Computational Continuum Mechanics
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Lecture – 21-22
Hyperelasticity-1
Lagrangian and Eulerian elasticity tensors

Welcome to the next module. So, in this module and in the next module, we are going to discuss the concept of Hyper-elasticity. We will develop the necessary equations and linearization associated with hyper-elastic material, ok.

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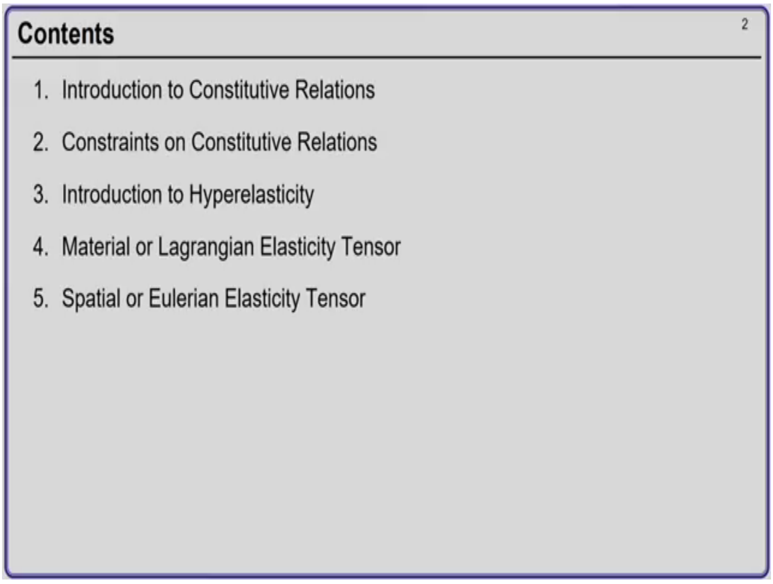

So, the contents for this module are as follows. So, we will first look into what is meant by constitutive relations, ok. And then, since this is not a course on continuum mechanics so, we have left out what is called thermodynamics part ok. So, we have to actually cover certain topics ok, so that we have a background on what is necessary to get certain constitutive relations, ok.

So, one type of constitutive relations that we are going to discuss in this particular course is on hyper-elasticity, but we will look into what are the certain constraints that are enforced on the constitutive relations, because of the thermodynamic considerations and there are some other consideration like objectivity, ok. And then, once we have dealt with the constraints on constitutive equations, we will go into introduction to hyper-elasticity, ok.

We will look into what is meant by hyper elastic material, ok. And then, we will derive the material or the Lagrangian elasticity tensor followed by the spatial or the Eulerian elasticity tensor, ok. So, these tensors will be necessary when we are going to linearize our virtual work principle, which we discussed in the previous lectures and there these concepts will be helpful.

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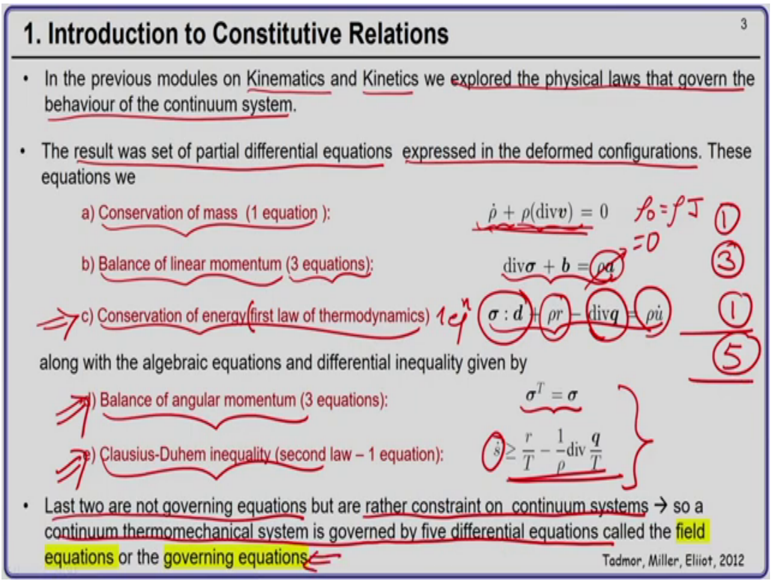

So, let us begin. So, in our previous modules on kinematics and kinetics, we had explored the physical laws that govern the behaviour of the continuum system. So, these physical laws resulted in a set of partial differential equations which were expressed in the deformed configurations, ok. What were these equations? So, the 1st equation was the conservation of mass. So, it was given by the material time derivative of the density plus the density times the divergence of the velocity field equal to 0.

This is essential the spatial form of the conservation of mass. So, the material form was rho 0 equal to rho J; so, that was the material form and this is the spatial form. So, this is one equation, which is in the density rho. So, next we had discussed balance of linear momentum and this led to the equation of motion, divergence of the Cauchy stress tensor plus body forces equal to rho times the acceleration.

For static problems, the acceleration part goes away and then what you get is the equilibrium equation divergence of the Cauchy stress tensor plus the body forces equal to 0. So, these were essentially 3 equations ok. So, you had 3 equations.

And this part, we did not cover because this is not a course on continuum mechanics, but if we look into the thermodynamics part, there the conservation of energy leads to or the first law of thermodynamics leads to following equation. So, this is 1 equation. So, you have the internal energy plus the rate of generation of heat inside the body minus the external heat flux should be equal to the rate of change of internal energy ok. So, that was 1 equation if you come from the thermodynamics aspect.

So, along with this, you had balance of angular momentum which were 3 equations which basically resulted in the symmetry of the Cauchy stress tensor, ok. And then, the second law of thermodynamics led to 1 equation where the rate of change of entropy s dot is given by the following expression. So, these are basically the equality and inequality equations, ok. So, essentially, the last two are not governing equations, but they are rather constraint on the continuum systems, ok.

So, these equations are not equations per se, but they are rather like a constraint on the continuum system, ok. So, a continuum thermo dynamical system is governed by five differential equations which are called the field equations, ok. So, you had 1 equation over here, you had 3 equation over here and you had 1 equation over here all in all you had total of 5 equations, and these 5 equations are called the field equations or the governing equations.

