Computational Continuum Mechanics
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Hyperelasticity - 2
Lecture – 23 - 24
Isotropic Hyperelasticity and Compressible Neo-Hookean Material

So, in the next two lectures we are going to deal with Isotropic Hyperelasticity and Compressible Neo-Hookean Material.

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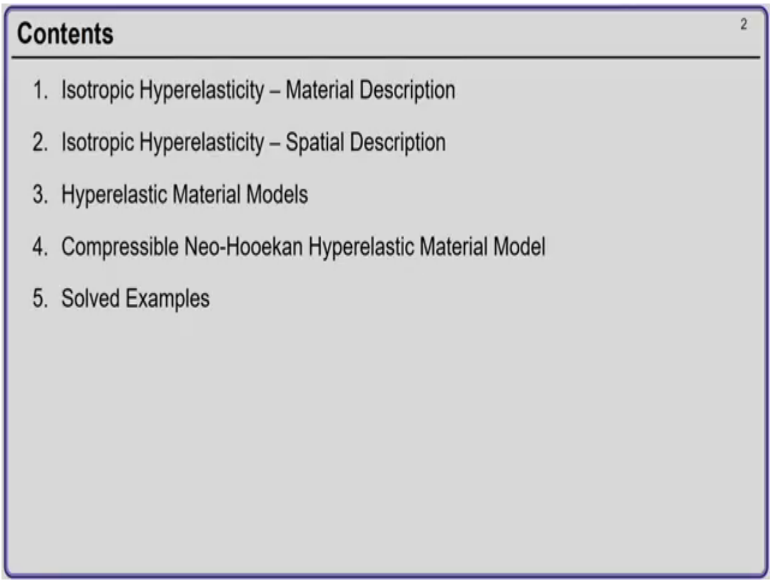

So, following is the content of the next two lectures. We will first discuss the isotropic hyper-elasticity in the material description, followed by in the spatial description, ok. And, then we will discuss some of the hyper-elastic material models that are usually employed. Finally, for one particular type of compressible hyper-elastic material model, that is

Neo-Hookean material model, we will derive the constitutive relation and also the material or the spatial elasticity tensor, which is required for the numerical implementation, ok.

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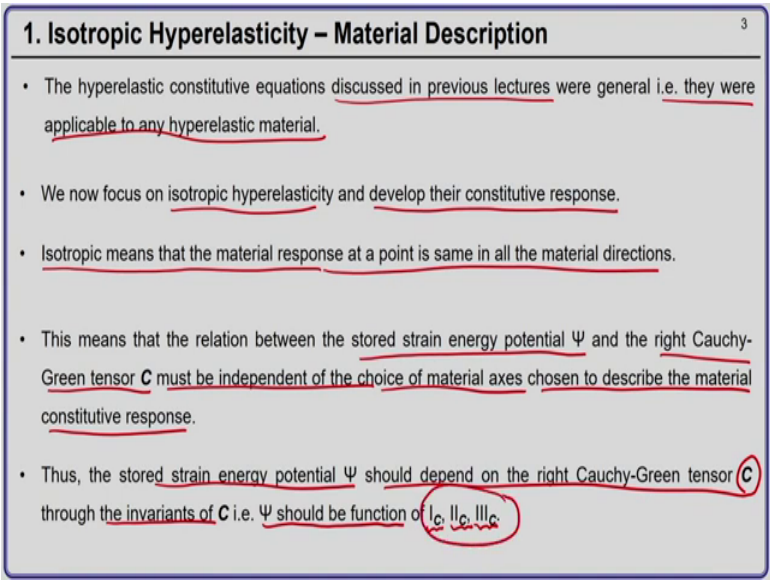

So, the hyper-elastic constitutive equations that we discussed in the previous lectures, they were all general ok, that is, they were applicable to any hyper-elastic material. But now, we focus on isotropic hyper-elasticity and we will try to develop their constitutive response; that is, the relation between the stress and the strain.

So, isotropic means that the material response at a point is same in all the material directions, ok. So, this means that the stored strain energy potential psi and the right Cauchy Green deformation tensor C must be independent of the choice of material axes ok, which is used to describe the material constitutive response, ok. So, the relation between psi and C must be independent of the choice of material axes.

What this actually means is, that the strain energy potential psi should depend on the right Cauchy-Green deformation tensor C through the invariants of C, that is psi should depend on the first invariant of the right Cauchy-Green deformation tensor, the second invariant of the right Cauchy-Green deformation tensor and the third invariant of the right Cauchy-Green deformation tensor. And, from our discussion, earlier that we had, we know that the three invariants of a 2nd order tensor ok, their magnitude do not change with the change in the coordinate system.

