Computational Continuum Mechanics
Dr. Sachin Singh Gautam
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Kinetics - 2
Lecture - 18-20
Work conjugacy, Different stress tensors, Stress rates

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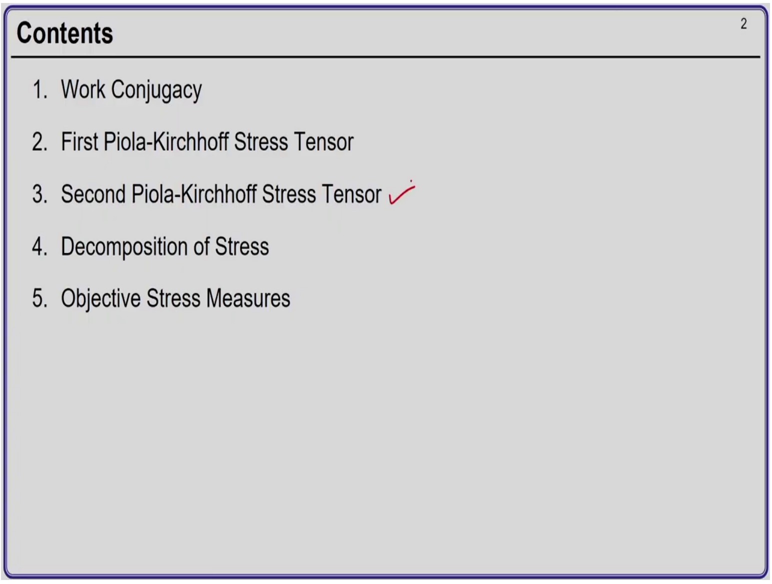

So, the next stress measure that we consider is the second Piola-Kirchhoff stress tensors ok.

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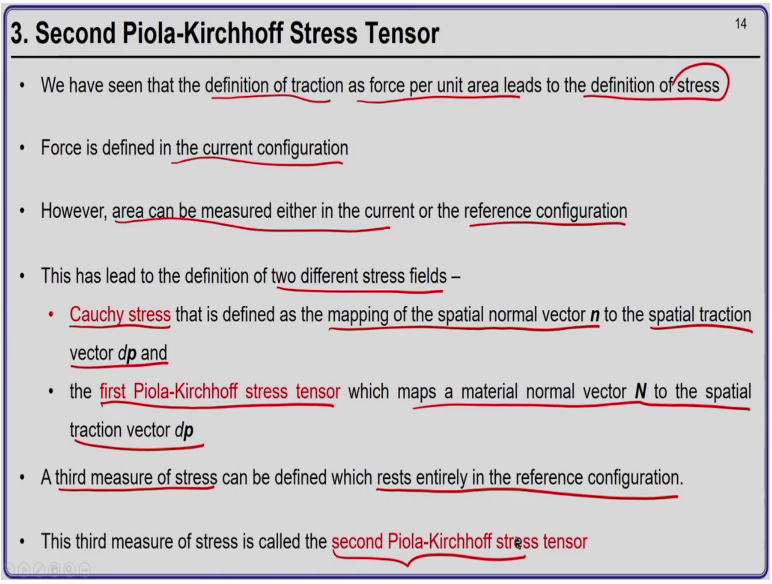

So, we have seen that the definition of traction as a force per unit area leads to the definition of stress ok. Now force is always defined in the current configuration; however, the area can always be defined either in the current configuration or in the reference configuration.

So, depending on which area we took we have defined two different stress fields; one of course, was the Cauchy stress which was defined as the mapping of a spatial normal vector n to the spatial traction vector dp ok. Again, if we took area in the reference configuration, we define what was called the first Piola-Kirchhoff stress tensor which maps a material vector N to the spatial traction vector dp.

Now, it so, happens that we can define a third measure of stress which rests entirely in the reference configuration. See Cauchy stress rests entirely in the spatial configuration and the second Piola-Kirchhoff has a part in the spatial configuration another part you can say is in the material configuration. So, there is a third stress measure that we can define which rests entirely in the reference or the material configuration. And this third kind of stress measure is what we call as the second Piola-Kirchhoff stress tensor.

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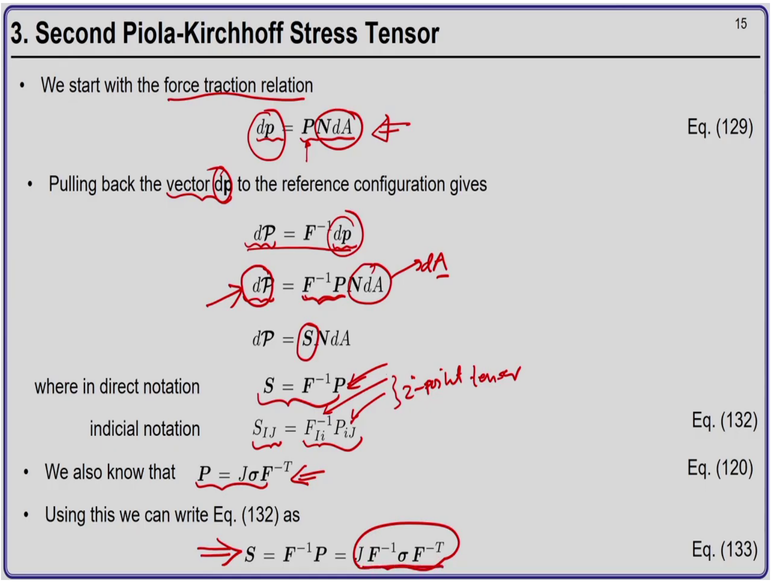

