So, virtual work is a function of this mapping, ok.

So, strictly we should write equation 5, which is here as the virtual, total virtual work is a function of the mapping psi and the virtual velocities is equal to the internal virtual work which is a function of psi and the virtual velocities, minus the external virtual work which is a function of psi and the virtual velocities del v, ok.

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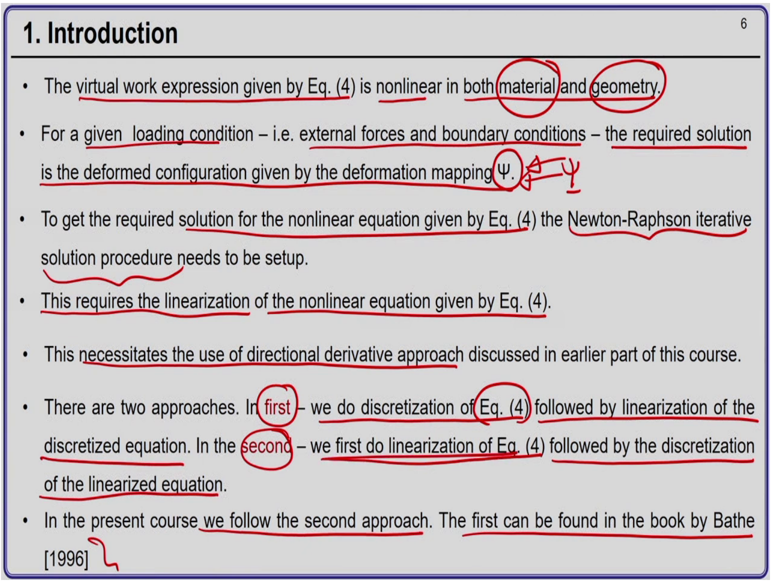

Now, we note that this virtual work expression given by equation number 4 is a highly non-linear expression. Non-linear because they are there can be both material and geometric non-linearities that may be present, ok. The material might be non-linear; in our case, we are taking a incompressible Neo hookean material, ok. So, it is like rubber and it can go to very large strength. And also, we have we have placed no restriction on the deformation. So, we may have material non-linearity and we may have the geometry non-linearity, ok.

So, for a given loading condition, that is external forces and the boundary conditions, ok. So, for particular problem you may be given the loading condition you may be given the body forces, you may be given the surface forces and also the boundary conditions, some displacement restrictions on the body might be given to you, ok. Then, the required solution will be the deformed configuration given by the deformation mapping psi. So, our objective is to find out this mapping psi, ok.

Now, to get the required solution that is to get this deformation mapping psi what we need to do is we need to solve these non-linear equation given by equation number 4 using certain numerical technique, ok.

If it is a 1 degree of freedom system its can be easily solved by hand, but in a practical situation you will have multiple degrees of freedom or system, where the number of degrees of freedom may be very high, may be close to million, 10 million like this. In that case, you need to setup some Newton-Raphson iterative solution procedure, to solve those system of non-linear equation, to get the required solution.

So, setting up the Newton-Raphson iterative procedure, ok, if you recall from our discussion on Newton-Raphson requires that we should have the tangent matrix associated with the Newton-Raphson procedure, ok.

So, getting the tangent matrix means that the Newton-Raphson procedure requires the linearization of the non-linear equation given by equation number 4, ok. So, you have a non-linear equation and to setup the Newton-Raphson iterative procedure you need to linearize those non-linear equation, then only you can get the tangent matrix, ok.

And linearization as you would recall from our previous discussions means that use of directional derivative approach, ok. You have to take the directional derivative of the non-linear equation in a certain direction, ok.

Now, to apply the directional derivative approach there are two ways. The first one, is we first do the discretization of equation 4. What it means is we construct the finite element setup, and then we do the finite element discretization of equation number 4 followed by linearization of the discretized equation, ok.