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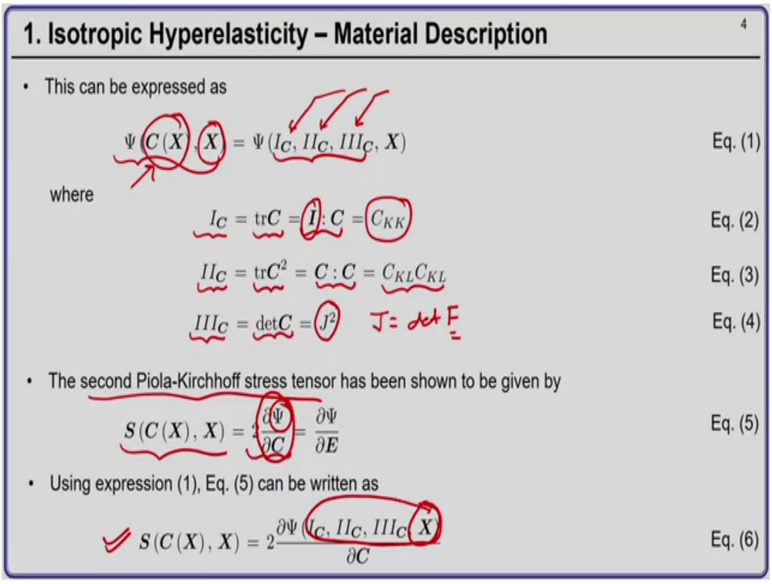

So, what it actually means is, mathematically psi which has to be a function of right Cauchy-Green deformation tensor, and the material particle X should indeed depend on C through its three invariants, ok. So, the three invariants here, contain C ok. So, we define the first invariant of right Cauchy-Green deformation tensor as the trace of right Cauchy-Green deformation tensor C or the double contraction of C with the 2nd order identity tensor. This in turn can be written in indicial rotation as C KK, ok.

The second invariant of the right Cauchy-Green deformation tensor is defined as trace of C square ok, which is nothing but the double contraction of C with itself, ok. And this in indicial notation can be written as C KL into C KL, ok. And, the third invariant of the right Cauchy-Green deformation tensor is nothing but the determinant of the right Cauchy deformation tensor and which is nothing but, J square; where J is the Jacobean and J is nothing but, the determinant of deformation gradient tensor.

So, the second Piola Kirchhoff stress tensor, we have already seen in our previous module that is can be derived as twice the partial derivative of strain energy density potential psi with respect to the right Cauchy-Green deformation tensor or del psi by del E, where E is the Green Lagrange strain tensor ok. So now, if you substitute equation 1 in equation 5 here, what you get? You get the second Piola Kirchhoff stress tensor is nothing but, twice of del psi by del C and psi is function of the three invariants and the material particles position.

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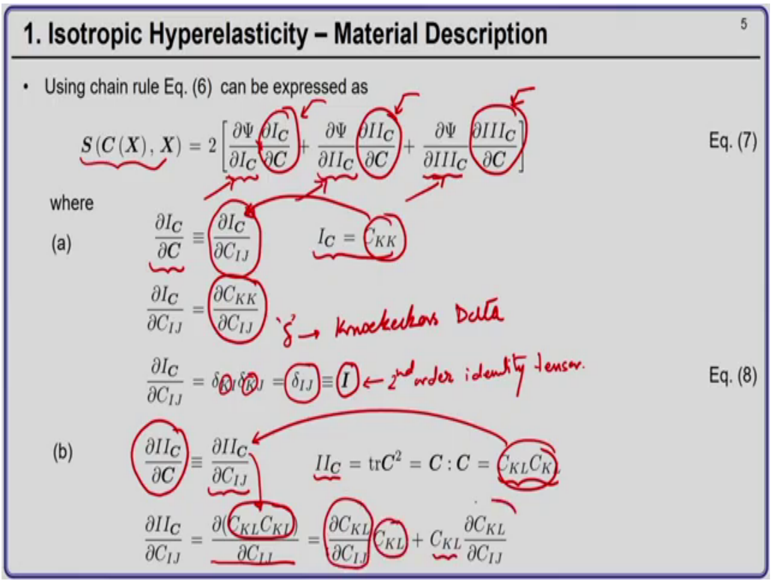

So, now I can use chain rule in equation 6 and then I can express the second Piola Kirchhoff stress tensor as twice of del psi by del I C ok. So, because psi depends directly on the first invariant, the second invariant, and the third invariant of the right Cauchy-Green tensor. Therefore, we will have del psi by del I C and into del I C by del C plus del psi by del 2 C into del 2 C by del C and del psi by del 3 C into del 3 C by del C, ok. This is the simple chain rule which has been applied, ok.

Now, because psi has been given in terms of I C, 2 C and 3 C that is the three invariants, it is easy to compute these terms, ok. These three terms can be easily computed. The problem is, we have to compute the derivatives of these three invariants of the 2nd order tensor C with respect to the tensor C itself, ok. Now, let us see how to do that, ok.