So, now let us see how we come up with the mathematical expression of the second Piola-Kirchhoff stress tensor. So, we know the force traction relation. So, this force is given in terms of the first Piola-Kirchhoff stress tensor and the material area vector by equation 129 ok.

Now, if we pull back the vector dp to the reference configuration and how we can do that? Ok. So, just like we can pull back the spatial vector d X to material vector d capital X by multiplying the spatial vector d X by F inverse similarly we can get the pullback of the vector dp and get its counterpart in the reference configuration. So, d capital P let us say is the vector corresponding to the spatial vector dp in the material configuration and this is given by dP equal to F inverse dp.

I can now substitute equation 129 in for the expression of the spatial vector dp ok. So, what I will get? The material vector dP is F inverse PN d A ok. Now NdA I can write as the material vector area vector d capital A and then on the left hand side we have something in the material configuration. So, whatever is here ok. So, this is d A. So, whatever is this quantity is mapping the area vector d A in the reference configuration to the vector d P in the reference configuration. Therefore, we can write F inverse p as S and S maps an area vector in the reference configuration to a vector dp in the reference configuration. So, S is a second order tensor which is given by F inverse P in indicial notation I can write S subscript capital IJ.

Why capital IJ? Because S resides totally in the material configuration, it is mapping something in the material configuration to something in the material configuration ok. So, area vector in the material configuration to vector dp in the material configuration. So, S would reside totally in the material configuration and in indicial notation this is S IJ is F inverse I i P i J. So, that is the and you note here both P and F are the 2-point tensors.

Now, we know that the relation between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor as J sigma F inverse transpose ok. So, I can substitute P in terms of the Cauchy stress in equation number 132 here and then I can get the second Piola-Kirchhoff stress tensor. So, S is called the second Piola-Kirchhoff stress tensor in terms of the Cauchy stress tensor as following expression. S equal to J F inverse sigma F inverse transpose.

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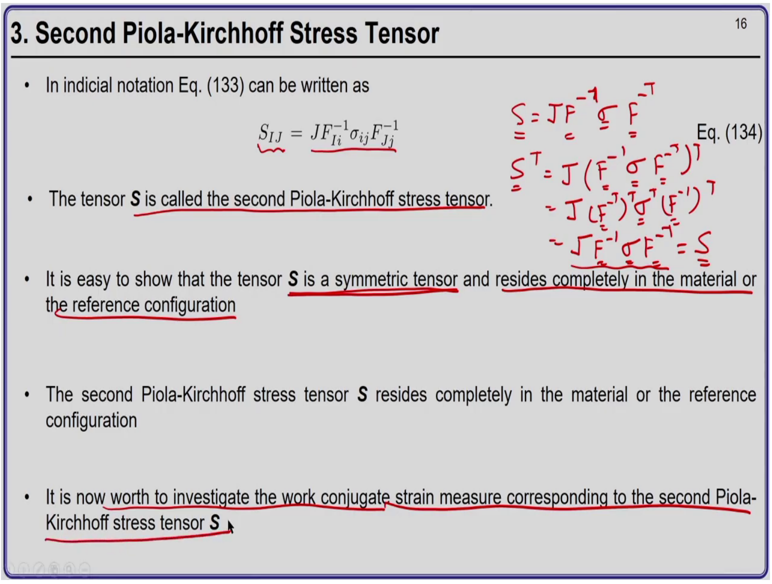

So, in indicial notation I can write S IJ is J F I i inverse sigma ij F inverse J j ok. So, that is the indicial notation for equation 33. So, this tensor s is called the second Piola-Kirchhoff stress tensor and you can show that S is a symmetric tensor. So, we can show because S is given by J F inverse sigma F inverse transpose.

So, if I take S transpose I will get J F inverse sigma F inverse transpose which becomes J F inverse transpose, transpose sigma transpose F inverse transpose which becomes J F inverse. And now because Cauchy stress is a symmetric tensor sigma transpose is same as sigma and then I get F inverse and this expression over here is nothing, but the expression for the second Piola-Kirchhoff stress tensor. So, this shows that S is a symmetric tensor and it resides completely in the material or the reference configuration.

