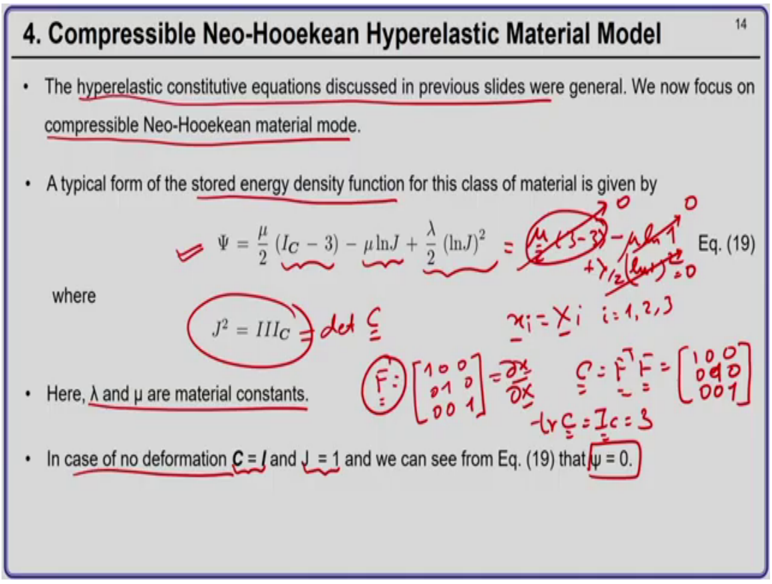
Computational Continuum Mechanics
Dr. Sachin Singh Gautam
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Hyperelasticity – 2
Lecture – 23-24
Isotropic Hyperelasticity and Compressible Neo-Hookean Material

So, the next we move to the derivation of constitutive response of a Compressible Neo-Hookean elastic Material.

(Refer Slide Time: 00:51)



So, the hyperelastic constitutive equations that we discussed in the previous slides were general ok. We now, focus on the compressible Neo-Hookean material model.

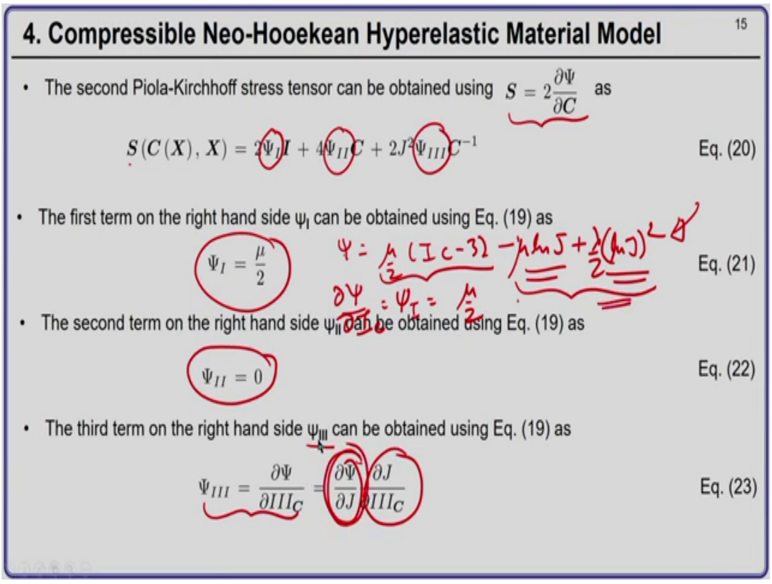
So, a typical form of the stored energy density function for this class of material is given by psi equal to mu by 2, the first invariant of the right Cauchy-Green deformation tensor minus 3 into mu times natural log of the Jacobian plus lambda by 2 into the square of the natural log of the Jacobian ok, where mu and lambda are material constant and J square equal to the third invariant of the right Cauchy-Green deformation tensor ok.

Now, as you can see in case of no deformation which means the deformation gradient tensor is identity, in case there are no deformation therefore, x 1 or let us say x i will be equal to X i, where i equal to 1 2 3.

Therefore, F which is nothing, but del x by del capital X will be nothing, but identity tensor in that case C will be F transpose F equal to identity. So, C will be identity tensor and J that is the determinant of the deformation gradient tensor will be equal to 1 and if we substitute ok.

So, C equal to 1 means trace of C equal to I C will be equal to 3 ok. So, if you substitute here you will get mu by 2 3 minus 3 minus mu l n 1 plus lambda by 2 l n 1 the whole square; so 3 minus 3 is 0. So, this term goes away natural log of 1 is 0. So, this also goes away and this term also goes away; so which gives you psi equal to 0. So, stored energy function will be equal to 0 which means this stored energy density function can be used for a hyperelastic material model.

(Refer Slide Time: 03:34)



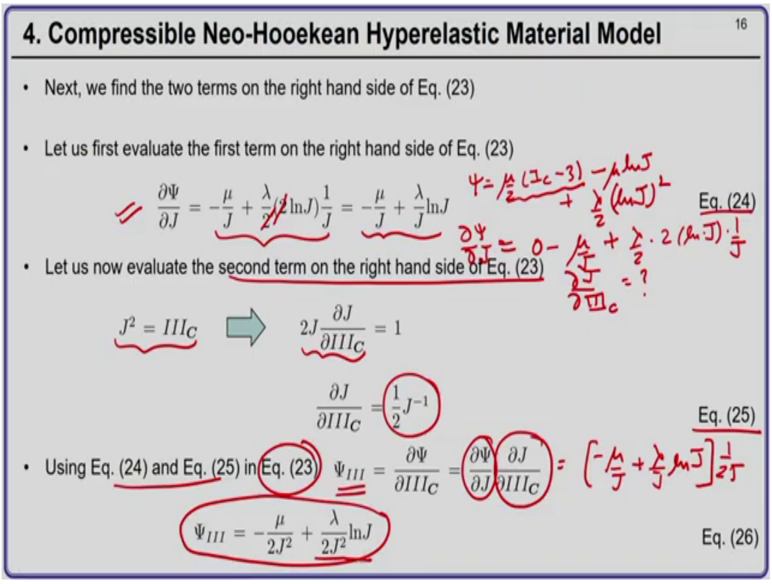
Now, the second Piola Kirchhoff stress tensor is nothing, but S equal to 2 del psi by del C and S we already have derived is nothing, but 2 psi 1 into I plus 4 psi 2 into C plus 2 J square psi 3 into C inverse and now, we have our expression for psi ok. We have the explicit expression for psi therefore; we can compute psi 1 psi 2 and psi 3 ok.

So, let us do it ok. So, psi 1 will be, because your psi is mu by 2 I C minus 3 minus mu l n J plus lambda by 2 l n J the whole square. So, del psi by del I C which is nothing, but psi 1 will be nothing, but mu by 2, because these two terms are independent of the first invariant. Only, the first term has the first invariant of the right Cauchy-Green tensor therefore, psi 1 will be equal to mu by 2 which is what we have here ok.

Now, we know that psi is independent of the second invariant ok, psi sorry if you see the expression over here there is no term for the second invariant which means that the derivative of the stored energy potential with respect to the second invariant will be equal to 0 ok.

Now, the third invariant we have the last two terms which are dependent on the third invariant of the right Cauchy-Green tensor. So, the third term psi 3 can be evaluated as del psi by del 3 C and using chain rule I can write del psi by del 3 C as del psi by del J into del J by del 3 C ok. Now, psi is explicitly ok. So, if you see here psi is explicitly a function of J therefore, the first term I can directly evaluate from the given stored energy density function ok.

(Refer Slide Time: 06:04)



So, we have to find the first term so del psi by del J ok. So, I write again, this is mu by 2 I C minus 3 minus mu l n J plus lambda by 2 l n J the whole square ok.

So, del psi by del J. So, the first term does not contain any J ok. So, that is 0 minus mu by J. So, del l n J by del J is mu by J plus lambda by 2 into 2 l n J into 1 by J ok. So, that is what you have here ok. So, if you simplify, this 2 goes away and then you have minus mu by J plus lambda by J into natural log of J ok.