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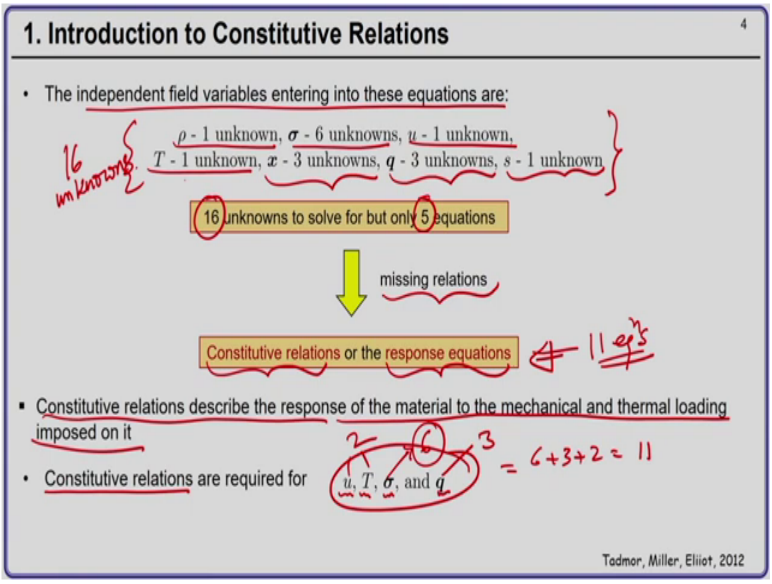

Now, let us see how many unknowns we have. So, the total number of unknowns that we have is 1 is density, there are 6 stresses, then you have the internal energy, you have the temperature, you have the positions, you have the heat flux and you have the entropy, ok. If you count total 1 plus 6 plus 1 plus 1 plus 3 plus 3 plus 1 so, all in all this gives you total of 16 unknowns, but you have only 5 equations to solve for ok. So, you have 16 unknowns, but you have only 5 equations to solve for.

So, to accurately or to precisely solve this equations, we need some extra set of equations ok. So, we have certain missing relations and these relations are obtained when we set up our constitutive relations which are also called the response equations. So, what are constitutive relations? Constitutive relations describe the response of the material to the mechanical and thermal loading which are imposed on the system. So, when we describe the constitutive relation, we will get the extra 11 equations which will help us to solve for these 16 unknowns.

Now, we need the constitutive relation for internal energy, temperature, stress and the heat flux ok. So, in general, we will need these many so, we will need 6 equations over here, you need 2 equations over here and you need 3 equations over here, ok. So, 6 plus 3 plus 2 equal to 11. So, these 11 equations when we get combined with the previous 5 field equations will result in the solution. But in this course, we are only dealing with isothermal cases, there are no heat flux into the system ok. So, in our case, only the stresses will be unknown.

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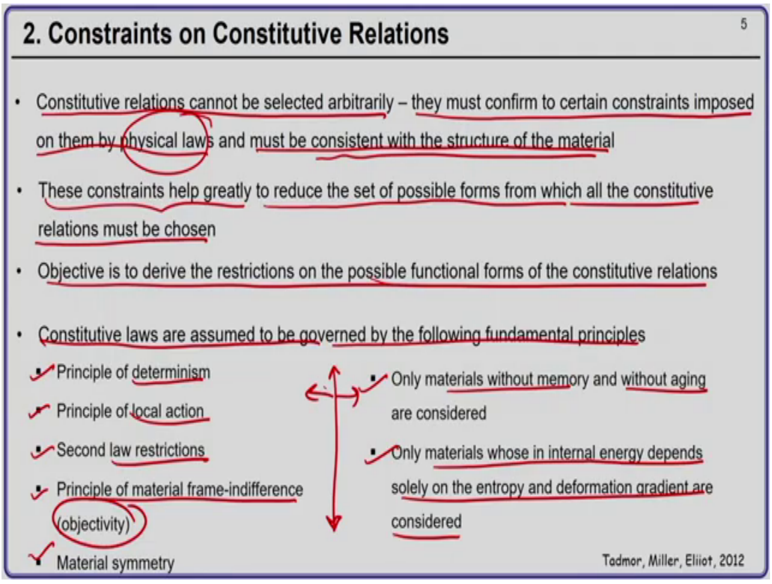

Now, these constitutive equations or the relations cannot be selected arbitrarily. So, they must conform to certain constraints imposed of them by the physical laws and they must be consistent with the structure of the material ok. So, you can just cannot select any constitutive relation. Those constitutive relations cannot be arbitrary, they have to be selected based on certain constraints which are imposed by the physical laws, and also the constitutive relations must be consistent with the material that you are having.

So, all this that we are covering right now is for the sake of completeness, ok. So, also these constraints help us to greatly reduce the set of possible forms from which all the constitutive relation must be chosen. So, there are many forms you can choose and with the help of these constraints, you will be able to narrow down on certain possible forms only.

So, the objective of these constraints is to derive the restriction on the possible functional forms on the constitutive relations and the constitutive laws or the relations are assumed to be governed by the following fundamental principles, which are the principle of determinism, principle of local action, restrictions which are imposed by the second law of thermodynamics, principle of material frame indifference; that is objectivity this we have already seen what is meant by objectivity, the material symmetry and for example, additional constraints can be material without memory and without ageing and materials whose internal energy depends solely on the entropy and deformation gradients, ok.

These are some of the other constraint that you can consider and based on these constraints, you can derive the constitutive relations, ok. So, we will look one by one into all these constraints in mostly one slide to complete our understanding on the constraints imposed on the constitutive relation.