So, now we have three stress measures, the Cauchy stress measures which resides totally in the spatial configuration. We have the second Piola-Kirchhoff stress tensor which reside totally in the material configuration. And we have first Piola-Kirchhoff stress tensor which resides somewhere in between it connects the spatial and the material configurations.

So, now it is worth to investigate the work conjugate strain measure corresponding to the second Piola-Kirchhoff stress tensor.

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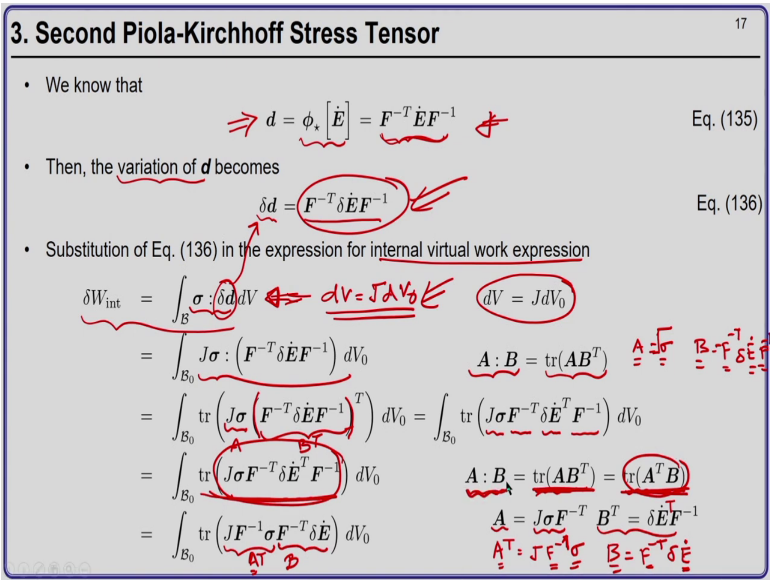

So, let us start first we recall that the rate of deformation tensor d was nothing, but the push forward of the material time derivative of the Green LaGrange strain tensor E and how this push forward was carried out? It was carried out by pre multiplying the material time derivative of Green LaGrange strain tensor by F inverse transpose and post multiplying it by F inverse ok. So, this is given in equation 135.

So, the variation of the rate of deformation tensor d now becomes del d equal to F inverse transpose del E dot F inverse. Now one thing you have to remember is that E that is Green LaGrange strain tensor is a symmetric tensor therefore, E dot and del E dot they both are symmetric quantities.

Now, let us derive what is the work conjugate strain tensor corresponding to the second Piola-Kirchhoff stress tensor ok. So, as usual we start with the internal virtual work expression and this is given by following expression ok. So, in the current configuration it is nothing, but the integration of the double contraction of these Cauchy stress tensor with the rate of deformation tensor and then integrated over the current volume.

Now, in this expression I can write d V as J d V 0 ok. So, this relation we already know. Using this relation I can transform this integral from integral over the reference current configuration to integral over the reference configuration. And also from equation 136 I know that the variation of rate of deformation tensor is given by this quantity over here. So, I substitute these two quantities in our integral and then finally, I get J sigma double contraction with the quantity in the bracket integrated over the reference configuration.

Now, we know that double contraction of two second order tensors can be written as trace of AB transpose. So, using this and realising that A in our case is sigma and B is F inverse transpose del E dot F inverse then I can write ok. So, A is J sigma. So, this is A and this is B transpose and now I can open up this bracket of the transpose and I can get J sigma F inverse transpose del E dot transpose F inverse.

Now, this is what we have and then what I can do now is again I know that trace of AB transpose is trace of A transpose B. So, this property we have already discussed when we were discussing the mathematical preliminaries that particular section.

So, now this quantity in the bracket if we look closely and I identify A as J sigma F inverse transpose and B transpose S del E dot transpose F inverse then I can identify that my integrand is trace of AB transpose that I can write as trace of A transpose B. So, doing that. So, this is trace of A transpose. So, A transpose will be J F inverse sigma transpose and now, sigma is a symmetric tensor becomes sigma and B transpose was this. So, B will become F inverse transpose del E dot. So, that is what we have here. So, this is A transpose and this is B.