So, the derivative of the first invariant of the right Cauchy Green tensor with respect to the right Cauchy Green tensor can be written in indicial form as del I C by del C I J. Where the first invariant is nothing but, the trace in indicial notation is nothing but C K K ok. So now, I can substitute this here, and then what I have is del C K K by del C I J.

Now, I know that, del C KK by del C IJ is nothing but, del KI into del KJ, where del is the kronecker delta, ok. So, this del is nothing but the, kronecker delta. So, now, I can use the substitution property of the kroneckers delta. So del KI by del KJ, ok. So, if I have this, so because K is common, if you remember the substitution property the common symbol drops out and we get delta I J. In direct notation delta I J is nothing but the, 2nd order identity tensor ok. So this I is nothing but the, 2nd order identity tensor.

Now, we come to the second term which is the derivative of the second invariant of the right Cauchy-Green tensor with respect to right Cauchy-Green tensor. So, this in indicial notation is del 2 C by del C I J ok. And, then 2 C in indicial notation is C KL into C KL ok. So, that is what we are going to do, we are going to substitute it here, ok. So, when you substitute you get this equation and then you take the derivative. So, the derivative of the, so because this is product, so you have the derivative of the first term into the second term plus the first term into the derivative of the second term.

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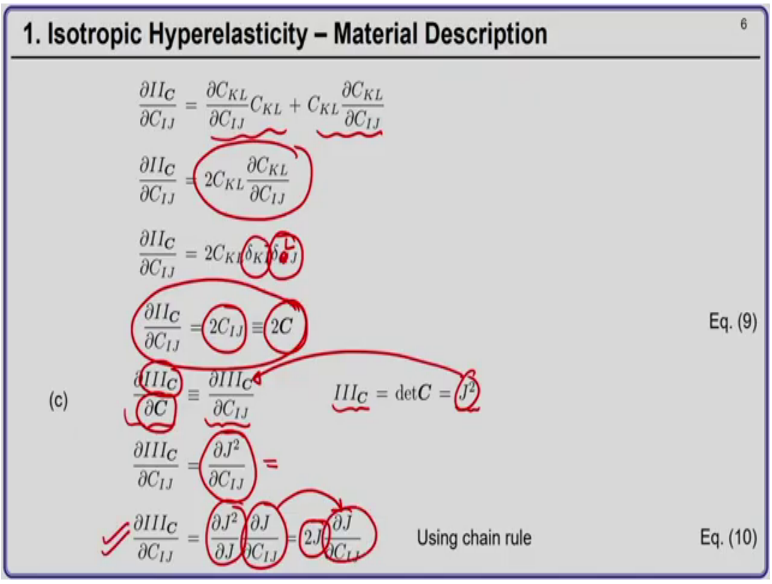

Then, del C K L by del C I J will be nothing but, ok. So, we can see that these two terms are both same, so I can just write it as twice of C KL into del C K L by del C I J, and we see that, del C KL by del C IJ from our previous slide is nothing but, delta KI into delta KJ, ok. And, then this becomes nothing but, delta sorry, so this is delta K I delta L J, ok.

And this becomes nothing but 2 C IJ ok, this become 2 C I J and in direct notation this is nothing but twice of the right Cauchy Green deformation tensor C. Now, we have to compute the third term which is the derivative of the third invariant of the right Cauchy Green tensor with respect to the right Cauchy Green tensor itself. So, this indicial notation is del 3 C by del C I J.

Now we know that 3 C is nothing but J square. So, we just substitute it here, and what we get is del 3 C by del C IJ is del J square by del C IJ, ok. So, this becomes nothing but, so I can take chain rule ok, I can apply chain rule here. So, that is what we have done, that is del J square by del J into del J by del C IJ. So, del J square by del J is nothing but, 2 J and then you have the second term. So, to evaluate this third term, we have to evaluate what is del J by del C I J, ok.

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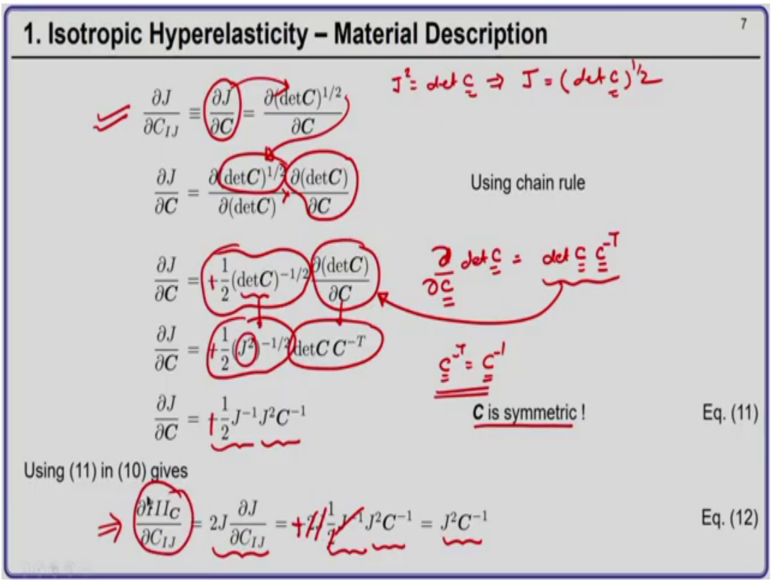