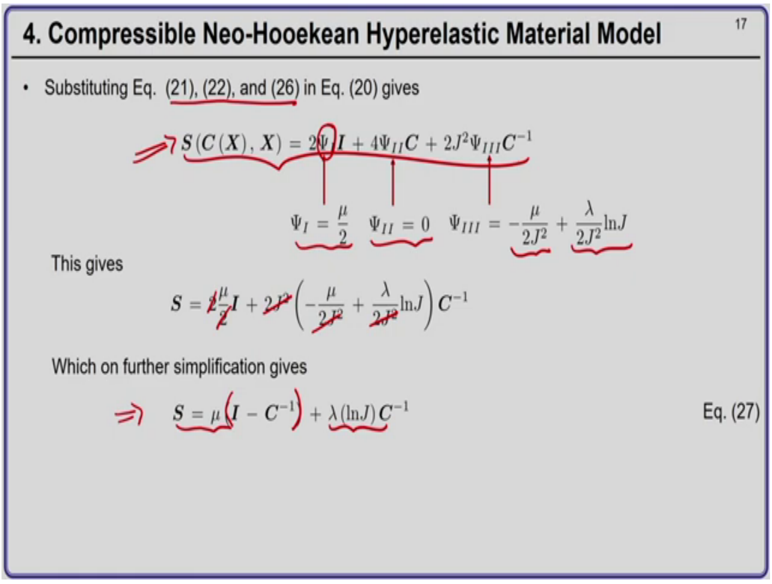
Now, let come to evaluation of the second term on the right hand side of equation number 23. The second term is nothing, but del J by del 3 C ok. Once, we have this term we will multiply equation 24 with that term and will get our expression for del psi by del 3 C ok.

So, let us see how to evaluate. We know that J square is 3 C. So, if I take derivative on both side with respect to the third invariant what I get? 2 J del J by del 3 C equal to 1 ok. So, therefore, del J by del 3 C will be nothing, but 1 by 2 J ok. So, now, if I use 25 and 24 in equation number 23, what I get?

The third derivative of the stored energy potential with respect to the third invariant which is del psi by del J which we just evaluated as minus mu by J plus lambda by J l n J into del J by del 3 C which is nothing, but 1 by 2 J.

So, once we do this we get psi 3 as mu by 2 J square plus lambda by 2 J square l n J the natural log of Jacobian J ok.

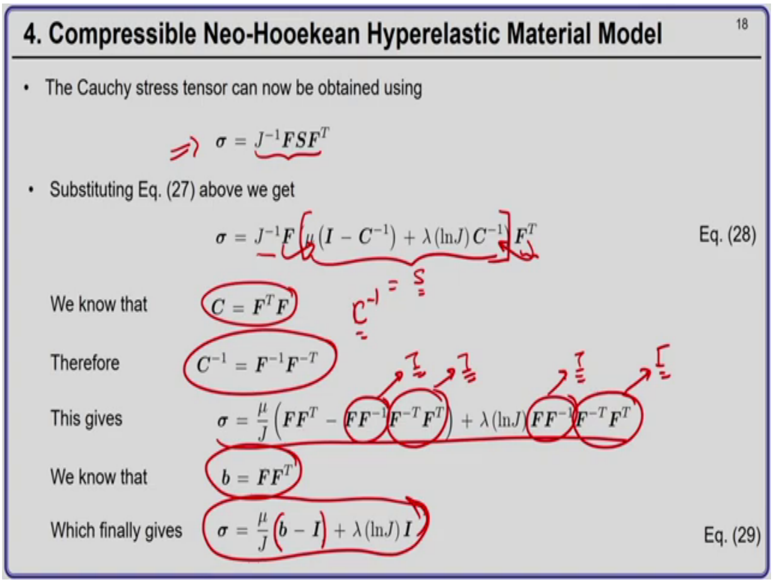
(Refer Slide Time: 08:41)



Now, we have all the three terms. So, what we do? We substitute 21, 22, and 26 in equation number 20 and equation number 20 is the expression for the second Piola Kirchhoff stress tensor and we get. So, psi by del I which is psi 1 is mu by 2 this term psi 2 is 0 and psi 3 is minus mu by 2 J square plus lambda by 2 J square natural log of Jacobian.

So, this we substitute and then we just simplify, take 2 J square inside and when we simplify we get the second Piola Kirchhoff stress tensor as mu times, the second order identity tensor minus C inverse plus lambda natural log of Jacobian J into C inverse. Once, we have the second Piola Kirchhoff we can; obviously, go ahead and compute the Cauchy stress tensor and that is what we are going to do next.

(Refer Slide Time: 09:49)



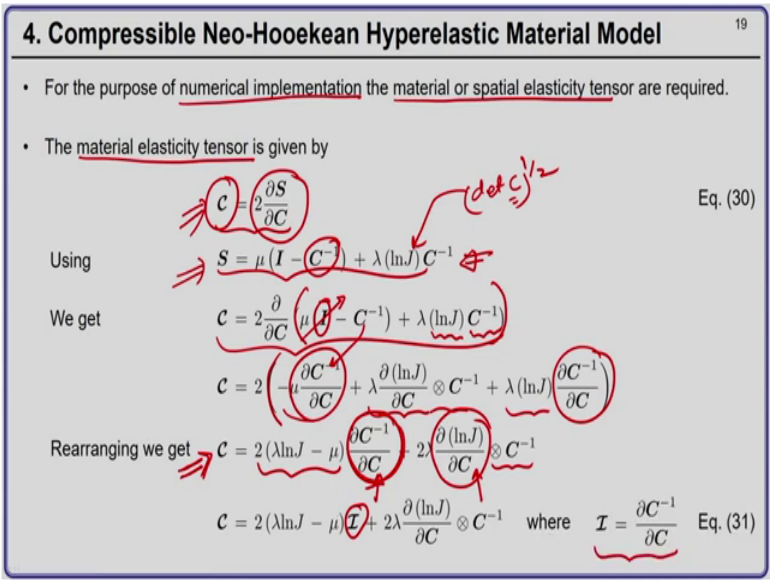
The Cauchy stress tensor can be obtained from the second Piola Kirchhoffs stress tensor using the following push forward operation. J inverse F S F transpose and we substitute S from the previous slide. So, this is your expression for S ok.

So, we get J inverse F into this is the expression for second Piola Kirchhoff stress tensor into F transpose and we take F inside and we also take F transpose inside and we know that C is F transpose F therefore; C inverse will be F inverse F inverse transpose ok.

So, what we get? This expression is what we get F F inverse is nothing, but identity F inverse F inverse transpose ok. F inverse transpose F transpose this is nothing, but identity this F F inverse is nothing, but identity and this F inverse transpose F transpose is nothing, but identity.

So, we simplify and we identify that the left Cauchy-Green deformation tensor is given by F F transpose. So finally, our expression for the Cauchy stress tensor is given by mu by J into b minus I plus lambda l n J into I ok. So, we can given the deformation we can directly compute the Cauchy stress tensor using equation 29 ok.

(Refer Slide Time: 11:40)



Now, for the purpose of numerical implementation ok. So, when you are going and implementing this material model on a computer you need to write a finite element code and you need to develop the Newton Raphson iteration technique. In, during that time we will be needing what is called the material or the spatial elasticity tensor. So, that is what we are going to derive next.

So, the material elasticity tensor is given by and remember this is a fourth order tensor is given by twice of the partial derivative of the second Piola Kirchhoff stress tensor with respect to the right Cauchy-Green deformation tensor. Now, we know the expression for the second Piola Kirchhoff stress tensor which is given here and, because this expression is a function of right Cauchy-Green deformation tensor, you directly have C inverse and J is nothing, but square root of determinant of C.

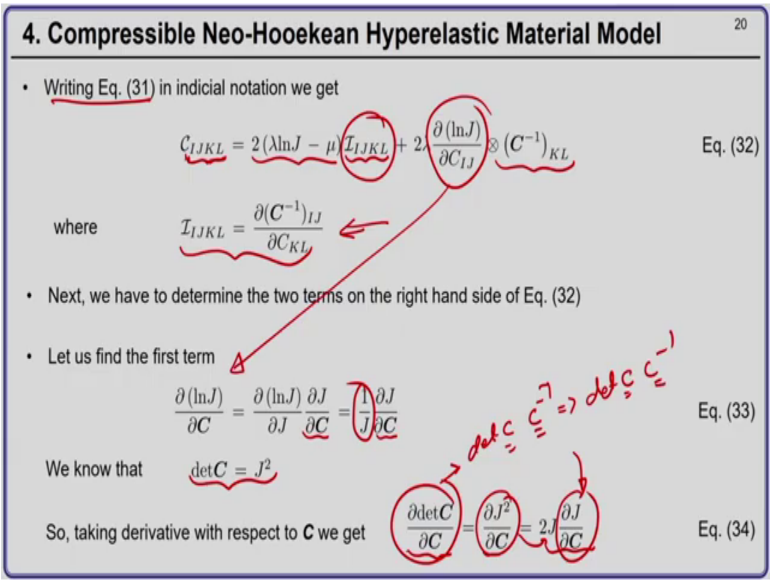
So, your right Cauchy-Green deformation tensor is present here. So, all you need to do is take the derivative of S with respect to right Cauchy-Green deformation tensor C to get the material elasticity tensor shown by equation number 30 ok.