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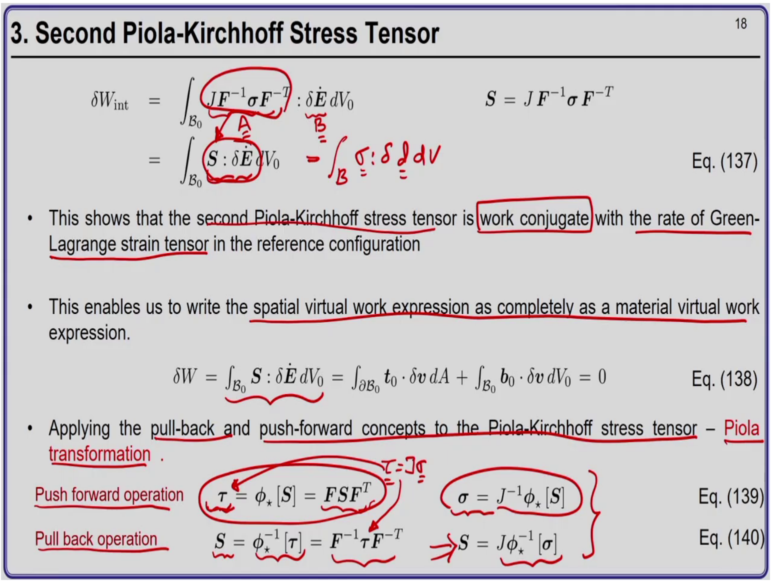

And now because trace of A transpose B is nothing, but A contracted with B I can write J F inverse sigma F inverse transpose contracted with. So, this is our A and this is our B ok.

Now, I can identify this quantity as nothing, but my expression for the second Piola-Kirchhoff stress tensor ok. So, I can write the integral over reference volume of the double contraction of the second Piola-Kirchhoff stress tensor with the rate of Green LaGrange strain tensor.

So, from this I can identify I can say that the second Piola-Kirchhoff stress tensor is work conjugate with the rate of Green LaGrange strain tensor in the reference configuration because this is same as B sigma del d V ok. So, Cauchy stress was work conjugate with the rate of deformation tensor in the current configuration and the second Piola-Kirchhoff stress tensor is work conjugate with the rate of Green LaGrange strain tensor in the reference configuration.

So, as a trivial exercise we can write the spatial virtual work expression completely as a material virtual work expression which means all the quantities are in the material configuration and the integration is also carried out in the material configuration. So, my virtual work expression becomes the internal virtual work written in the material configuration is equal to the external virtual work written in the material configuration.

So, now I can define what is called the Piola transformation which is nothing but the pullback and push forward operations which are carried out on the Piola-Kirchhoff stress tensor and this is called the Piola transformation. So, first I look into the push forward and I know this relation. The Kirchhoff stress tensor is nothing, but FS F transpose ok. Therefore, because tau is J sigma therefore, if I substitute j sigma here I can write sigma as J inverse push forward of the second Piola-Kirchhoff stress tensor.

So, the Kirchhoff stress tensor is nothing, but the push forward of the second Piola-Kirchhoff stress tensor therefore, sigma is nothing but J inverse times T push forward of the second Piola-Kirchhoff stress tensor. And now I can invert this relation I can simply invert this relation over here and get the pullback operation.

So, the second Piola-Kirchhoff stress tensor is nothing, but the pullback of the Kirchhoff stress tensor and this is carried out in this manner therefore, the second Piola stress tensor is the J times the pullback of the Cauchy stress tensor because I can substitute sigma here and then this is nothing, but the pullback of the Cauchy stress tensor. So, these two transformations are called the Piola transformation and they will be visited again when we are discussing the objective stress rates which will come in the later slides.

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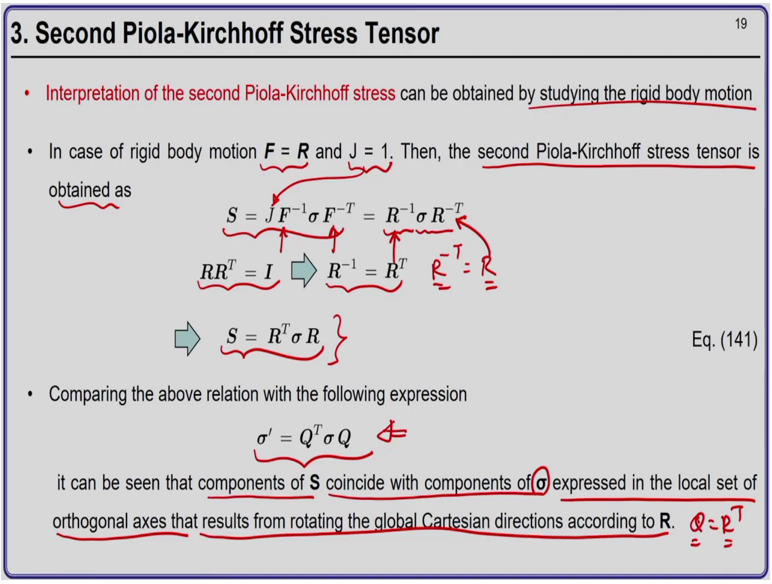

Now, let us look into the interpretation of second Piola-Kirchhoff stress tensor ok. So, we have provided an interpretation for the Cauchy stress which can be considered like a true stress, while the first Piola-Kirchhoff stress tensor can be considered as the engineering stress or the nominal stress.