So, now let us evaluate del J by del C I J which in direct notation is nothing but, del J by del C. Now, I know that J square is determinant of C therefore, J is nothing but, square root of determinant of C. So I just substitute square root of determinant of C. So, my term becomes del by del C of square root of determinant of C, ok. So, now I can again apply chain rule. So, this term over here is del by del by del determine C of square root of determinant of C into del by del C of determinant of C, ok.

So, now, the first term becomes minus 1 by 2 determinant of C into minus 1 by 2 and del by del C of determinant of C, ok. Now, determinant of C is nothing but J square, and we know that the partial derivative of the determinant of a tensor with respect to the tensor itself is nothing but, determinant of the tensor times the tensor inverse transpose ok. And now, this is what goes here, ok. So, you have this coming from the second term and this one coming from the first term.

So eventually, and also because C is symmetric therefore, C inverse transpose nothing, but C inverse. Because C is symmetric therefore, C inverse itself is symmetric therefore, C inverse is nothing but, C inverse transpose. So, if you use this then we get minus 1 by 2 J inverse into J square C inverse. And, then when you substitute all this you get del 3 C by del C I J ok.

So, this should be plus sorry this should be plus, so you get del 3 C by del C IJ is 2 J del J by del C IJ ok. And this is nothing but, twice of J into 1 by 2 J inverse J square C inverse. So, this J this J inverse cancels out this 2 and this 2 cancel out, and what you are left with is J square C inverse, ok. So, now you have evaluated the third term, ok.

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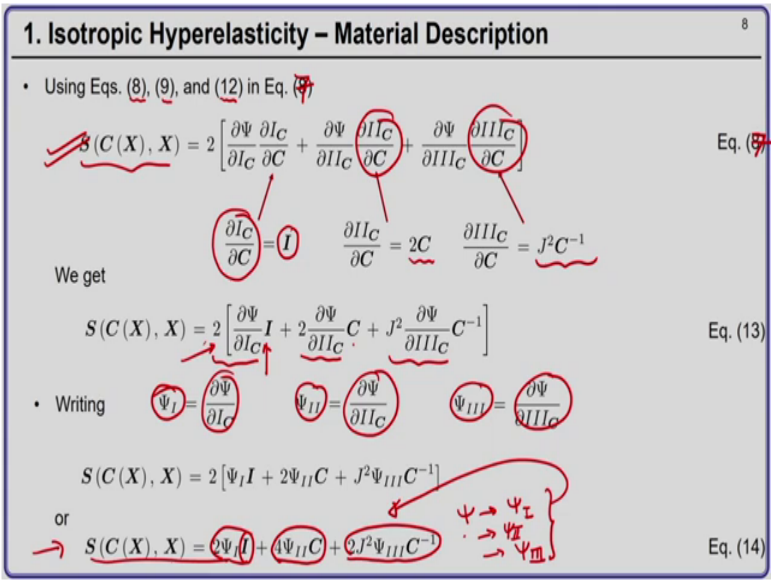

So, now we substitute equation 8 equation, 9 equation, 12 in equation 8, ok. So, so we already have evaluated, ok. Let us see ok, so this equation 8 equation, 9 and our equation 12. These three we substitute in equation number 7. So, this is your equation number 7 sorry. So, once you substitute it here ok, so del I C by del C is nothing but, 2nd order identity tensor. This term is nothing but, 2 C and this third term is nothing but, J square C inverse.

So, what you get? You get twice del C by del I C into identity tensor plus twice del psi by del 2 C into C plus J square del psi by del 3 C into C inverse. In short I can write, if I write the first term ok. So, first derivative in short I can write psi subscript 1, the second term I can write psi subscript 2, in the third term I can write psi subscript 3.

If I use this notation, and if I open up the bracket, take 2 inside, what I get? The second Piola Kirchhoff stress tensor is given by 2 psi 1 into I plus 4 psi 2 into C plus 2 J square psi 3 into

C inverse. So, you have three term which come in the expression for second Piola Kirchhoff stress tensor.