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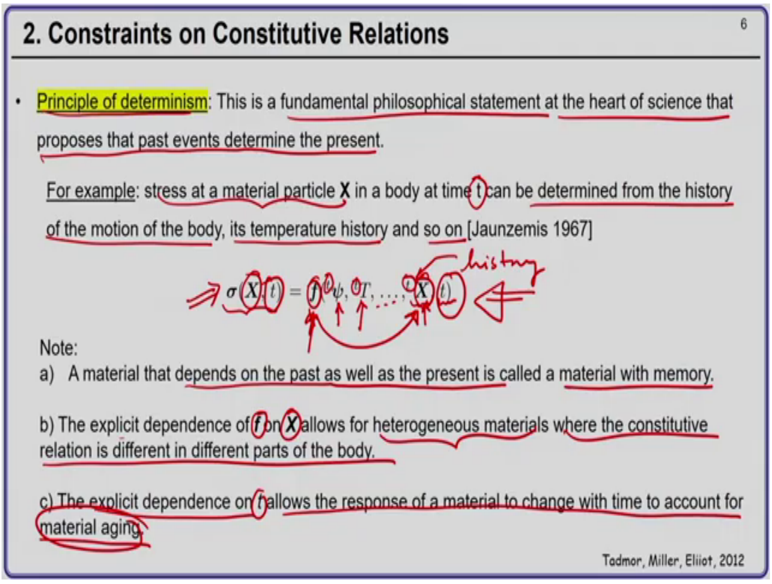

Now, the first principle is the principle of determinism and this is the fundamental philosophical statement at the heart of science that proposes that the past events determine the present ok. So, what it means is if you know the past deformation of the body or the continuum system, it should be possible to determine the current configuration of the body.

If you know everything about the past, it should be possible to tell about the present. So, this is called the principle of determinism, ok. Say for example, the stress at a material point X in a body at time t can be determined from the history of motion of the body, say for example is temperature history and so on, ok. So, as I have written here, the stress at a point material point X at time t is a function of its deformation mapping, say the temperature and there may be many other variables and say its current position explicitly and also explicitly it can depend on time.

So, if you look closely, there is a superscript t here which denotes that these quantities psi, T, x they are dependent on the history ok. So, time t means you have known these quantities from the time t equal to 0 when the process actually started. So, that is what denotes the, this t denotes the history. So, a material that depends on the past as well as the present is called the material with memory and the explicit dependence of this function f on X, you see this f depends explicitly on X here, that is the material coordinate X.

So, this allows for the what is called heterogeneous materials where the constitutive relations can be different in different parts of the body ok. So, explicit dependence on X is for heterogeneity. And, the explicit dependence on time t allows for the response of the material to change with time to account for material aging, ok. You might have come across material which over the period of time if you do not apply you apply stress and leave it there, after a certain period of time they will develop cracks, they will deform in a certain way, ok.

So, the material is said to be aged, ok. So, this explicit in in dependence on time t is for material aging, ok. So, this statement for example, says that from principle of determination that the stress at any material point X at time t can be determined if you know all these; so, it can be determined if you knew the history of all these variables.

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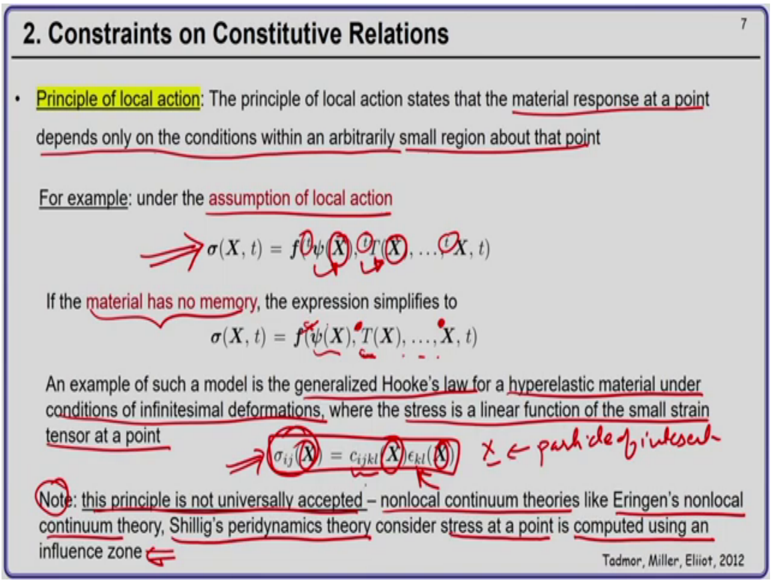

Now, there is a principle of local action which states that the material response at a point depends only on the conditions within an arbitrary small region around that point, ok. So, if you look closely psi here was not considered a function of X, but now from the principle of local action, the same relation that we had in the previous slide is now dependent all the variables are now local means they are all pertaining to the material point X. So, psi corresponds to material point X, it does not correspond to any other material point, temperature corresponds to the current material point like that.

So, now, if your material has no memory, what it means? It means that there is no so this time dependence which was here goes away you see there is no time dependence here it goes away. So, if your material does not have memory, so it means it does not know the past, it only knows the present which means that psi the deformation mapping psi, temperature and all such quantities only correspond to the current position at current time t such materials have no memory.

So, example of such a model is the generalized Hooke's law for hyper-elastic material under conditions of infinitesimal deformation, where the stress is a linear function of small strain tensor at a point, ok. So, sigma ij is c ijkl epsilon kl which is the standard Hooke's law generalized Hooke's law. And all the stresses, the material constitutive tensor c and the strain they all depend on X, that is the particle of interest.

For example, strain is at that particular point. The strain is not obtained from strains of some other points. So, one point to note is this principle is not universally accepted, ok. For example, in non-local continuum theories like the Eringen's nonlocal continuum theory, the Shillig's peridynamic theory, the stress at a point is considered to be computed using an influence zone so, rather than sigma here for example, being computed based on only at point X sigma will be computed based on a certain influence zone. So, we discuss now further on this, but the principle of local action is not universal.