Now, let us see what is the interpretation of the second Piola-Kirchhoff stress tensor and that can be obtained by studying the rigid body motion ok. So, if there is a rigid body motion then we know that the deformation gradient tensor is nothing but the rotation because the right stress tensor U will be equal to 1 and by right polar decomposition which is equal to R U, U is

1 that is identity matrix therefore, the deformation gradient tensor F will be nothing, but the rotation tensor R and then the Jacobian will be one because under rigid body motion the volume does not change. So, Jacobian is 1 ok.

In that case the second Piola-Kirchhoff stress tensor can be obtained as. So, this is S equal to J F inverse sigma F inverse transpose and now because F is R therefore, and J is equal to 1 I can get R inverse sigma R inverse transpose and now I know that RR transpose is identity. I know that R is an orthogonal tensor. So, RR transpose is identity. So, I can write R inverse as R transpose therefore, R inverse transpose will be equal to R ok.

So, I can substitute this here and I can substitute this here and then I can show that the second Piola-Kirchhoff stress tensor is R transpose sigma R and now if I compare this relation I compare this relation with the following relation sigma dash is Q transpose sigma Q. So, this is the relation for the component of the Cauchy stress how they transform when the coordinate system itself is rotated.

So, this is the relation and then I can see that the components of the second Piola-Kirchhoff stress tensor coincide with the components of the Cauchy stress expressed in the local set of orthogonal axes that results from rotating the global coordinate directions according to R.

So, if I do that in that case my Q is R transpose if I just rotate then I get the similar expression.

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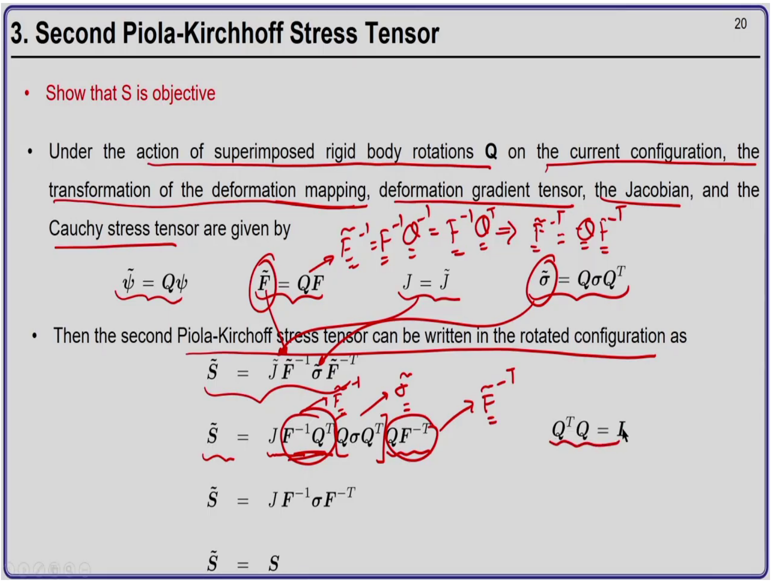

Now, another point to look into is to inquire whether the second Piola-Kirchhoff stress tensor is an objective quantity. Objective means under the super impose rigid body rotation how does its component transform and because its a second order tensor and that as an exercise you can do to show that indeed S is a second order tensor, we want to see whether the components of S or the S itself transforms according to the relation for an objective second order tensor.

So, under the action of super impose rigid body rotations Q on the current configuration, the transformation of the deformation gradient tensor the deformation mapping the Jacobian and the Cauchy stress ok. So, this we had already discussed and they are given by psi tilde is Q psi, F tilde is Q F, J is J tilde because its a scalar quantity. So, it does not change and sigma tilde is Q sigma Q transpose ok. In that case the second Piola-Kirchhoff stress tensor in the rotated configuration can be written as S tilde is J tilde F tilde inverse sigma tilde F tilde inverse transpose.

Now, I can substitute for F tilde here sigma tilde from here and J tilde from here and I can write S tilde is J times F inverse Q transpose and then Q tilde is Q sigma Q transpose. So, this is sigma tilde this relation over here is nothing but F tilde inverse because if you take F tilde inverse it will be F inverse Q inverse and because Q is orthogonal it is equal to Q transpose.

So, that is what we get here and then F inverse transpose. So, if I take F inverse transpose here F tilde inverse transpose, it will be equal to Q transpose transpose which is Q and F inverse transpose. So, that is what I get here. So, this is nothing but F tilde inverse transpose and now I noticed that Q transpose Q is identity. So, in this relation over here let me rub all this.