So, if we substitute this is what we are going to get ok. So, this is what is your S if I open up the bracket I get twice of minus of the first term is independent of C. So, that does not contribute the second term gives you minus mu del C inverse by del C plus the third term. So, this term is a function of C and this also is a function of C.

So, first we have lambda l n J lambda del by del C of l n J tensor product C inverse plus lambda l n J del C inverse by del C ok. So, I can now take the terms of del C inverse by del C as one term and we get the fourth order material elasticity tensor as twice of lambda l n J minus mu del C inverse by del C plus 2 lambda del by del C of l n J tensor product C inverse.

Now, if I have to explicitly get the expression for the fourth order material elasticity tensor C, then I need to compute this term over here, and also need to compute this term over here ok. Let this term del C inverse by del C denoted by a fourth order tensor I ok. So, this I is del C inverse by del C.

(Refer Slide Time: 15:01)



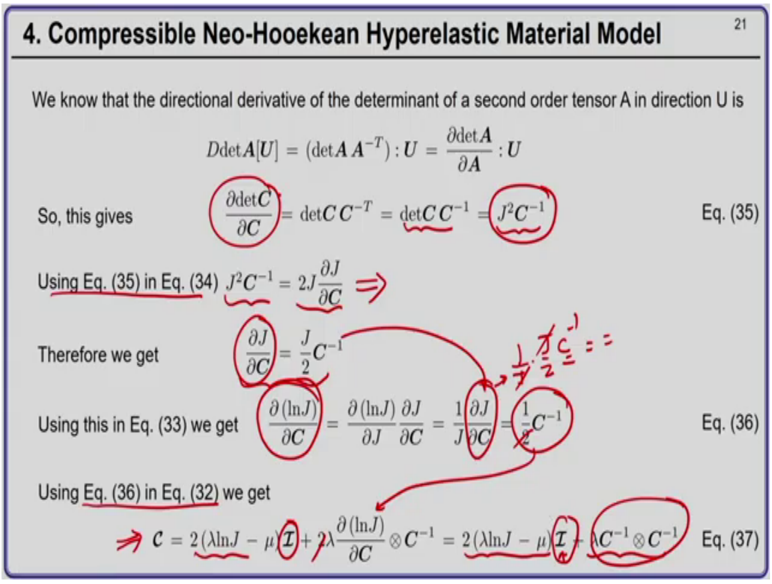
Now, I can write equation number 31 in indicial notation ok. Remember, it is a material elasticity tensor therefore, all the indices will be in the upper case. So, and it is also a fourth order tensor. So, there will be four indices.

So, we have C I J K L equal to twice of lambda l n J minus mu I I J K L plus 2 lambda del by del C I J of l n J tensor product C inverse K L, where the fourth order tensor I in indicial notation is given by this expression. So, we have to determine these two terms ok. These two terms need to be determined so that we can have the explicit expression for the fourth order material elasticity tensor.

Now, let us take this term; so del by del C of l n J is nothing, but del by del J of l n J into del J by del C. So, this is when we are applying the chain rule and del by del J of l n J is nothing, but 1 by J and then you have del J by del C and we know that determinant of C is J square therefore, if we take derivative on both side with respect to C, we have del by del C of determinant of C equal to del J square by del C and this is nothing, but 2 J del J by del C.

So, this is what we want to find out ok. So, to find out del J by del C, we need to compute del by del C of determinant of C and this we have already seen this is nothing, but determinant of C into C inverse transpose and, because C is symmetric, this will be determinant of C into C inverse ok.

(Refer Slide Time: 17:03)

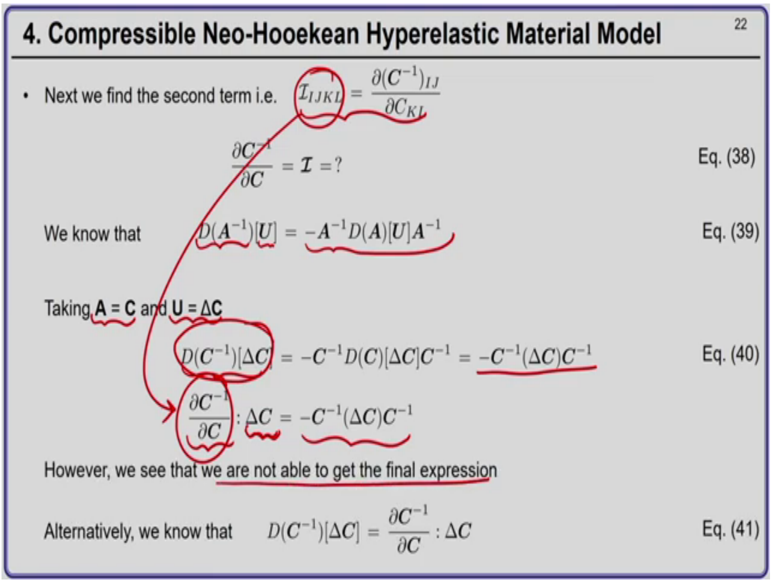


So, that is what we have ok. So, del by del C of determinant of C is nothing, but determinant of C into C inverse and determinant of C is nothing, but J square therefore, we have del by del C of determinant of C equal to J square C inverse. Using this equation in 34 we get J square C inverse is 2 J del J by del C and from here, I can directly compute del J by del C as J by 2 C inverse ok.

Now, I have del J by del C and then I can compute del by del C of l n J as 1 by J del J by del C and this I can substitute here. So, 1 by J into J by 2 C inverse which is; so J cancels out we get 1 by 2 C inverse ok. So, when you use equation number 36 in equation 32 you get the fourth order material elasticity tensor as twice of lambda l n J minus mu into fourth order tensor I plus 2 lambda del by del C l n J tensor product C inverse as twice of lambda l n J minus mu into fourth order tensor I plus lambda C inverse tensor product C inverse ok.

So, this here is 1 by 2 C inverse. So, 1 by 2 cancels out and this is what you are left. Now, we have the second term completely identified. We only are left to identify the fourth order tensor I and this is a little involved process.

(Refer Slide Time: 18:58)

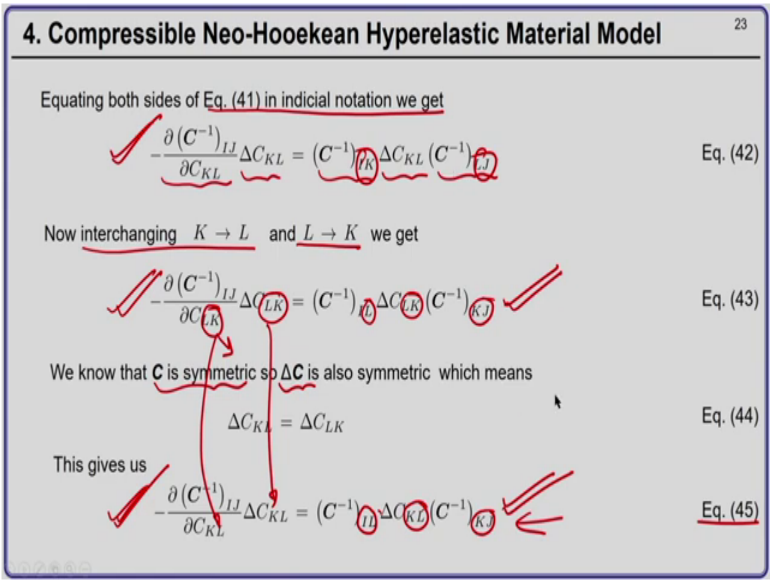


So, now let us concentrate on the second term ok. So, indicial notation the fourth order tensor I is nothing, but del C inverse I J by del C K L ok. So, we if you remember from our discussion, when we are discussing the mathematical basis of this course the directional derivative of the inverse of A tensor ok. In a general, direction U is nothing, but minus of A inverse directional derivative of the tensor A in the direction U multiplied by the inverse of the tensor A ok.

Now, if I take A as right Cauchy-Green tensor C and the general direction U is the change delta C of the right Cauchy-Green tensor, then I can identify the directional derivative of C inverse in the direction delta C as nothing, but minus of C inverse delta C C inverse and this directional derivative of C inverse in directional delta C can be written as del C inverse by del C double contracted with direction delta C and this is equal to minus of C inverse delta C into C inverse ok.