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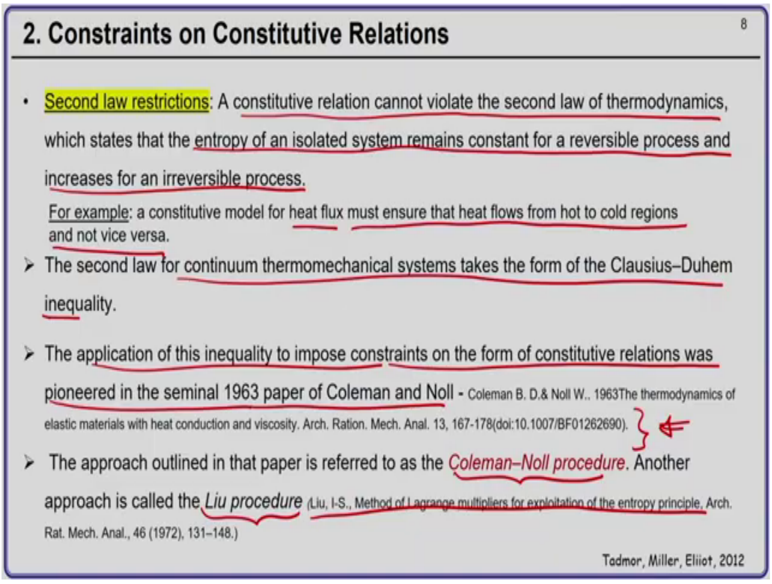

Now, coming to the second law of restrictions. So, constitutive equation cannot violate the second law of thermodynamics and so second law of thermodynamics states that the entropy of an isolated system remains constant for a reversible process and increases for a irreversible process, ok. So, for example, a constitutive model for heat flux must ensure that the heat flows from hot to cold region and not vice versa.

So, if you are proposing constitutive relation for heat flow inside a material, then your constitutive relation for heat flux should be such that it should always result in flow of heat from the hotter regions to the colder region, because heat cannot flow from cold to hot region. So, any constitutive relation or the model for heat flux which shows the opposite that is flow of heat from the cold to the hot region will not be accepted because it violates the second law of thermodynamics, ok.

So, the second law of thermodynamics takes the form of Clausius-Duhem inequality. And, the application of this inequality to impose constraints on the form of constitutive relations was pioneered in the 1963 paper by Coleman and Noll ok. So, this is that paper for the sake of completeness I will give given it here ok.

So, the approach that was outlined in this paper is called the Coleman-Noll procedure, ok. There are other procedure for example, the Liu procedure which is given in this paper other than Coleman-Noll procedure to apply this kind of second law of thermodynamics constraint.

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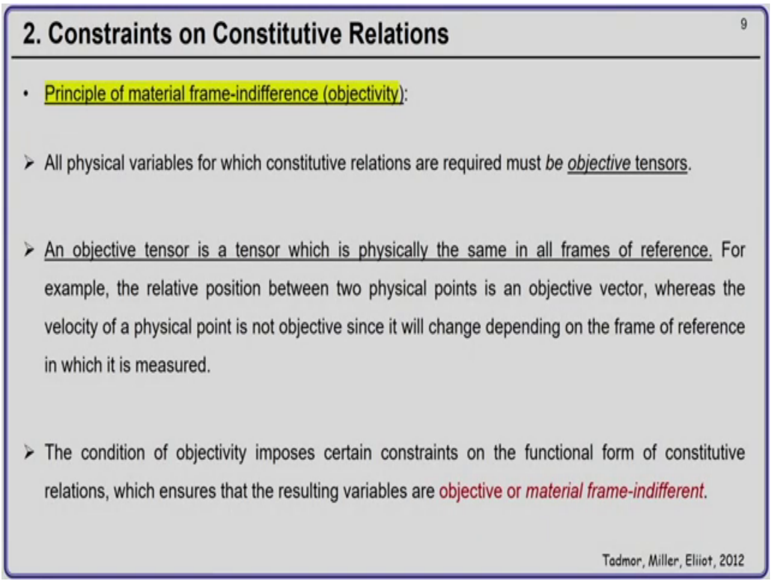

Now, the next constraint is the principle of material frame indifference or the objectivity criteria, ok. So, all the physical variables for which the constitutive relations are required must be objective tensors. So, an objective tensor is a tensor which physically remains the same in all frames of reference. For example, the relative position between two physical points is an objective tensor whereas, the velocity of a physical point is not an objective since it will change depending on the frame of reference in which it is measured.

So, the condition of objectivity imposes certain constraints on the functional form of constitutive relation, which ensures that the resulting variables are objective or material frame in different. So, the objectivity criteria or the principle of material frame indifference will impose certain constraints, ok. So, in our hyper-elastic constitutive relations, we will see what kind of constraint is imposed by principle of material frame in difference ok. We will come to it in next few slides.

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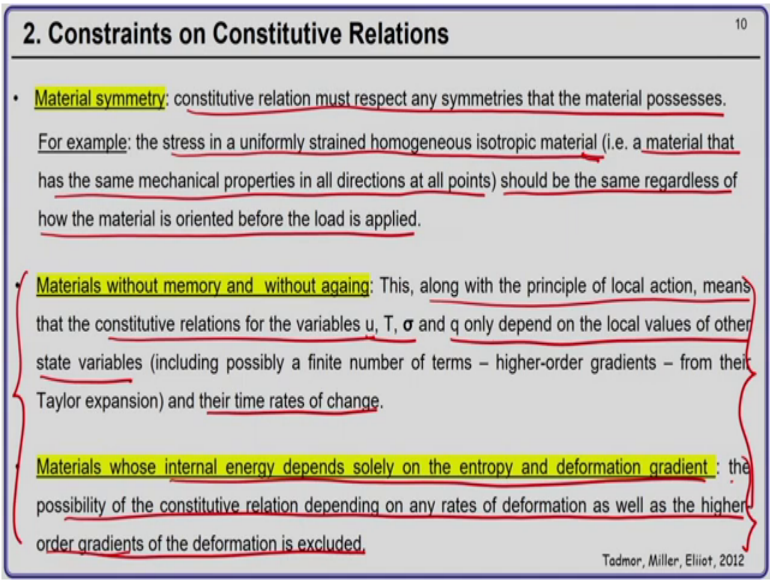

Next comes the material symmetry. So, the constitutive relations must respect any symmetries that the material possesses, ok. So, for example, the stress in a uniformly strain homogeneous isotropic material, that is a material that has the same mechanical properties in all directions, at all point should be the same regardless of how the material is oriented before the load is applied.