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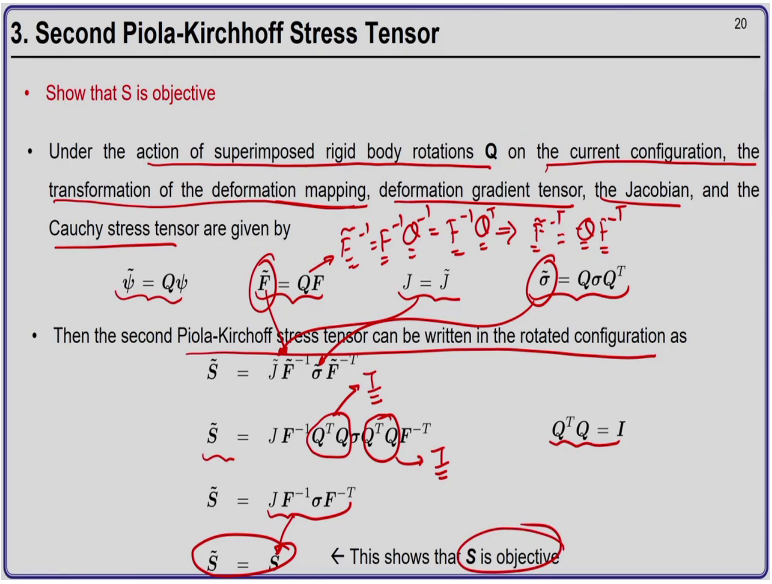

So, in this relation over here Q transpose Q this is identity and also this Q transpose Q will be identity; therefore, what I get is J F inverse sigma F inverse transpose which is nothing, but the expression for the second Piola-Kirchhoff stress tensor.

So, what we see here that the second Piola-Kirchhoff stress tensor has no effect when a super impose rigid body rotations are applied. So, this shows that F S indeed is an objective tensor. Now coming to the decomposition of stresses ok.

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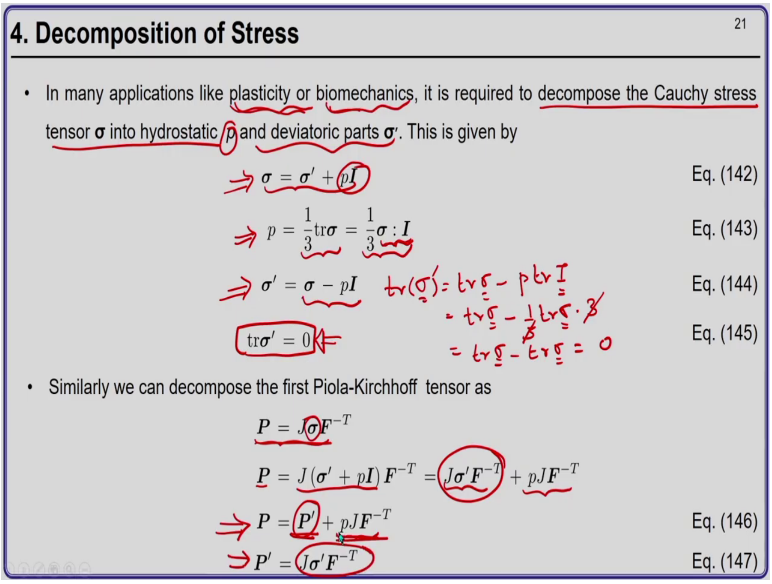

So, in many applications for example, in plasticity and in biomechanics and also in incompressible hyper elasticity which unfortunately we are not doing in this cours;e because of the time constraint, but if you apply the concepts of this course for compress incompressible hyper elasticity, there you will have to deal with decomposition of stress.

So, we are just discussing how the stresses can be decomposed and not only in incompressible hyper elasticity, there are many application like plasticity biomechanics where you are required to decompose the Cauchy stress tensor into what is called a hydrostatic part and a deviatoric part sigma dash.

So, sigma the Cauchy stress can be decomposed as deviatoric part sigma dash plus a hydrostatic part which is p times identity tensor I where p is the pressure and then p is defined as 1 by 3 trace of the Cauchy stress tensor ok. So, it is also trace of sigma can be written as sigma contracted with the second order identity tensor I ok. So, this is 1 by 3 sigma double contacted with I therefore, the deviatoric part of the stress can be obtained as sigma minus the hydrostatic part and also you can show that the trace of the deviatoric part is equal to 0.

So, if you take trace on both the sides you get trace of sigma minus p trace of I and then trace of I ok. So, trace of sigma and p is nothing but defined as 1 by 3 trace of sigma and trace of I is nothing, but 3. So, 3 cancels out and you have trace of sigma minus trace of sigma which is equal to 0. So, the trace of the deviatoric part of the Cauchy stress tensor will be equal to 0.

Similarly, we can decompose the other two stress measures that is the first Piola-Kirchhoff stress tensor and the second Piola-Kirchhoff stress tensor. So, we know that the first Piola-Kirchhoff stress tensor is written I written as J sigma F inverse transpose as we have seen here.

Now, I can substitute sigma given by equation 142 and then I can write P as j sigma dash plus p I F inverse transpose when I open up the bracket the first term becomes J sigma dash F inverse transpose and then the second term is p J F inverse transpose ok. I can write then the part which contains the deviatoric part of the Cauchy stress as P dash which is the deviatoric part I define that as a deviatoric part of the first Piola-Kirchhoff stress tensor P dash plus p J F inverse transpose.