So, this here is what is our fourth order tensor I ok, but still we have this delta C which is not known ok. So, somehow we have to get rid of it and for this what we do is we see that we are not able to get the final expression.

(Refer Slide Time: 20:36)



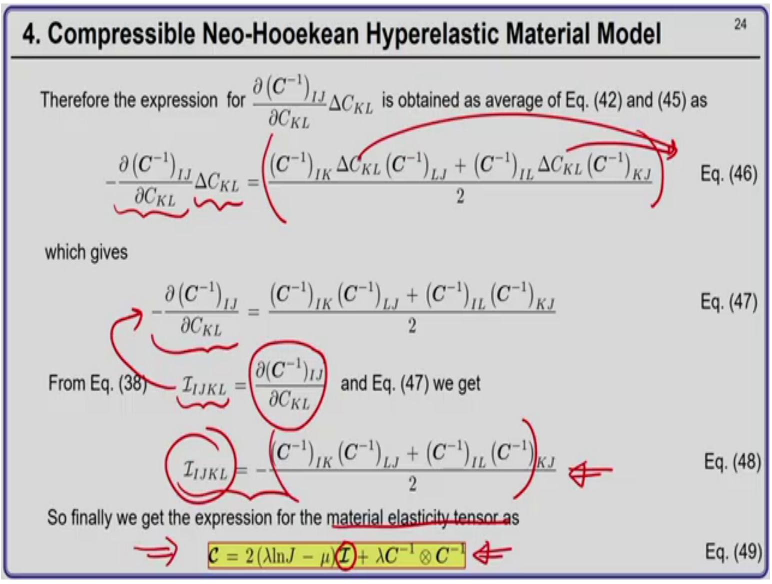
So, what we do now is we write both sides of equation 41 in indicial notation ok. So, we write minus of del C inverse I J by del C K L into delta C K L is C inverse I K delta C K L C inverse L J ok. Now, if I interchange K with L and L with K on both the sides ok, if I just interchange I get so, I have just interchanged K with L and L with K, you can see here I have interchanged ok.

Now, I know that C is symmetric and so, does the chain delta C will also be symmetric therefore, this C L K will be equal to C K L and delta C L K will be equal to delta L K ok. So, therefore, this again becomes K L and this again become K L and this becomes K L.

So now, I have another expression, before the first expression now just by interchanging K with L and L with K I get another expression given by equation number 45 ok. So, equation

42 and 45 are both same on the left hand side and we just have I L here, we have I K here, we have K J here, and we have L J ok.

(Refer Slide Time: 22:22)



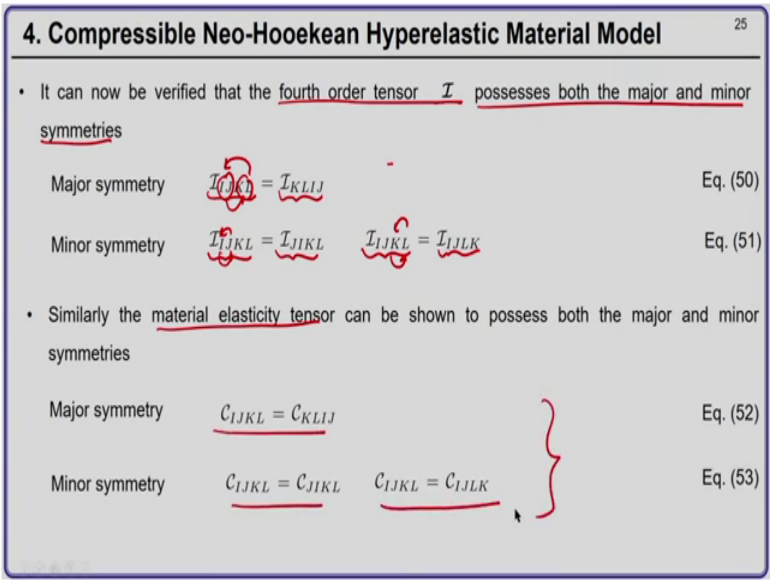
Which means if I add equation 42 and 45 and divide by 2 I can get minus del C inverse I J by del C K L into delta C K L as this term and then delta C K L can be taken out from both the sides.

So, it can be taken out and I get the expression for minus of del C inverse by del C as the average of C inverse tensor product C inverse.

Now, from equation 38 I know that the fourth order tensor I is nothing, but del C inverse by del C ok. So, this is nothing, but my minus of I ok. So, my I is nothing, but minus of this term, average of C inverse tensor product C inverse. So, finally, my expression for the material elasticity tensor will be given by C equal to 2 lambda l n J minus mu into fourth order tensor I plus lambda C inverse tensor product C inverse where I is given by following expression ok.

So, I can be computed if you already know C, you can compute C inverse and then once you have computed C inverse your fourth order material elasticity tensor can be computed ok.

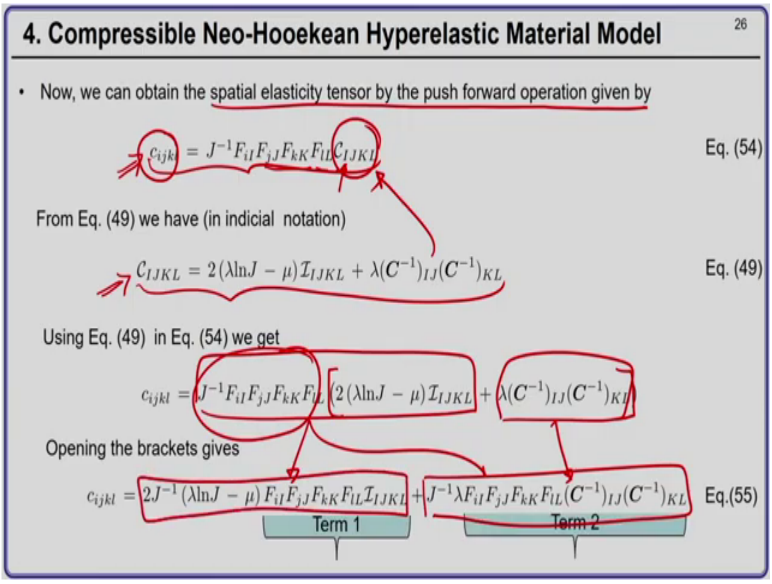
(Refer Slide Time: 24:00)



Now, we can show that the fourth order tensor I possesses both the major and minor symmetries. Major symmetry means I I J K L is nothing, but I K L I J ok. So, I J is brought here and K L is brought here this is called the major symmetry. It also posses the minus symmetry which means I I J K L is equal to I J I K L and I I J K L is equal to I I J L K ok.

So, here we are just in the first, we are just replacing I with J J with I and here we are replacing K with L and L with k ok. So, it can be shown that the fourth order tensor I possesses both the major and minor symmetry. Similarly, because I posses both the major and minor symmetry you can show that the material, fourth order material elasticity tensor C also posses both the major and the minor symmetries ok, which are shown in equation 52 and 53 ok.

(Refer Slide Time: 25:15)

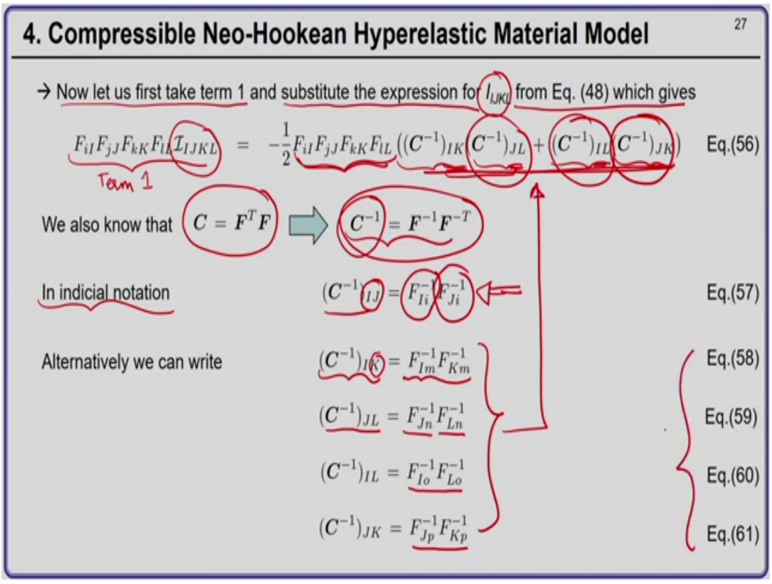


Now, we can obtain the spatial elasticity tensor by the following push forward operation ok. So, this relation we have already derived in our previous module, where the fourth order spatial elasticity tensor C is nothing, but J inverse F i I F j J F k K F l L into the fourth order material elasticity tensor C I J K L and now, I can substitute C I J K L here and then I can simplify to get my final expression for c i j k l ok.