In the second, approach we do the linearization of equation 4 first and then we follow it up with the finite element discretization of the linearized equation, ok. So, in the second approach is used in the current course because it is more suitable for the solid continuum, ok. And the first approach you can find out in the book by bathe, ok.

We follow the second approach which means we will first do the linearization of our equations. As a next step we will do the discretization of the linearized equation, ok.

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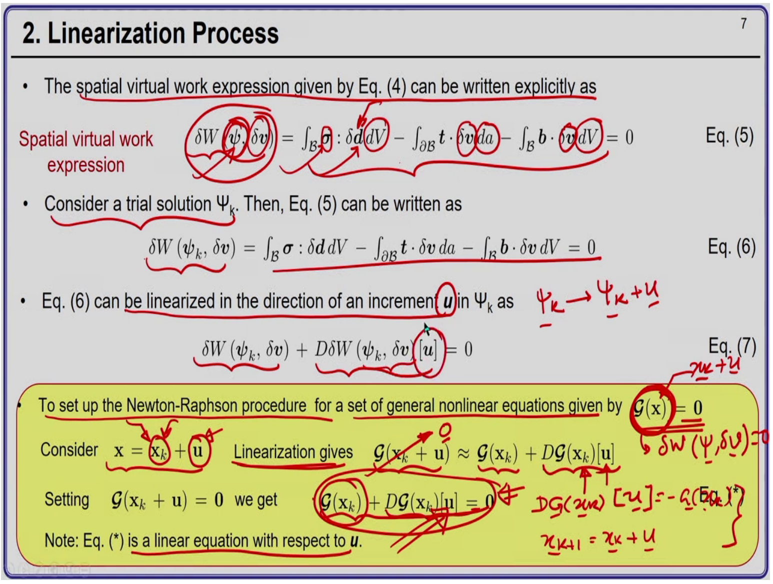

So, the spatial virtual work expression which is given by equation 4 can be written explicitly as, ok. So, now, I can write explicitly the virtual work as a function of the mapping psi in the virtual velocities, ok. So, psi is hidden somewhere here, in sigma and d, ok.

So, psi is hidden here and also psi is hidden in this area and the current area and the current volume, ok. You know that Nanson's formula and the relation between the current volume and the initial volume, ok. So, dV is j dV 0, so j is a function of deformation gradient tensor. The deformation gradient tensor itself is derived from the deformation mapping psi, ok.

So, similarly, the Cauchy stress tensor is the function of the deformation mapping psi and the rate of deformation tensor itself is a function of the deformation mapping psi. So, the expression on the right-hand side is a function of both the deformation mapping psi and the virtual velocities del v, ok. So, del v occurs directly here, ok. In the last equation and hidden in the virtual rate of deformation tensor in the first term, ok.

Now, we just recall the Newton-Raphson procedure, that we discussed earlier, but for the sake of discussion we again have a small recall, ok. So, remember that to setup the Newton-Raphson procedure for a general set of non-linear equation which are given by G of x equal to 0. We first consider a trial solution x k, and then we consider a new solution x as the trial solution plus a increment in x k given by u, and then we linearize our general non-linear equation, which is given here, where x is x k trial solution plus the change u, ok.

So, this linearization of the general non-linear equation G x k plus u is nothing, but G x k plus the directional derivative of G evaluated at x k in the direction u, ok. And now, we set the left-hand side equal to 0, ok. When we set the left hand side equal to 0 we get the general non-linear equation evaluated at x k plus the directional derivative of G evaluated at x k in the direction of u equal to 0, ok.

So, note that this equation is a linear equation with respect to u. And, assuming that you can solve this equation the Newton-Raphson iteration iterative procedure means that D directional derivative of G at x k in the direction u should be equal to minus of G x k and x k plus 1 equal to x k plus u, ok. So, this is the Newton-Raphson procedure.