So, the first Piola-Kirchhoff stress tensor can be decomposed into what is called a deviatoric part and a volumetric part where P dash is a deviatoric part is J sigma dash F inverse transpose.

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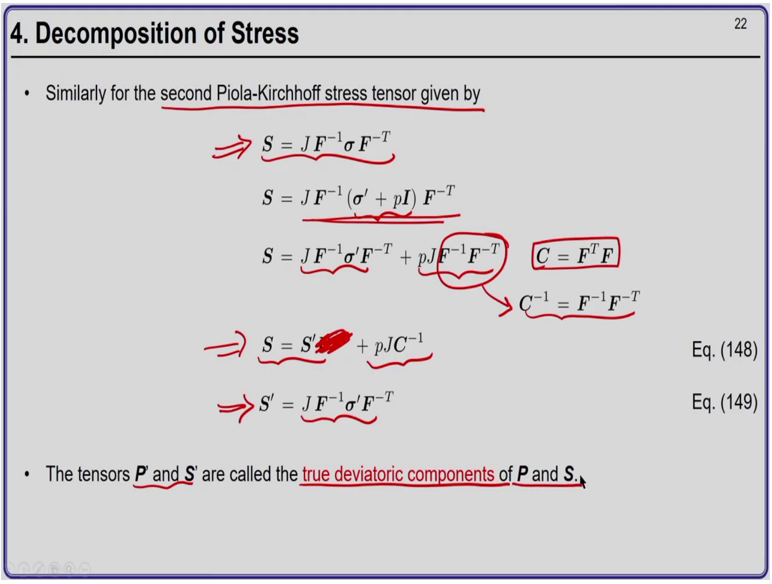

Similarly, for the second Piola-Kirchhoff stress tensor, I can start from the definition of the second Piola-Kirchhoff stress tensor which is given I J F inverse sigma F inverse transpose. I can substitute the decomposition of the Cauchy stress tensor a sigma dash plus p I and I can open up the bracket ok. I can open up the bracket and I can write J F inverse sigma dash and F inverse transpose plus p J F inverse F inverse transpose.

Now, I know that the right Cauchy green tensor is F transpose F or C inverse is F inverse F inverse transpose and if I see here this is F inverse F inverse transpose which is nothing, but C inverse. So, I can write S as ok so, this should not be here. So, S as the deviatoric part plus a volumetric part where S dash is nothing, but J F inverse sigma dash F inverse transpose and this is called the deviatoric part of the second Piola-Kirchhoff stress tensor.

So, these tensors P dash and S dash are called the true deviatoric components of P and S.

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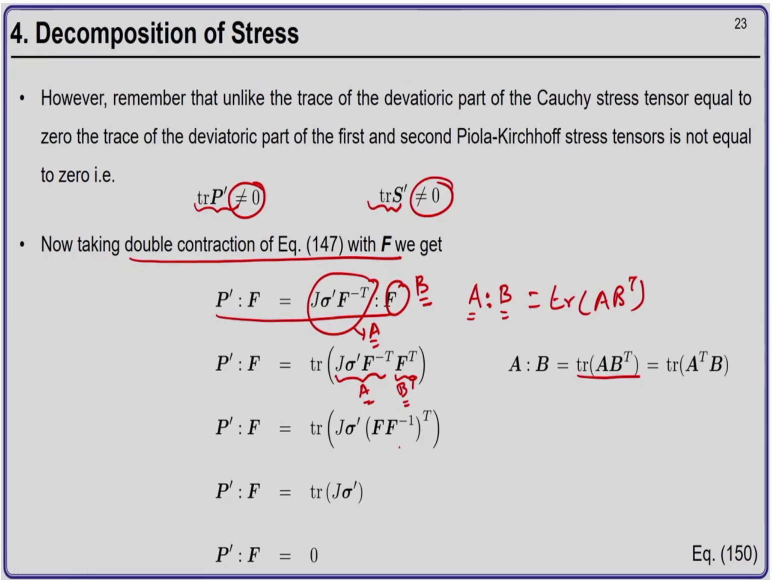

However, unlike we have seen that the trace of the deviatoric part of the Cauchy stress equal to 0, the trace of the deviatoric part of the first Piola-Kirchhoff stress tensor and the second Piola-Kirchhoff tensor stress tensor are not equal to 0 ok. So, this is one difference with the Cauchy stress.

But if I take the double contraction of the deviatoric part of the first Piola-Kirchhoff stress tensor which is P dash with the deformation gradient tensor F, I can write J sigma dash F inverse transpose double contracted with F and now A contracted with B is nothing, but trace of AB transpose trace of AB transpose and this is my A and this is my B. So, this is A into B transpose.