So, our isotropic material has this kind of symmetry, ok. So, if you take different specimens and then, if you load them, it they response with the same force, in the same direction it does not depend the response of the material will not depend on how the two materials were oriented before the forces were applied. If you apply the force in x direction, the response of the material will be same for both the samples, because it has that symmetry built into it. So, the your constitutive relation which will describe these kind of material behaviour that is the isotropic material behaviour should be respected, ok.

The other is the materials without memory and without aging, ok. So, if you have this restriction into in place which means that along with the principle of local action, the constitutive relations for internal energy temperature and stress and the heat flux only depend on the local values of the state variables and their time rates of change.

And, materials whose internal energy depends solely on the entropy and deformation gradient ok. So, for this, the possibility of the constitutive relation depending on any rates of deformation as well as higher order gradients on the deformation will be excluded. So, these are certain.

So, the last two constraints are very specific, but the first five are usually be constrained which have to be respected. The other two here are the constraints which are specific. So, if you have a specific kind of material, these constraints have to be respected. With these kinds of constraints, one can derive the constitutive relation, because this is not a course on continuum mechanics, we will directly deal with hyper our hyper-elastic material for which all these constraints are already satisfied, ok.

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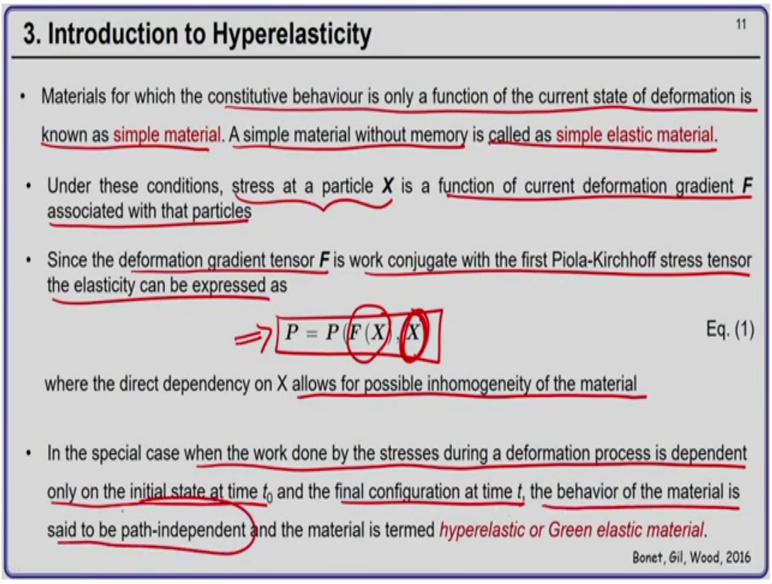

So, material for which the constitutive behaviour is only a function of the current state of deformation is known as a simple material, ok. So, what is the simple material? Its current state is the function of current deformation. And, now a simple material without memory is called a simple elastic material. So, if the material does not have any memory and it state depends the constitutive behaviour of the material depends only on the current state of deformation, then that material will be called a simple elastic material.

So, under these condition, the stress at a material point X is a function of the current deformation gradient F associated with that particular particle, ok. And since, the deformation gradient tensor F is the work conjugate with the first Piola-Kirchhoff stress tensor, so the elasticity can be expressed by following equation, ok.

So, the first Piola-Kirchhoff stress tensor is a function of the local value of the deformation gradient tensor as well as the current position of the particle, where the current position of the particle is there to allow for possible in homogeneity of the material. So, if I have; if you have inhomogeneous response, then this particular term explicit dependence on the particle position helps.

So, in the special case, when the work done by the stresses during a deformation process is dependent only on the initial state at time t 0 and the final configuration at time t, then the behaviour of the material is set to be path-independent and the material is referred to as hyper-elastic or green elastic material, ok. Suppose your work done by the stresses depends only on the current state and the initial state, and it does not depend on how you achieve the current state from the initial state that is the response of the material is path independent, then such kind of materials are called the hyper-elastic material or green elastic material.

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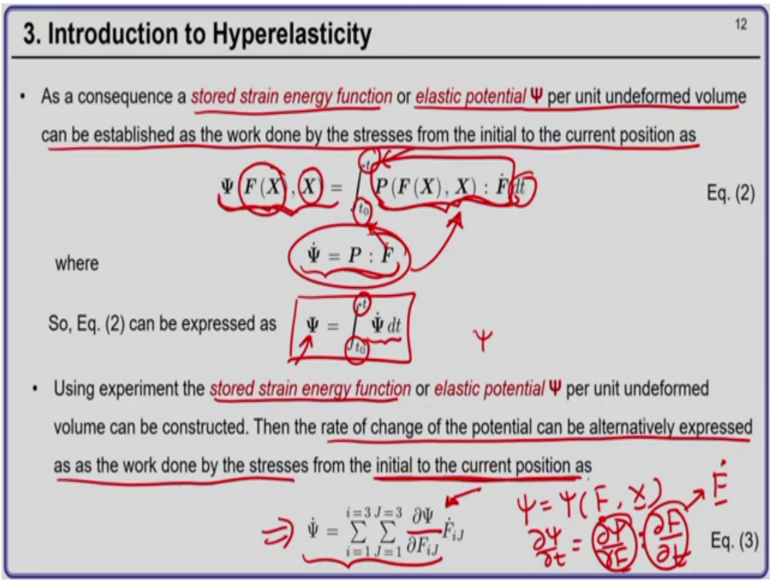

So, as a consequence of this, the stored energy strain energy function or the elastic potential psi per unit the undeformed volume can be established as a work done by the stresses from the initial to the current position as psi equal to P double contracted with the rate of deformation gradient tensor integrated over time from initial time to the current time.

So, the total strain energy that is stored inside the body is nothing but the, strain energy density or the work done by the stresses from the initial configuration to the final configuration, ok. So, psi here is the stored energy per unit undeformed volume, ok. And now, this depends psi depends on the current value of the deformation gradient and also the current position ok.

So, we can identify the material time derivative of the stored strain energy density function psi dot as p double contracted with F dot ok. So, if you use this in this particular expression, the total stored strain energy per unit undeformed volume is given by integration of psi dot from initial to final configuration.