To do this first notice, this is the indicial notation for the fourth order material elasticity tensor. Now, if I substitute it here I will just have to simplify ok. So, that is what I do I substitute equation 49, in equation 54 and then I will just have to simplify ok.

So, if I just open up the brackets so, this is the first term ok. So, this is the first term and then this term into this term over here gives me the second term. So, I have two terms which I have to simplify. So, this is term 1 and this is term 2 and these two terms I have to simplify now.

(Refer Slide Time: 26:54)



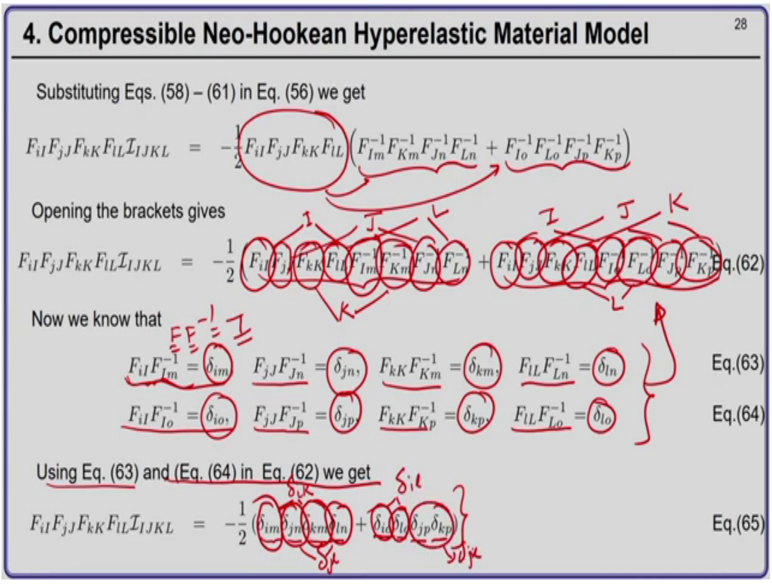
Now, let us take term one and then let us now substitute the expression for the fourth order tensor I I J K L from equation number 48 ok. So, this was our term 1 and when we substitute I J K L from equation 48 so, we have minus 1 by 2 C inverse I K C inverse J L plus C inverse I L C inverse J K and this is the four four deformation gradient tensors.

Now, we know that in direct notation the right Cauchy-Green tensor C is given by F transpose F ok. So, C inverse can then be written as F inverse, F inverse transpose. So, let us now write this expression in indicial notation ok. Remember, C is a material tensor which means it will have upper case indices when written in indicial notation. So, C inverse I J will be F inverse I i F inverse J i ok. Now, notice we have C inverse I K and C inverse J L C inverse I L C inverse J L.

So, in equation 57 what we have to do is we have to make equation 57 consistent with these four expression for C inverse ok. So, the first one is C inverse I K. So, equation 57, if we replace J by K then our expression for C inverse I K becomes F inverse I m F inverse K m then C inverse J L which is here will become F inverse J n F inverse L n then C inverse I L which is here will become F inverse I o F inverse L o.

And then finally, the last expression C inverse J K will become F inverse J p F inverse K p ok. So, now, these four expressions for C inverse we can substitute in equation number 56. So, 58 to 61, we substitute in equation number 56 and then let us simplify ok.

(Refer Slide Time: 29:35)



So, that is what we have substituted. So, now, what we do is take these four deformation gradient tensors inside the bracket ok. So, will open up the bracket and will take these four deformation gradient tensors inside and then we will get this long expression.

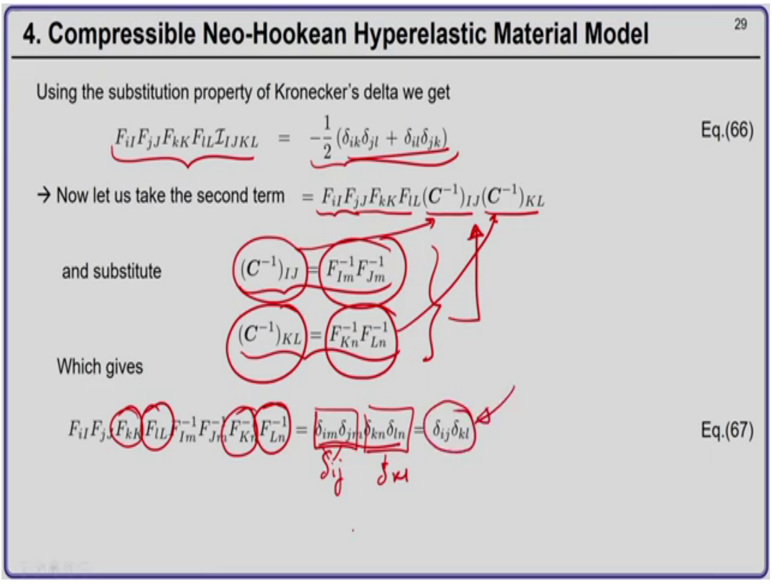
So, these they seem to be pretty long expressions and pretty intimidating, but; however, if you look closely you will find that you have F i I F inverse I m. So, these two indicial notations have I common ok.

Then if you see; so, here you have I common if you see F j J and if you see F inverse J n, you have J common, if you see F k K and F k F inverse K m, you have capital K common and if you say F l L and F l inverse you have L common ok. Similarly, here you have F i I F inverse I o again you have I common, you have F j J, you have F inverse J p you have J common, you have F k K, you have F inverse K p, you have capital K common and if F l L F inverse L o you have L common ok.

So, if you write these terms now, F i I F inverse I m indirect notation this is nothing, but F F inverse which is nothing, but identity. So, therefore, F i I F inverse I m will become nothing, but delta i m that is Kronecker delta, F j J F inverse J n will become delta j n, F k K F inverse K m will become delta k m, F l L F inverse L n will become delta l n, F i I F inverse I o become delta i o, F j J F inverse J p becomes delta j p, F k K F inverse K P becomes delta k p, F L K F inverse L o becomes delta l o ok. So, from equation 63 and 64 we substitute in equation 62 ok.

So, using equation 63 and 64 in equation number 62 what we get we get minus 1 by 2 delta i m delta j m delta k m delta l n plus delta i o delta l o delta j p delta k p ok. Now, again we notice that the following expression can be further simplified, if we use the substitution property of Kronecker delta which means if you look delta i m and delta k m this will become nothing, but delta i k and delta j m delta l n this will become delta j l, delta i o delta l o will become delta i l and delta j p delta k p will become delta j k ok.

(Refer Slide Time: 33:10)



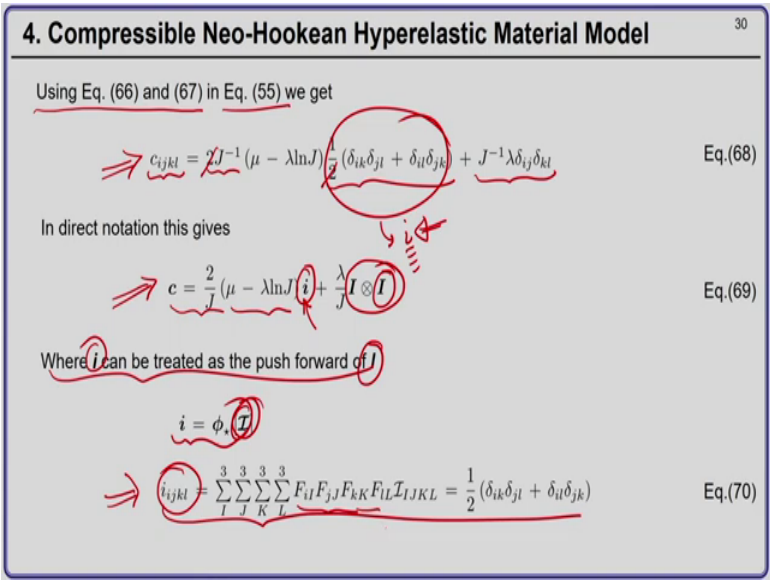
So, further simplifying we get our first term entirely in terms of Kronecker delta as minus 1 by 2 delta i k delta j l plus delta i l delta j k ok. Now, we take the second term. Now, the second term is F i I F j J F k K F l L C inverse I J C inverse K L ok. Now, again I can write C inverse in indicial notation ok.