So, given, psi you can compute, what is psi 1, what is psi 2 and what is psi 3 ok. Because psi will be given in terms of I C 2 C and 3 C. So, you can compute all these three expressions and you can substitute it in equation number 14 and then finally, you will get the final expression, for the second Piola Kirchhoff stress tensor. Once we have this, the expressions for second Piola Kirchhoff stress tensor, we can compute the Cauchy stress tensor.

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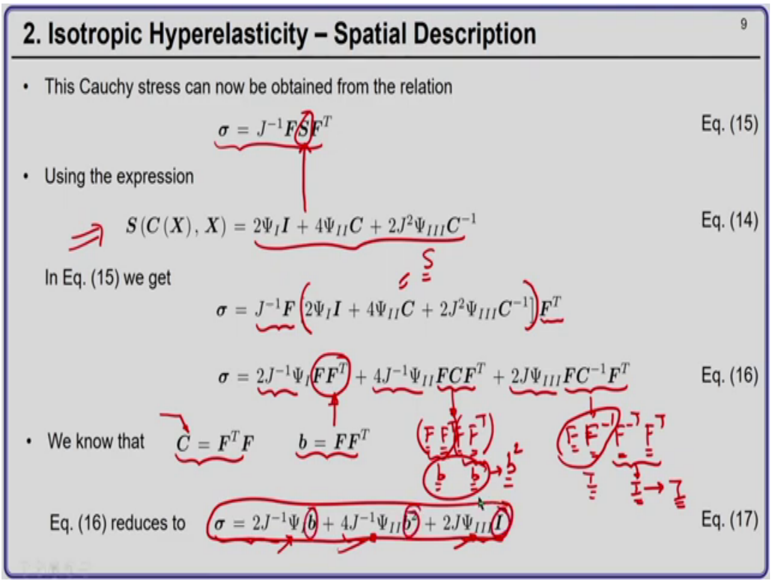

To compute the Cauchy stress tensor, we know the relation between the Cauchy stress and the second Piola Kirchhoff stress tensor which is given by J inverse F S F transpose ok. So, from equation 14, I can substitute the expression of second Piola Kirchhoff stress tensor S here, and then what I get?

The second Piola Kirchhoff stress tensor is this expression over here. This is already we had in our previous slide. So, this I can substitute here, ok. So, what I get? J inverse into F into this expression for second Piola Kirchhoff stress tensor into F transpose, ok.

Now, I can take F and F transpose inside the bracket I can open up the bracket I get 2 J inverse psi 1 F F transpose plus 4 J inverse psi 2 F C F transpose plus 2 J psi 3 into F C inverse F transpose. So, we now know that C is F transpose F, ok. So, the right Cauchy Green deformation tensor is F transpose F and the left Cauchy Green deformation tensor is nothing but, F F transpose, ok.

So, this term over here is nothing but left Cauchy Green deformation tensor. This term over here is nothing but, F and C is F transpose F, so I have F transpose F into F transpose. And, now I can recognize these terms, this term is nothing but left Cauchy Green deformation tensor. This is also left Cauchy Green deformation tensor and this over here is F and C inverse is nothing but, F inverse F inverse transpose into F transpose.

So, F F inverse is nothing but, identity and F inverse transpose F transpose is nothing but, again identity, ok. So, identity into identity gives me 2nd order identity tensor. So, and this b into b is nothing but, b square, ok. So, if I use all these in equation 16, what I get? Equation 16 reduces to sigma equal to 2 J inverse psi 1 into b plus 4 J inverse psi 2 into b square plus 2 J psi 3 into 2nd order identity tensor I.

So, with this I can compute my Cauchy stress tensor at a material location, ok. So, all I need to get is psi 1, psi 2, psi 3, but remember these psi 1, psi 2 and psi 3 are derivative of psi with respect to the invariants of right Cauchy Green deformation tensor C. But, we have left Cauchy Green deformation tensor here, ok.

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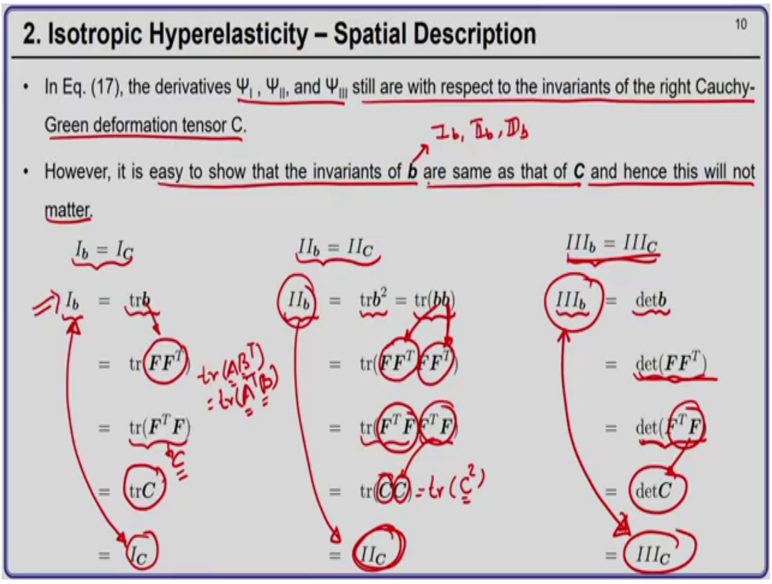