Now, if you can somehow set up experiments ok, to get this stored energy density function ok. So, you using some experiments, you can determine the functional form of psi for example. Then, the rate of change of potential can be alternatively expressed as the work done by the stresses from the initial to the current position, ok.

So, psi dot; so, psi dot will be nothing but, see psi is a function of F. So, del psi by del t will be equal to del psi by del F contracted with del F by del t and this is nothing but, F dot and this is nothing, but del psi by del F. So, in indicial notation I can write psi dot as summation over i and J del psi by del F iJ F dot iJ, ok. So, this is the expression for the work done by the stresses from the initial to the current position.

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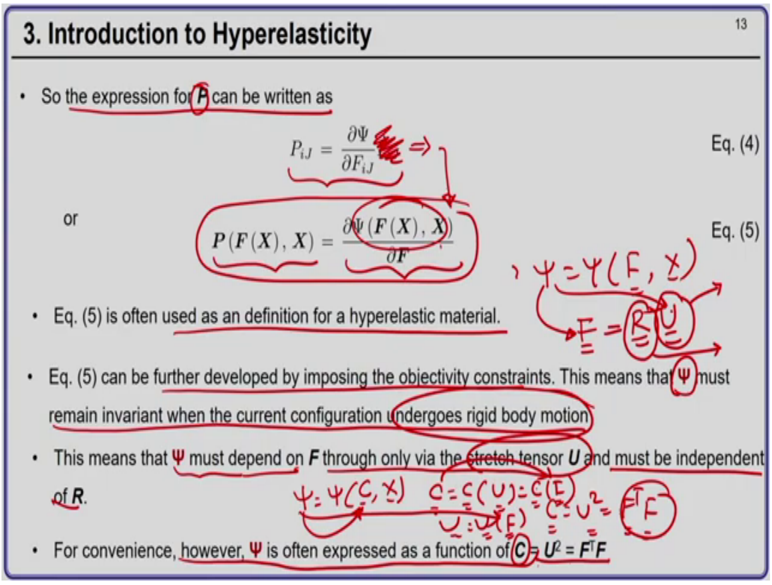

Now, the expression for the first Piola-Kirchhoff stress tensor can now be written as P equal to del psi by del F into F dot. So, as a direct notation, I can write P equal to del psi by del F sorry this will not be here, del psi by del F ok. So, this relation equation number 5 is used as a definition of hyper-elastic material. So, you can further develop equation 5, now using the objectivity constraints.

This means that, what does objectivity constraint imply? This means that the stored strain energy density potential psi must remain invariant when the current configuration undergoes rigid body motion, ok. For example, if there is rigid body rotation, then the psi must remain invariant and how can this be insured? This means that the psi you remember, psi here if you see here, psi is a function of F, ok. So, psi must depend on F that is given, but it can depend on F only through the stress tensor U, ok. And, it must be independent of R which is the rotation tensor.

Remember, the deform using right polar decomposition, deformation gradient tensor F can be decomposed into one which is the rotation part, and the another which is the stress part. So, R corresponds to pure rotation and U corresponds to your stretch. Now, if the effect of rigid body motion or the rotations have to be taken away, then my psi has to depend on F through U, ok.

It cannot directly, psi cannot directly depend on F, it can depend on U and U can only be obtained from F. So, in a way there is a relation between psi and F, but in between you have this stress tensor U and then, because psi depends on U, we have taken out the effect of rotation R.

So, for convenience, however, psi is often expressed as a function of right Cauchy-Green tensor C which is given by U square which is nothing but, F transpose F. So, our psi will be function of C and X and C is a function of U and U is a function of F ok, but from this relation, C can be also be said as a function of F, ok.

So, psi cannot directly depend on F, psi as to depend on say C or U and then, C depends on F so, that is how this dependence on; dependence of the strain energy density potential on the deformation gradient tensor is obtained.

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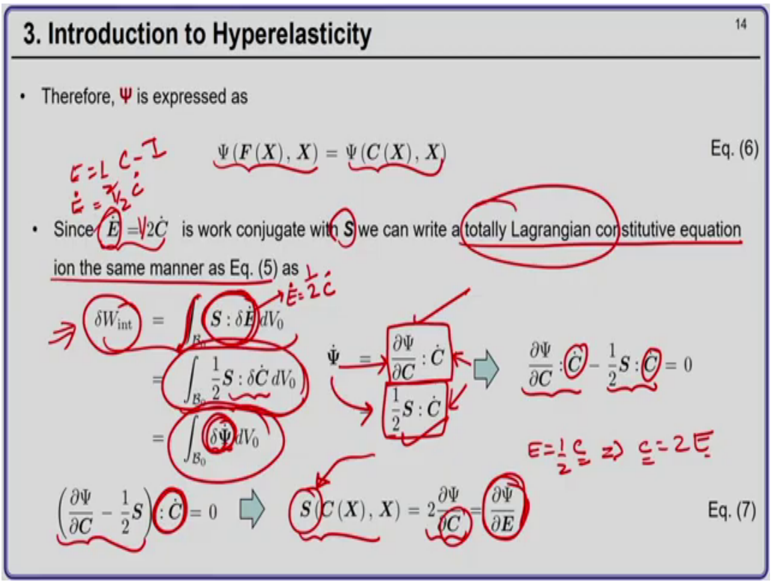

Therefore, psi which is has to be a function of deformation gradient tensor in the current position has to be a function of right Cauchy-Green tensor and the current position. Now, we know that the material time derivative of the Green-Lagrange strain tensor is twice the material time derivative of the right Cauchy-Green tensor.

Therefore, because the Green-Lagrange strain tensor is work conjugate with the second Piola-Kirchhoff stress tensor, we can totally write a Lagrangian constitutive equation in the same manner as equation 5 ok. So, this was our equation 5, which was the constitutive relation for a hyper-elastic material in terms of the first Piola-Kirchhoff stress tensor and the deformation gradient tensor.