So, C inverse I J which is here is nothing, but F inverse I m F inverse J m and C inverse K L is nothing, but F inverse K n F inverse L n ok. So, this is here. So, now, we substitute these two expressions here and then what we get? We get following expressions.

So, this is already there and this comes from the first C inverse I J and this comes from C inverse K L and now we notice that F i I F inverse I m will become delta i m F j J F inverse J m that will become delta j m and F k K F inverse K n will become delta k n and F inverse L l and F inverse L n that will become delta l n ok.

Now, in the following expression we can again use the substitution property. So, if you look these two Kronecker deltas, this will become delta i j, because m is common and this will become delta k l, because n is common. So, finally, term two becomes delta i j delta k l ok.

(Refer Slide Time: 35:12)

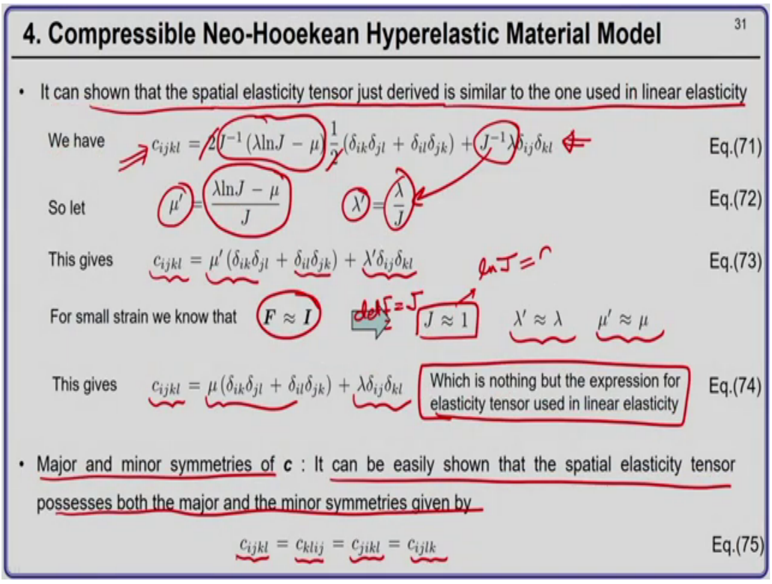


So, now if we use equation 66 and 67 in equation 55, we get our fourth order spatial elasticity tensor c i j k l as 2 J inverse mu minus lambda l n J into 1 by 2 delta i k delta j l delta plus delta i l delta j k plus J inverse lambda delta i k delta i j delta k l.

So, this 2 cancels out or we can write this itself as a fourth order tensor i that is the fourth order identity tensor i. So, in that case two will remain and then indirect notation I can write my fourth order, spatial elasticity tensor c as 2 by J mu minus lambda l n J fourth order identity tensor i plus lambda by J and tensor product of second order identity tensor I.

Now, where we can treat this i that fourth order identity tensor has it push forward of the fourth order tensor I ok. So, this i is the push forward of capital I and indicial notation this is given by following expression ok. So, i i j k l is nothing, but F i I F j J F k K F l L I I I I J K L equal to 1 by 2 delta i k delta j l delta i l delta j k.

(Refer Slide Time: 37:00)



So, now we have shown that the spatial elasticity tensor we have just derived [nose] is similar to 1 which is can used in linear elasticity ok. So, what it means is the spatial elasticity tensor that we have used we have derived here under spatial condition will give you the fourth order elasticity tensor that is usually used in linear elasticity ok. So let us now let us see how that can be done ok.

So, remember in indicial notation the fourth order spatial elasticity tensor is given by following expression. Now, let us say we denote mu lambda l n J by J as mu dash and lambda dash as lambda by J ok. So, this becomes lambda dash and this term over here ok. So, now this term over here becomes mu dash ok.

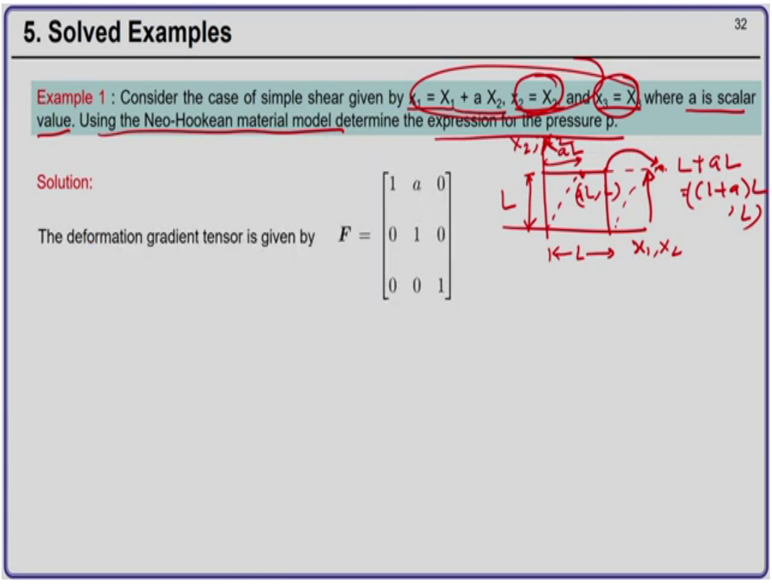
So, then we can write c i j k l as mu dash delta i k delta j l plus delta i l delta j k plus lambda dash delta i j delta k l. Now, suppose you are dealing with small strain and that is what linear elasticity means. Linear elasticity means your deform and the undeformed configurations are very close to each other which means the fourth that the deformation gradient tensor F is nearly equal to identity ok.

If F is nearly equal to identity this implies that the Jacobian which is nothing, but determinant of F ok, determinant of deformation gradient tensor F is nearly equal to 1 ok. If J is nearly equal to 1 it means l n J nearly becomes 0 and this J is nearly equal to 1 which means mu dash will be equal to mu and lambda dash, because J is nearly equal to 1 then lambda dash becomes equal to lambda ok. Now, if you substitute this both here then we get the fourth order spatial elasticity tensor as mu lambda delta i k delta j l plus delta i l delta j k plus lambda delta i j delta k l and now, here mu and lambda as your Lame's constant ok. They are Lame's constants and then if you recognize this equation over here, equation 74 has a same form as that used in linear elasticity ok.

So, the Neo-Hookean material model unders very small strain is similar to linear elasticity. So, that is what we have written here equation 74 is nothing, but the expression for elasticity tensor used in linear elasticity. Now, we notice that the fourth order spatial elasticity tensor c possesses both the major and minor symmetries ok.

So, it can be easily shown that the spatial elasticity tensor possesses both the major and minor symmetries which mean that c i j k l equal to c k l i j that is the major symmetry and c j i k l is c i j l k that is the minor symmetry ok. So, these major and minor symmetries help you to reduce the number of independent elastic constants.

(Refer Slide Time: 40:53)



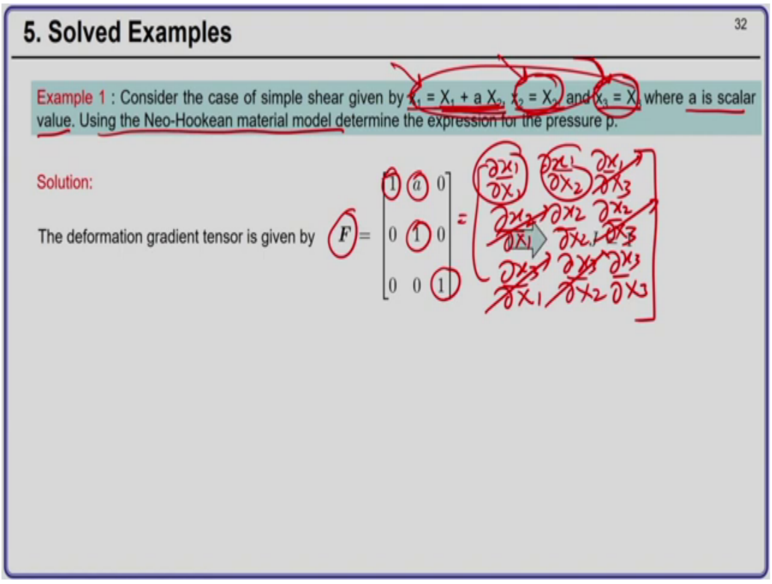
Now, let us see one example ok. So, now, consider the case of a simple shear which is given by x 1 equal to capital X 1 plus a into capital X 2 small x 2 is capital X 2 ok, x 3 is capital X 3 and where a is a scalar value ok. So, this simple shear; so you can identify this if you have a block. So, you have a block of certain length say L and if you apply this simple shear ok.