So, this derivatives psi 1 psi 2 psi 3 are still with respect to the invariants of right Cauchy Green deformation tensor C. But we can easily show that the invariants of b, let us say the invariants of b are 1 b say 2 b and 3 b. So, the invariants of b that is the left Cauchy Green deformation tensor are same as that of C that is the right Cauchy Green deformation tensor. And hence, this will not matter whether you are given psi in terms of the invariants of b or invariants of C.

What it means is, that the first invariant of b is same as the first invariant of C the second invariant of b is same as the second invariant of C the third invariant of b is in same as third invariant of C. So, let us show ok. So, let us show the first case ok, then the first invariant of b is nothing but, trace of b ok; b is nothing but F F transpose. So, I substitute F F transpose ok. And, now I know the property of trace, the trace of A B transpose is same as trace of A transpose B, ok.

That we discuss already in the mathematical preliminaries that we had in the initial half of the present course. So, now trace of A B transpose is trace of A transpose B. So, A here is F, B here is F transpose. So, what we get is trace of F transpose F and F transpose F is nothing but, your right Cauchy Green deformation tensor C which is nothing but, trace of C and trace of C is nothing but, the first invariant of right Cauchy Green deformation tensor.

So, you can see the first invariant of left Cauchy Green deformation tensor is same as the first invariant of the right Cauchy Green deformation tensor ok. Similarly, I can show for the second invariant. So, I start similarly trace of the second invariant of b is nothing but, trace of b square which I can write trace of b into b now b is nothing but, F F transpose, ok.

So, this is first b and this is second b and now I can use this same property, that trace of a b transpose is nothing but, trace of a transpose b and I can write trace of F transpose F into F transpose F and F transpose F is nothing but, C and the second F transpose F is also equal to C and this is nothing but, equal to trace of C square which nothing but, is the second invariant of the right Cauchy Green tensor C. So, we have shown that the second invariant of the left Cauchy Green deformation tensor b is same as the second invariant of the right Cauchy Green deformation tensor C, ok.

Coming to the third invariant, which is nothing but the determinant ok. So, the third invariant of the left Cauchy Green deformation tensor is nothing but the determinant of b, b is nothing but, F F transpose and trace of determinant of F F transpose is same as determinant of F transpose F and this F transpose F is nothing but, C ok. So, this is nothing, but determinant of C and this is nothing, but the third invariant of the right Cauchy Green deformation tensor, ok.

So, we have shown that the third invariant of the left Cauchy Green tensor is same as the third invariant of the right Cauchy Green tensor, ok. So, what it means is it does not matter if in equation 17 whether you are given psi in terms of invariants of b or invariants of C you just have to compute psi 1, psi 2, psi 3 and then simply substitute it here and you will get the expression for the Cauchy stress tensor, ok.

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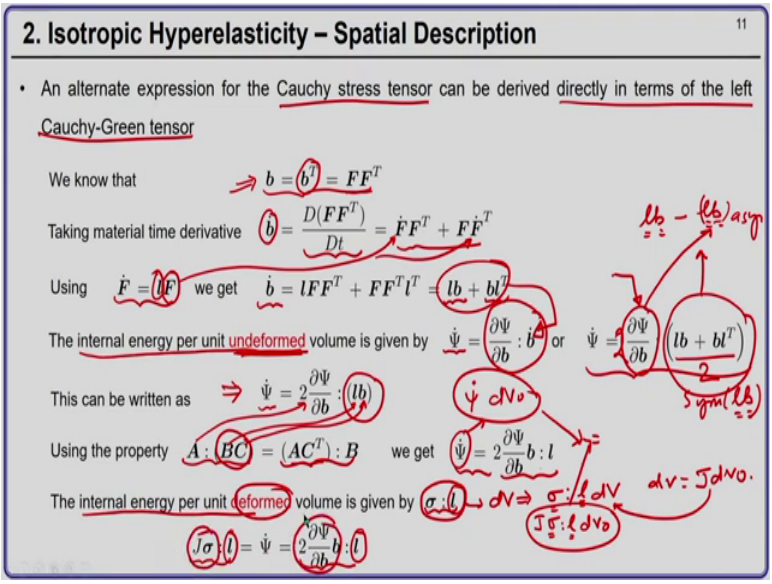

So, now we can derive an alternative expression for the Cauchy stress tensor directly from the terms from the left Cauchy Green tensor. So to do that, we start with the following expression, we note that the left Cauchy Green deformation tensor is symmetric; which is b equal to b transpose and it is nothing but, equal to F F transpose. Now, if I take the material time derivative of b which is nothing but, the material time derivative of F F transpose, ok.