But, P and F in equation 5, are two-point tensor. So, a part of them also resides in the spatial configuration to get a totally so, to get a totally Lagrangian constitutive relation, what we can do is we can convert the, we can obtain a totally Lagrangian constitutive relations by converting the first Piola-Kirchhoff, and the deformation gradient into second Piola-Kirchhoff and the Green-Lagrange strain tensor, ok.

So, if you see the internal virtual work expression that was integration of the work done by the second Piola-Kirchhoff and the Green-Lagrange strain tensor work done by the second Piola-Kirchhoff stress strain integrated over the material configuration. Now, if I substitute E dot as so this is 1 by 2, because E is 1 by 2 C minus I therefore, E dot will be 1 by 2 C dot, ok. So, instead of 2 here, we should have 1 by 2, ok.

So, E dot here, if I substitute 1 by its C dot, I will get 1 by 2 S double contracted with variation of the material time derivative of right Cauchy-Green tensor integrated over the material configuration, and this I can the total internal virtual work will be the integration of the stored virtual strain energy over the reference configuration, ok.

So, psi dot is nothing but, del psi by del C double contracted with C dot and it is also will be equal to 1 by 2 S double contracted with C dot. So, if we equate both these so, psi dot is also equal to this, it is equal to this so, these two expressions have to be equal. So, del psi by del C contract double contracted with C dot minus 1 by 2 S double contracted with C dot should be equal to 0.

So, because C dot is common ok, in both the expression I can take them take it outside and I get del psi by del C minus 1 by 2 S double contracted with C dot equal to 0, and this implies that S and if this is equal to 0 for all C, then S or the second Piola-Kirchhoff stress tensor should be equal to twice of del psi by del C, ok.

And, now because E is 1 by 2 C which means C is 2E and if I use this here, I can get the second Poila-Kirchhoff stress tensor as the derivative of the stored strain energy density potential with respect to the Green-Lagrange strain tensor, ok. So, for hyper-elastic material, the second Piola-Kirchhoff stress tensor can be obtained by taking the derivative of the stored strain energy density potential with respect to the Green-Lagrange strain tensor, ok.

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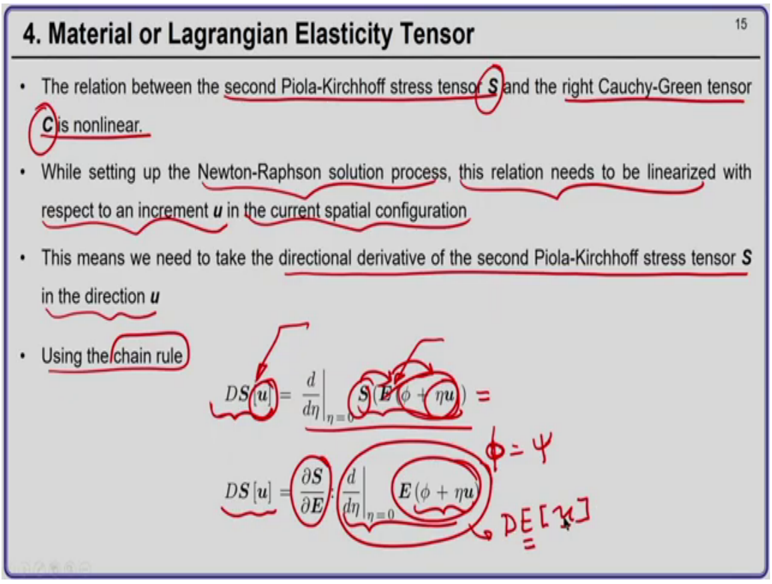

Now, this relation between the second Piola-Kirchhoff stress tensor and the right Cauchy-Green tensor C will be a non-linear expression. So, when we come to our neocon model, you will see that psi is a non-linear function in C or E. Now, if it is a non-linear function in C, you will get a non-linear relation between the second Piola-Kirchhoff stress tensor and the right Cauchy-Green tensor.

And, when you have to set up the Newton-Raphson solution process, for studying the deformation of a hyper-elastic material under external loads, you have to linearize this relation between the second Piola-Kirchhoff stress tensor and the right Cauchy-Green tensor and this has to be linearized with respect to an increment u which is usually the displacement in the current spatial configuration, ok.

So, this means you have to take the directional derivative or the second Poila-Kirchhoff stress tensor in the direction of displacement u and how can you do this? You can use the chain rule, so the directional derivative of the second Piola-Kirchhoff stress tensor in the direction of u what it actually means is if you have a small change in the current configuration given by u, then what is the change in the stresses?

So, this change in the stress by this much change in the displacement from the current configuration is nothing, but the directional derivative and the way you can compute is d by d eta of this first Piola-Kirchhoff tensor which is a function of Green-Lagrange strain tensor which in turn is a function of the current configuration plus your eta times u. So, the directional derivative and now, you can use the chain rule S depends on Green-Lagrangian strain tensor and Green-Lagrangian depends on u, ok.

So, d by d eta of S is nothing but, d by dE of S double contracted with d by d eta of the Green-Lagrange strain tensor evaluated at the configuration phi plus eta u. Remember, phi same as psi, ok. I am using phi here because psi was is used here for the strain energy density potential, ok. So, to remove that confusion, I am using phi here, ok. So, now, if you look closely this relation ok, this expression over here is nothing but, the directional derivative of the Green-Lagrange strain tensor in the direction of displacement u, ok.