So, what will happen it will deform something like this ok. So, this will become a times if say this is equal to L. Let a square, this will be a times L and this will go here and this will be nothing, but your L plus a times L which is 1 plus a L ok.

So, this point coordinate becomes a L into L and coordinate of this point becomes 1 plus a into L comma L. So, there is no deformation in the second direction, because x 2 is capital X 2 and there is no deformation in the third direction, because x 3 is capital X 3 ok.

Now, if you have a Neo-Hookean material model you have to determine, you have been asked to determine the expression for the pressure p ok. So, how do you start the first thing we have to do is we have to compute the. So, let me rub this. So, the first thing you have to compute is the deformation gradient tensor F ok.

(Refer Slide Time: 43:02)

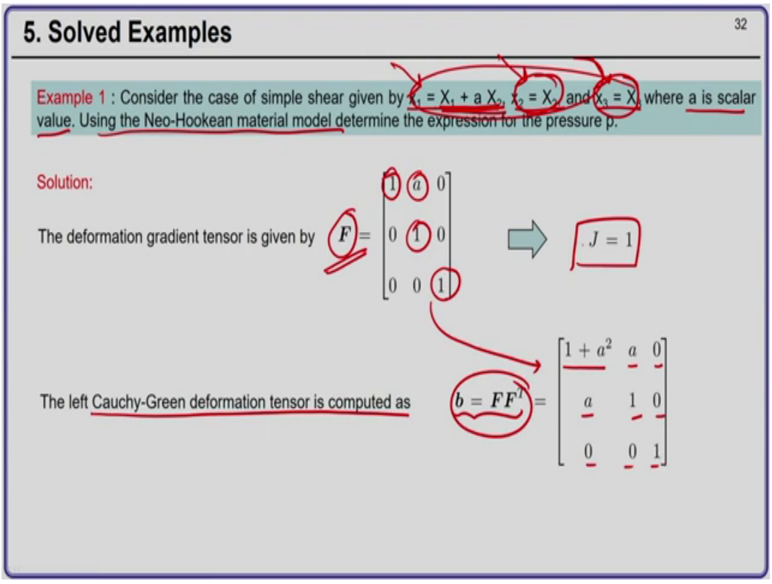


So, deformation gradient tensor is nothing, but del x 1 by del capital X 1 del x 1 by del capital X 2 del x 1 by del X 3 del x 2 by del X 1 del x 2 by del X 2 del x 2 by del X 3 del x 3 by del X 3 ok.

So, del x 3 by del X 1 del x 3 by del X 2 and del x 3 by del X 3 and you have been given small x 1 small x 2 small x 3 this is the spatial coordinates in terms of the material coordinates x 1 x 2 x 3. So, del x 1 by del X 1 is nothing, but 1, del x 1 by del X 2 from this is a and rest all 0. Similarly, this term is 0 and del x 2 by del X 2 is 1, del x 3 by del X 3 is 0, del x 3 by del X 1 is 0, del x 3 by del X 2 is 0 and del x 3 by del capital X 3 is equal to 1. So, once you have the deformation gradient tensor, you can compute the Jacobian ok.

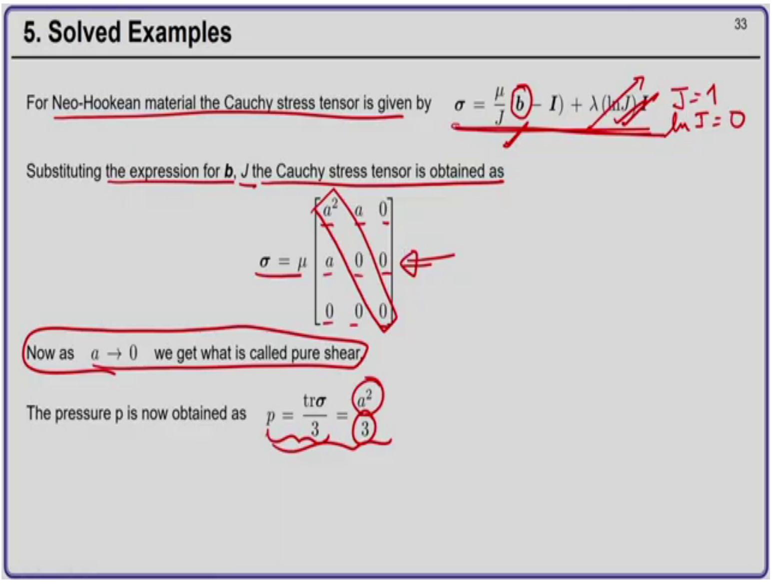
If you compute the Jacobian, if you take the determinant of F, you see that Jacobian is equal to 1 which means a simple shear is a isochoric deformation ok.

(Refer Slide Time: 44:38)



So, next you can determine the left Cauchy-Green deformation tensor which is nothing, but b and is equal to F into F transpose. We already have computed F. So, we can compute F into F transpose which is nothing, but 1 plus a square a 0 a 1 0 0 0 1.

(Refer Slide Time: 45:10)



So, now, once you have the left Cauchy-Green deformation tensor, I know my relation for the Cauchy stress tensor and the relation for the Cauchy stress tensor is given by following relation sigma equal to mu by J b minus I lambda l n J into second order identify tensor I ok.

So, remember J is equal to 1 therefore, l n J will be equal to 0. So, this term will drop out and I have already computed b so, I can substitute b and I can simplify. We substitute for b and J and we can get the Cauchy stress tensor as sigma equal to mu times a square a 0 a 0 0 0 0 0 that is your Cauchy stress tensor.

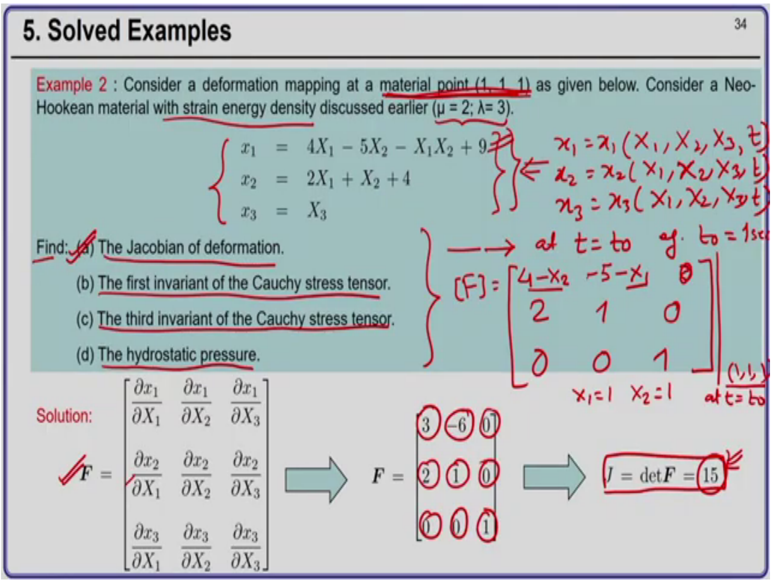
Now, pressure is nothing, but so, pressure is nothing, but trace of sigma by 3 ok. So, I can take the trace is nothing, but sum of the diagonal component which is a square plus 0 plus 0 which is nothing, but a square divided by 3. So, pressure will be equal to a square by 3 that is our answer ok.

We can also note that as a approaches 0, we get what is called pure shear ok. So, with this example you could see that as soon as you are given the deformation mapping you can compute the deformation gradient tensor F. In the finite element context you will be given the deformed coordinates and using the deformed coordinates and the initial coordinates you can determine the deformation gradient tensor F at the particular Gauss point.

Once, you have determined the deformation gradient tensor F you can determine the Jacobian which is nothing, but the determinant and also you can determine b, the left Cauchy-Green deformation tensor which is nothing, but b equal to F F transpose and then using the relation for the Cauchy stress, you can directly compute the Cauchy stress tensor by substituting for left Cauchy-Green deformation tensor b and the Jacobian J and once you have substituted you will get the Cauchy stress tensor at a particular Gauss point.

Once you have the Cauchy stress tensor you can compute the internal forces and once you have the internal forces you can calculate the residual which is required for the Newton Raphson iteration ok.

(Refer Slide Time: 47:48)



So, the next example that we consider is as follows. So, you consider a deformation mapping at material point 1 1 1 as its given here.

Now, considering a Neo-Hookean material; that we have derived constitutive relation just now with the strain energy density potential, with the material constants mu and lambda given by 2 and 3 respectively ok. With this information in hand you have to find the Jacobian of deformation, the first thing variant of the Cauchy stress tensor and the third invariant of the stress tensor and also you have to find the hydrostatic pressure ok.