Now, if we compare this equation, that we have, general non-linear equation G of x with equation number 5 we see that G of x is nothing, but del W psi del v equal to 0, ok. So, our spatial virtual work, total spatial virtual work is what is our non-linear equation, ok. And, we have to solve this non-linear equation.

First, we have to have some trial solution and then we have to linearize our spatial virtual work at the trial solution in a certain direction u, and then we can setup the Newton-Raphson iterative procedure, ok. So, consider a trail solution, psi k, then at this trial solution your spatial virtual work will be del W psi k comma del v equal to the right-hand side, ok.

Then, our discussion that we have here we can linearize our virtual work spatial virtual work in the direction of an increment u in psi k, ok. So, now, psi k goes to psi k plus u, ok. So, if you compare this equation over here, so our G is nothing but del W. So, we get this particular equation. So, the virtual work, the total virtual work at psi k comma del v plus the directional derivative of the virtual work spatial virtual work evaluated at psi k the trial solution comma del v in the direction of increment u, ok.

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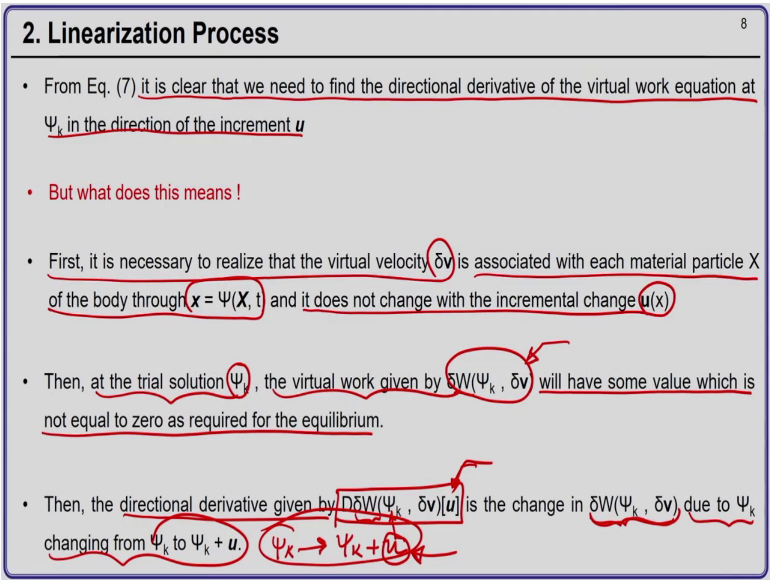

Now, let us think, more into it, ok. So, equation 7, it is clear that we need to find the directional derivative of the virtual work equation at psi k in the direction of the increment u to setup the Newton-Raphson iterative procedure. So, what does this actually means? We have to step back and pause for a moment and we have to ask what does directional derivative of the virtual work, equation at the trial solutions psi k in the direction of the increment u actually means, ok.

So, it is first necessary to realize that the virtual velocity, del v is associated with each material particle x of the body through this deformation mapping, ok. And also, it is worth remembering that this virtual velocity does not change with the increment u, ok. So, the a change u in psi does not change the virtual velocity, ok. So, this is first thing you have to remember.

Then, at the trial solution psi k, the virtual work given by del W psi k comma del v, ok. So, the total virtual work at psi k trial solution psi k comma del v will have some value which is not equal to 0 as required for the equilibrium, ok.

So, because psi k is a trial solution we do not know whether its actually the solution, then this spatial virtual work expression will have some nonzero value it may be 0 by chance, ok, but if you are not lucky enough if psi k is not the actual solution then your virtual work will not be equal to 0 as required by the equilibrium. So, equilibrium requires that the virtual work should be equal to 0.

Now, if psi k is a trial solution the virtual work is not equal to 0, what this means is that the directional derivative given by this expression, the directional derivative or the virtual total virtual work spatial virtual work evaluated at psi k comma del v in the direction del u will give you the change in the virtual work total virtual work evaluated at psi k comma del v, due to psi k changing from psi k to psi k plus u, ok.