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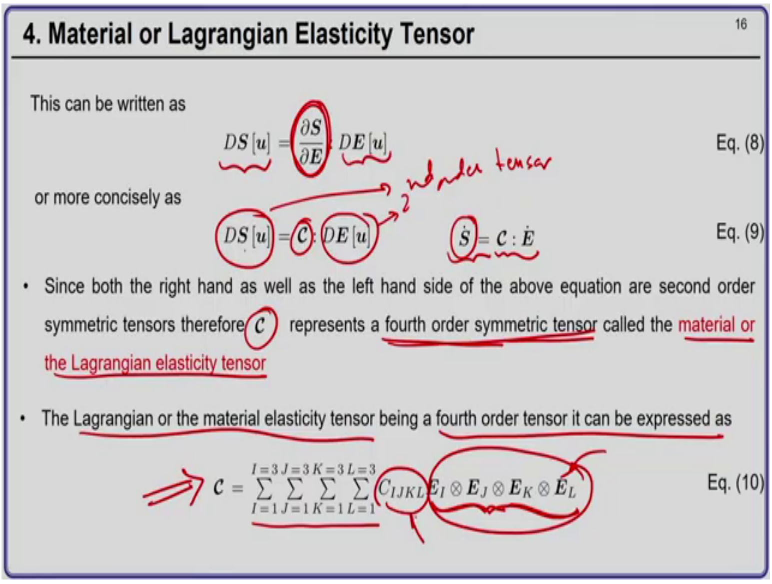

So, using this, I can write the directional derivative of the second Piola-Kirchhoff stress tensor in the direction u is nothing, but del S by del E double contracted with the directional derivative of the Green-Lagrange strain tensor in the direction u. Now, I can denote del S by del E by a symbol C, ok. And then, the directional derivative of S in the direction u is C double contraction with directional derivative of Green-Lagrange strain tensor in the direction u and we know the relation between the directional derivative and the material time derivative, ok.

So, they are one and the same. So, directional derivative of S in the direction of velocity v will be nothing but the, material time derivative S dot, ok. So, I can write this expression instead of u if you take the velocity v, then I get the material time derivative of second

Piola-Kirchhoff stress tensor is C double contracted with the material time derivative of Green-Lagrange strain tensor, ok.

Now, since directional derivative of S and the directional derivative of E, these both are second order ok, they both are 2nd order tensors therefore, this quantity C is a fourth order symmetric tensor, because both S and E are symmetric tensor. This you would remember from our kinematics discussion. So, because both S and E are symmetric and they are second order therefore, C has to be a fourth order symmetric tensor and it is also called the material or the Lagrangian elasticity tensor.

So, this Lagrangian or the material elasticity tensor because it is a fourth order tensor, it can be expressed in terms of its basis and because both S and E are in the original configuration or the undeformed configuration therefore, we use the basis in the undeformed configuration which is given by capital E, ok. So, C will be summation over I, J, K, L, C IJKL E I tensor product E J, tensor product E K, tensor product E L.

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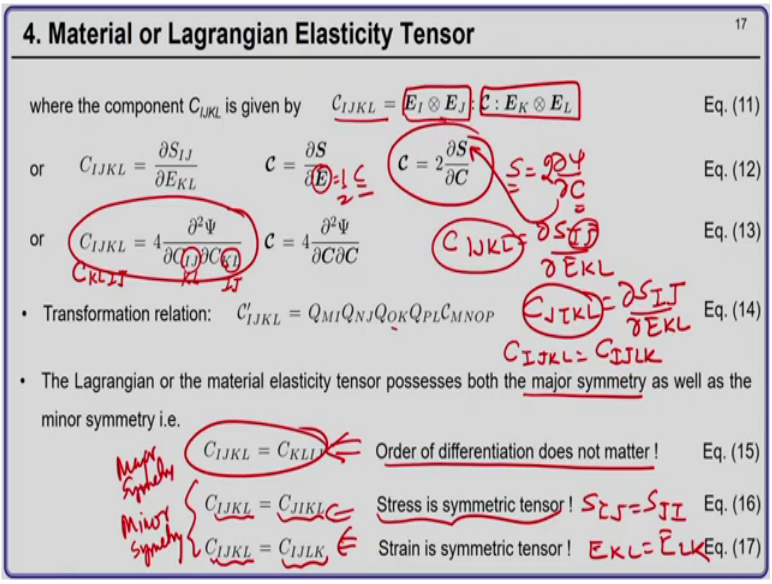

So, what will be C IJKL? C IJKL you can obtain and when we are discussing how to get the component of a fourth order tensor, this can be obtained by taking the double contraction of C with respect to E k, E L, E K tensor product E L and then, taking the double contraction of the resulting second order tensor with E I tensor product E J.

More precisely, C IJKL ok is nothing but, del S IJ by del E KL this is a direct notation, ok. And, because E is 1 by 2 C therefore, the material elasticity tensor C is also written as two twice of del S by del C, ok. And, now because S is del psi by del C, if I substitute this here, what I get? The material elasticity tensor is 4 times of so, this is twice so, 4 times of del square psi by del C del C or C IJKL is nothing but, 4 del square psi del C IJ del C KL ok. So, this is the; so, this is the direct notation, and this is the indicial notation.

So, you can determine the material or the Lagrangian elasticity tensor C if you know the functional form of psi and psi depends on C and the current position X if it is heterogeneous material. Therefore, if you know this relation, it should be possible for you to get the expression for S if you take del psi by del C and to study the response of the material, you need the fourth order elasticity tensor C and which you can obtain by 4 times of del square psi by del C del C, ok. So, given psi both the second Piola-Kirchhoff stress tensor and the Lagrangian material elasticity tensor can be known.

Now, because C are the material elasticity tensor is a fourth order tensor, the transformation relation is given by C dash IJKL is Q MI Q NJ Q OK Q PL C MNOP, ok. So, therefore, transformation tensors Q 1, 2, 3, 4. And, the another point to note if you see this relation over here ok, so I will rub first of all this. If you see this relation over here, if I just interchange IJ and KL, I make this KL and make this as IJ. So, on the left-hand side it becomes KL IJ. So, the result will not change.

So, you will get the same expression for the material constitutive tensor. So, this is called the major symmetry, and because both so C IJKL is del S IJ by del EKL because S is a symmetric tensor so, if I replace IJ by JI I can get C JIKL as del S JI by del E KL and because S is symmetric, S JI same as S IJ, so what I see is C IJKL is same as C JIKL similarly because strain is symmetric, I can show that C IJKL is same as C IJLK.

So, these two, ok. So, the order of differentiation does not matter implies you have what is called the major symmetry C IJKL is C KLIJ. The symmetry of the second Piola-Kirchhoff stress tensor implies what is the minor symmetry C IJKL is C JIKL and the symmetry of the strain implies that C IJKL is C IJLK, because S IJ is same as S JI and E KL is same as E LK ok. So, these are called the minor symmetry and this called the major symmetry, ok.

Thank you.