So, I get F dot F transpose plus F into F dot transpose. So, the material time derivative of the deformation gradient tensor is nothing but the velocity gradient tensor F l into the deformation gradient tensor F. So, I can substitute this here F dot is l F I can substitute here, and I can get the material time derivative of the left Cauchy Green tensor as the velocity gradient tensor times the left Cauchy Green tensor into the left Cauchy Green tensor times the transpose of the velocity gradient tensor.

Now, the internal energy per unit undeformed volume, ok. Note that I am writing undeformed volume is given by the material time derivative of psi into del psi by del b double contracted with b dot or if I substitute b dot as l b plus b l transpose here I get psi dot as del psi by del b double contracted with l b plus b l transpose.

Now, del psi by del b is a symmetric 2nd order tensor. And if you notice, if I add this divided by 2 and multiply by 2 this term over here then becomes the symmetric part of a 2nd order tensor. And, because del psi by del b is the symmetric tensor; therefore, the I can write this as l b minus the anti-symmetric part ok, because this is the symmetric part of tensor l b.

So, the symmetric part will be nothing but the tensor l b minus the anti-symmetric part. And, because del psi by del b is a symmetric tensor 2nd order tensor. So, it is double contraction with this anti symmetric part will be equal to 0 and therefore, we will be left with only the tensor itself 2nd order tensor itself which is nothing but, l b.

So, the material time derivative of the internal energy per unit undeformed volume is psi dot equal to 2 del psi by del b double contracted with l b, ok. Now, if I use the property this property that where A B C are 2nd order tensor a double contracted with the product of two 2nd order tensor B and C is nothing but, A C transpose double contracted with B ok.

Now, my A here is del psi by del b, my B here is l and my C here is b, ok. So, if I use this then I can write psi dot as 2 del psi by del b into b double contracted with velocity gradient tensor l. Now, I can write the internal energy per unit deformed volume in terms of the Cauchy stress tensor sigma and the velocity gradient tensor l as; so, the internal energy per unit deformed volume is given by sigma double contracted with l, ok.

Therefore, now this psi dot is the internal energy per unit undeformed volume. So, for a undeformed volume d V 0 this will be total internal energy will be psi dot into d V 0 ok. And, this is for the same volume, but in the deformed configuration, so the internal energy for volume d V will be nothing, but sigma double contracted with l d V, ok. And, now we know that d V is J d V 0; I can substitute this here and I can get the same energy as J sigma l d V 0, ok.

So, the internal energy in the undeformed or the deformed configuration whether you express in undeformed or the deformed configuration is both same, therefore I can equate this term over here with this term over here. So, these two terms can be equated because they both are same they represent the same energy.

So, therefore, I have J sigma double contracted with the velocity gradient tensor is equal to psi dot equal to 2 del psi by del b, b double contracted with l, ok. Now, if I see there is a double contraction with l on both the side therefore, this term over here should be equal to this term over here.

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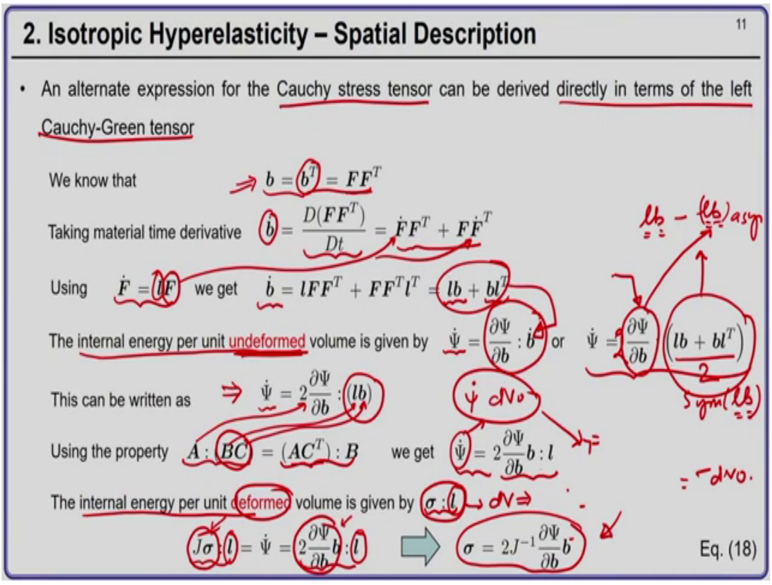

So, if I do this, if I just equate the two terms on the left hand side of the double contraction symbol, I get my Cauchy stress tensor sigma directly in terms of the left Cauchy Green tensor b which is nothing but, 2 times J inverse del psi by del b into b. So, with this, we now have our expression for the Cauchy stress tensor directly in terms of the left Cauchy Green tensor.