So, as the trial solution changing from psi k to psi k plus u your directional derivative will give you the change in the total virtual work because of this particular change in the deformation mapping, in the direction u.

And, if you can judiciously compute this direction u then you can make this change go to 0, or in other words, you can make the value of the total virtual work go to 0, ok. So, as a total virtual work goes to 0, which means that you are satisfying the equilibrium equation which means you are getting closer to the solution, ok.

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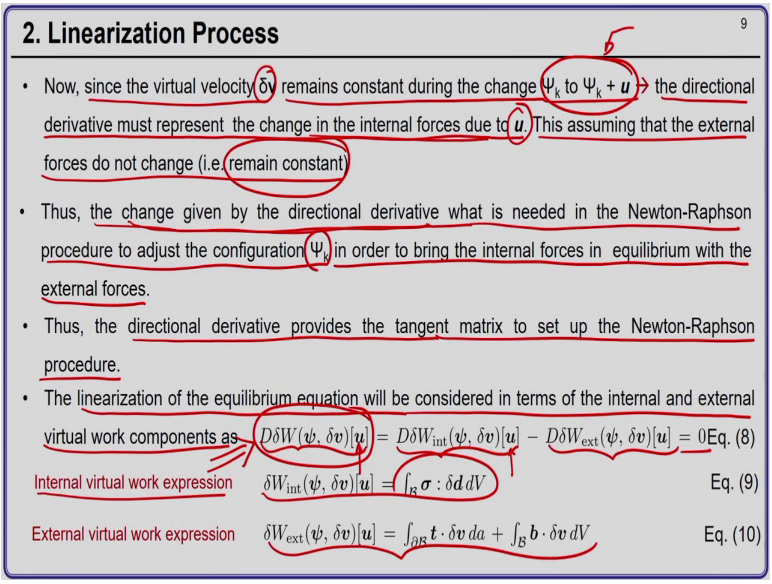

Now, since the virtual velocity del v remains constant during the change psi k to psi k plus u, the directional derivative therefore, must represent the change in the internal forces due to u, ok. Because the virtual velocities are constant as we change the trail solution, therefore, the change in the total virtual work which is given by the directional derivative should come from the change in the internal forces, because of this direction u, ok. u we are using because u we will also be used for displacement, ok.

So, this is only assuming that the external forces do not change with the virtual, external forces do not change as you change the trial solution psi k to psi k plus u, which means that the external forces or the external body forces and the surface traction remain constant, ok. Thus, the change given by the directional derivative is what is needed in the Newton-Raphson procedure to adjust the configuration psi k in order to bring the internal forces in equilibrium with the external forces, ok

So, this change in psi k leads to the change in the directional derivative and this is what is needed in the Newton-Raphson iterative procedure, so that once you change from psi k to psi k plus u the change gets you closer to the equilibrium, solution or equilibrium position which means that the internal forces are balancing out the external forces, ok. Thus, the directional derivative provides the tangent matrix which is needed to set up the Newton-Raphson solution procedure, ok.

Now, the linearization of the equilibrium equation will be considered in terms of the internal and the external virtual work expression, ok. So, we will do the linearization of the internal and the external virtual work expression separately, ok.

So, writing explicitly the directional derivative, because we want to compute this directional derivative of the total virtual work, in the direction u. So, this total directional derivative of the total virtual work will be equal to the directional derivative of the total internal virtual work in the direction u minus the directional derivative of the external virtual work in the direction u, ok.

And how did this happen? We have to recall the first property of the directional derivative that we discussed when we first discussed the directional derivative, ok. So, you need to go back and call the first property if you have a function which is sum of two functions, then the directional derivative of a function will be equal to directional derivative of the first function plus the directional derivative of the second function. And that is what we have used here, ok.

So, the internal virtual work expression if you recall is given by following expression, and the external virtual work if you recall was given by following expression, ok.

Thank you.