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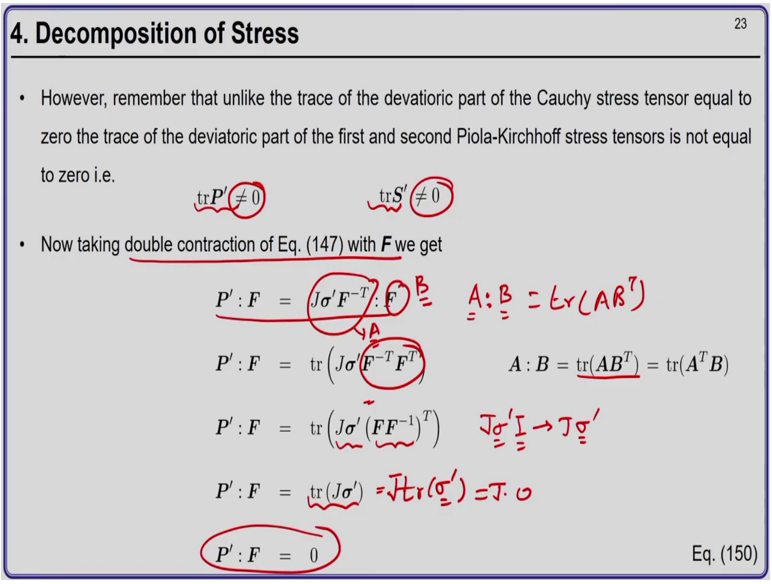

Now, I can concentrate now on the these two expressions and I can take the transpose outside and I can write J sigma dash F, F inverse transpose. Now FF inverse is nothing, but identity and therefore, J sigma dash identity which becomes J sigma dash ok. So, I get trace of J sigma dash which is nothing but j times of trace of sigma dash and now trace of sigma dash with a trace of the deviatoric part of the Cauchy stress tensor is equal to 0. So, J into 0 and this gives me the double contraction of the deviatoric part of the first Piola-Kirchhoff stress tensor with the deformation gradient tensor is equal to 0.

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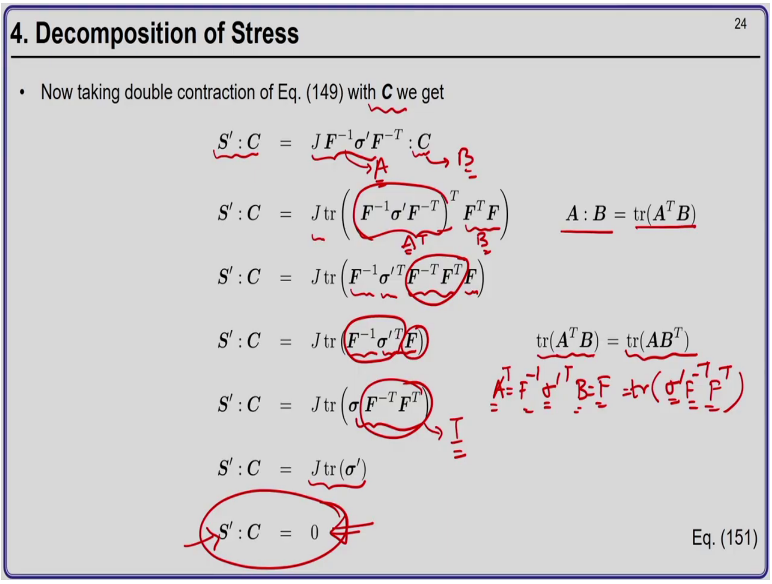

Similarly, I can take the double contraction of the deviatoric part of the second Piola-Kirchhoff stress tensor with the right Cauchy green tensors C ok. So, if I take this I get J F inverse sigma dash F inverse transpose contracted with C and now because this is A and this is B.

So, A contracted with B is nothing but trace of A transpose B I can write this as J trace of A transpose and this is B ok. So, this completely is A transpose B. So, I can open up this bracket, take the transpose I get F inverse sigma dash transpose F inverse transpose F transpose F. So, F inverse transpose F transpose is nothing but identity and now I get j trace of F inverse sigma dash transpose F ok.

Now, if I realize that now I realize that A transpose B is same as trace of AB transpose where A for me now is F inverse sigma dash transpose and B here for me is F. So, A transpose. So, A transpose for me is here F inverse sigma dash transpose and B is F therefore, A transpose B is trace of AB.

So, I can write trace of A becomes sigma dash F inverse transpose and B transpose becomes F transpose ok. So, I get sigma dash F inverse transpose F transpose and we have already shown that F inverse transpose transpose is nothing but identity and this gives us J trace of sigma dash ok.

And now trace of the deviatoric part of the Cauchy stress tensor is 0. Therefore, we have shown now that the double contraction of the deviatoric part of the second Piola-Kirchhoff stress tensor which is S dash with the right Cauchy green tensor C gives us 0 ok. So, they are you can say they are orthogonal to each other.

So, the next we have to discuss is the objective stress measures.