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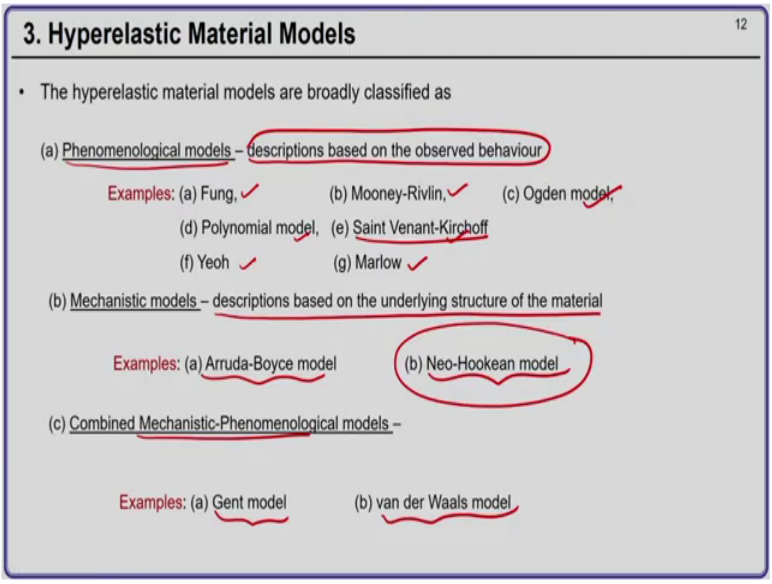

Now, before we move onto our derivation of constitutive relation for Neo-Hookean material, it is good to look into an overview of different kind of hyperelastic material models which are available, ok. So, this is not an exhaustive list, but this tests gives you the most popular ones which are out there in the literature.

So, the first type of models are what is called phenomenological models, which are based on descriptions on the observed behavior ok. So, you observe the behavior of the material and you develop these models. So, some of the examples are the Fung model, the polynomial model, Yeoh model, Mooney Rivlin model, Saint Venant Kirchhoff model, Marlow model, Ogden model ok.

So, Saint Venant Kirchhoff, we already have seen in our previous module. The other type of models are mechanistic models. In mechanistic models they are descriptions based on underlying structure of the material, ok. So, one observes the actual structure of the hyperelastic material and then these models are prepared or proposed.

So, two examples are the Arruda Boyce model and the Neo-Hookean material model. And, the third kind of models are the combination of the phenomenological models and the mechanistic models and example are gent model and van der Waals model. So, there are different kind of hyper-elastic models which are available, in the present course we will deal with a compressible Neo-Hookean material model, ok.

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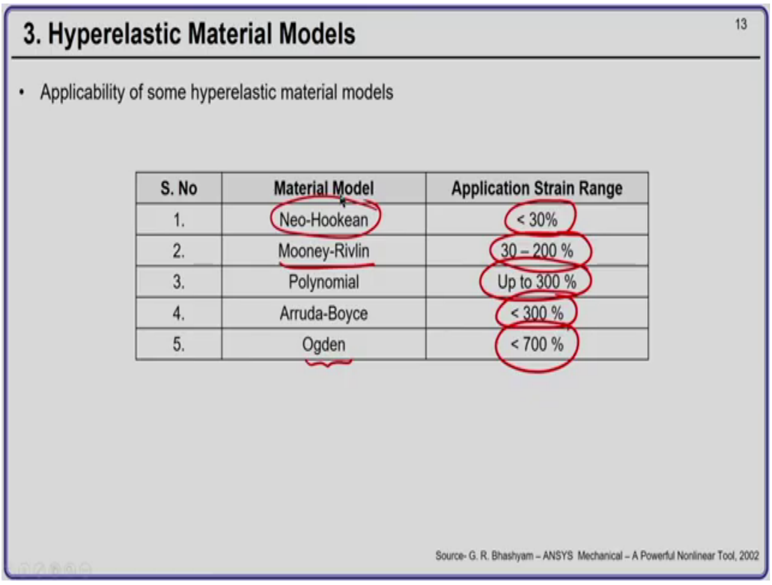

And then what are the applicability of some of these hyper-elastic material models. For example, the Neo-Hookean material model will give you appreciable result if the strain range is less than 30 percent, ok. If the strain range is between 30 to 200 you can use Mooney Rivlin, polynomial models up to 300 percent of the strain range.

Arruda Boyce models for all model all strain range less than 300 percent and Ogden model can accurately predict the hyper-elastic material behavior for strain range up to 700 percent. So, with this we now move to our discussion on the constitutive relation for a compressible Neo-Hookean material model ok, which is shown over here, ok.

Thank you.