So, the significance of this problem is the steps that will carry out to find out a, b, c, and d are exactly what we need to implement inside a finite element code. So, the first thing when you have a mapping is to find out the deformation gradient tensor ok.

So, for the first solution you have to find out the deformation gradient tensor and that is given by del x 1 by del capital X 1 del x 1 by del capital X 2 del x 1 by del capital X 3 del x 2 by del X 1 del x 2 by del X 2 del x 2 by del X 3 del x 3 by del X 1 del x 3 by del X 2 del x 3 by del X 3 ok. So, now, from this deformation mapping you have been given x 1 as a function of material coordinates X 1, X 2, X 3 and all though not shown here this deformation mapping may also be a function of time ok.

So, you have x X 1, X 2, X 3 material coordinates and time possibly, then x 3 is a function of X 1, X 2, X 3 and time. So, suppose you are given a problem where there was also a term containing time or the time was explicitly present in the deformation mapping. At that time you would be asked to find out all these quantities at a particular time t equal to t 0 for example, you might be given t 0 equal to 1 second or 1. So, at that time t equal to 1 and for material point say 1, 1, 1 you need to find out all these values ok.

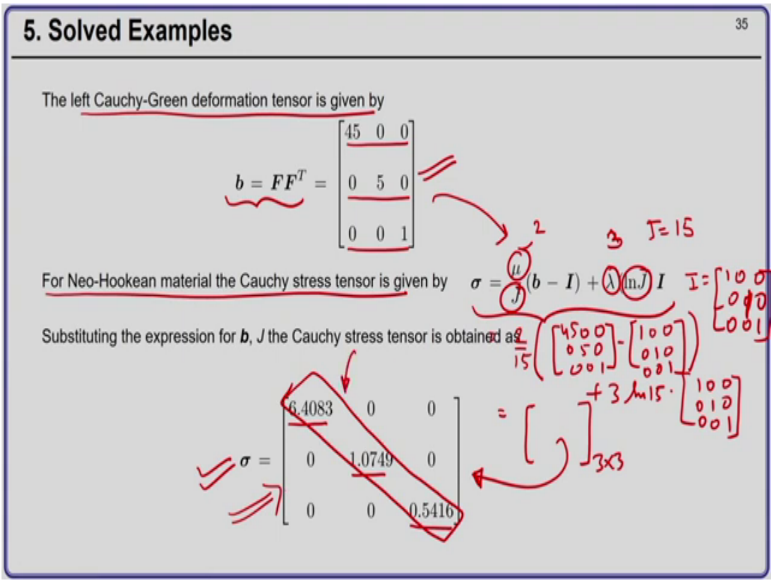
So, first thing is you find out this deformation gradient tensor and this deformation gradient tensor, you can see will be del x 1 by del X 1 which is 4 minus X 2 then del x 1 by del X 2 will be minus 5 minus X 1 then del x 1 by del X 3 is 0, because this term is independent of the third spatial direction x 3 for this is 0, then you have del x 2 by del X 1, del x 2 by del X 1 is 2, del x 2 by del X 2 that is equal to 1, del x 2 by del X 3 equal to 0, del x 3 by del X 1 0, del x 3 by del X 2 0 and del x 3 by del X 3 equal to 1 and now you have been given the material particle. So, the material particle or a position at 1, 1, 1.

So, you have to evaluate this at 1, 1, 1. Now, if you are given time also you would have to evaluate this at that particular material particle at that particular time instance t 0. So, you have to both substitute the material point as well as the time t. .

Now, if you substitute so, you have been given X 1 as 1 and X 2 as 1 we do not require X 3, if you substitute X 2 as 1 here, you get 3. If you substitute X 1 as 1 here, you get minus 6, you get 0 ok. You get 2 1 0 0 0 1 that is your deformation gradient tensor. So, once you have deformation gradient tensor you can compute the Jacobian. So, the Jacobian is given by determinant of F and you can compute the determinant of F which comes out to be 15. It will never be this high in actual situation, but this is a hypothetical case.

So, say it is that high of 15. So, the first point is solved. Now, we need to find out the first invariant and the third invariant of the Cauchy stress tensor in the hydrostatic pressure and all these three invariably depend on the Cauchy stress tensor ok. They have to be derived from Cauchy stress tensor. So, what we need to first find out is the Cauchy stress tensor and for a compressible Neo-Hookean material we have already derived the expression for the Cauchy stress tensor in terms of the left Cauchy-Green deformation tensor b, b is given by F F transpose. So, we have already calculated F here ok.

(Refer Slide Time: 53:58)



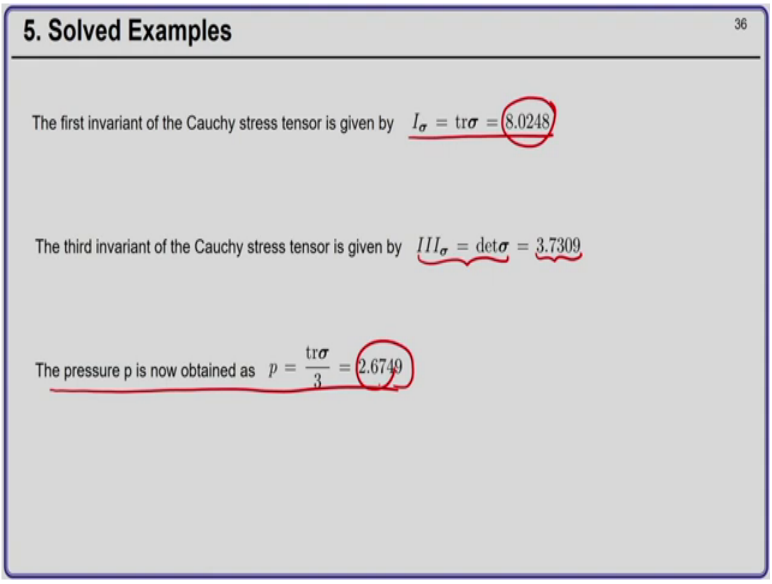
So, the next thing is we compute the left Cauchy-Green deformation tensor and which is given by b equal to F F transpose and we when we compute this we get 45 0 0 0 5 0 0 0 0.

Once, we have computed the left Cauchy-Green deformation tensor, we can compute for this Neo-Hookean material the Cauchy stress tensor from this particular equation, where mu and lambda are the material constants which are given to us mu is 2 lambda is 3, J we have already computed ok. So, J was so, this is 3, J was 15, I is a second order identity tensor. So, this will be 1 0 0 0 0 1 0 0 0 1 0 0 0 1 and b is here.

So, all we need to do is just substitute all these values here. So, will be 2 by 15 in the bracket 45 0 0 0 5 0 0 0 1 minus 1 0 0 0 1 0 0 0 1 plus 3 times natural log of 15 into 1 0 0 0 1 0 0 0 1. So, once you do this you will get the Cauchy stress tensor as a 3 by 3 matrix, you will get all the six components ok. So, once you do this, you get all the six components as 6.408000 1.0749000 0.5146.

So now, you have the Cauchy stress tensor and now the three parts that we have to evaluate are nothing, but the first invariant which is nothing, but the trace of sigma which means it is the sum of the diagonal components. The third invariant is nothing, but the determinant of the Cauchy stress tensor and, because this matrix is a diagonal matrix then the determinant is product of the diagonal term and the pressure is nothing, but the trace of sigma divided by 3 ok.

(Refer Slide Time: 56:31)



So, the first invariant is nothing, but trace of sigma and which comes out to be 8.0248. The third invariant is nothing, but the determinant of sigma which is the product of the diagonal component, because our sigma was a diagonal matrix and this comes out to be 3.7309 and the pressure is now evaluated as trace of sigma by 3, which is the first invariant divided by 3 which is 2.6749 ok.

So, in a similar way when you are coding in a finite element setup at every Gauss point so what will happen in a finite element code instead of that material point, you will have to take the Gauss point coordinate and at time you have to take the that particular instance of time when you are have to compute all these quantity.

So, the way you will approach is from the coordinates you have to first find out the deformation gradient tensor F, then you check or you compute the Jacobian J and you check for whether Jacobian is less than a certain small number and then you go on to compute the Cauchy stress tensor and you compute all the quantities ok. You also have to compute the internal force that is what we will discuss in the next module ok.

So, with this we come to the end of this module and next module we move to linearization of the equilibrium equation and taking the directional derivative of the principle of the virtual works expression which will be essential for getting the tangent matrices for setting up the Newton Raphson iterative procedure ok.

Thank you.