Computational Continuum Mechanics
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Kinetics - 1
Lecture – 15-17
Cauchy stress tensor, Equilibrium equations, Principle of virtual work

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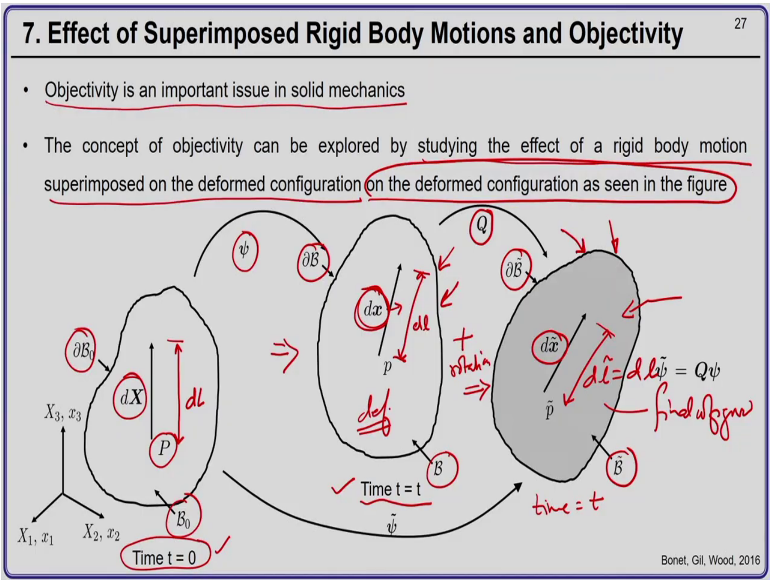

Today, we are going to look into some other concepts in Kinetics ok. So, last class we had done till the concept of Cauchy stress tensor. And today before going into proof that Cauchy stress tensor is an objective quantity and I will come to it what is meant by objective, first let us study and understand the effect of rigid body motions and what is meant by objectivity ok.

So, the title of the slide as you can read is effect of super imposed rigid body motions and objectivity. So, before we discuss and dwell into the mathematical details, physically what can happen during a deformation when you are going from say one-time instance to another time instance the body is deforming. Now there might be situations that the body between two-time step may undergo no deformation and only rigid body rotations or rigid body translations.

It may also so, happen that going from one-time instance to another the body may undergo rotations translations both rigid body as well as some deformation ok.

Now you can very well understand that when a body under goes rigid body motion that is rotation or translation, the body cannot tell up stresses there will be no deformation. So, no deformation means the body cannot develop any stresses ok. So, how do we deal with these kind of situations? Ok. So, we will focus only on the current subject which is focused more on solid mechanics part ok. So, objectivity is a very important issue in solid mechanics ok.

Now you can understand and explore the concept of objectivity by studying the effect of a rigid body motion which is super imposed on the deformed configuration on the deformed configuration as shown in the figure ok. Now you see at time t equal to 0 the body is occupying a certain configuration B 0 bounded by surface del B 0 and you have a material vector dX at point P ok.

Now it is a body undergoes deformation given by this deformation mapping psi ok. So, it occupies a volume in space B bounded by surface del B ok time t that is the configuration that the body is occupying and this material vector deforms to this spatial vector dx ok.

Now, till now what we have considered is between t equal to 0 to t equal to t, the body has undergone purely deformation there is no rigid body motion. But practically in physical situation you may have the body undergoing deformation as well as rotation.

Now as I said earlier the stresses can be developed only because of deformation not because of rotation. So, let us say we had the final configuration ok. So, we had the final configuration which is shown here ok. So, this is obtained. So, this shaded region is obtained ok. So, this is also at time t ok.

But this final configuration is actually what is called the deformation plus rotation equal to this is your final configuration ok. Now between this intermediate configuration and this final configuration you had body undergoing pure rotation which is a rigid body motion therefore, there should not be any stresses develop inside the body ok.

So, let Q be the transformation or the rigid body rotation tensor which causes this body at time t to rotate and occupy this position. So, the body is occupying the position configuration B tilde ok.

So, tilde we above a symbol where we put tilde to show that quantity is obtained by taking rotation and del B tilde is the surface and then this vector dx deforms to dx or rotates to d x tilde ok. Now, between the reference configuration and the current configuration that is this intermediate for example, this vector has changed its length ok.

So, d l d capital L goes to d small l, but when the vector rotates when this vector dx rotates the length of the vector let us say dl tilde will be same as the length of the vector before the rotation because rotation does not change the distance between any two point let us now try to put things in mathematical framework ok.

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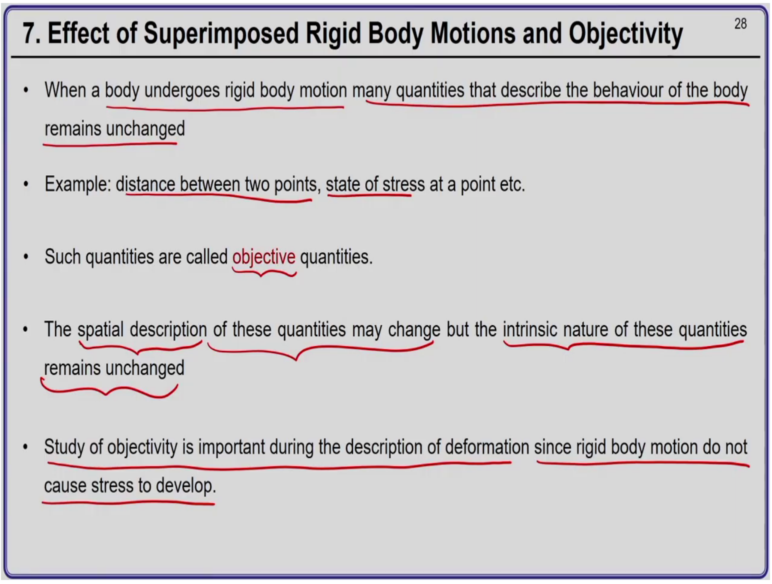

So, as I have written here when a body undergoes rigid body motion there are many quantities that describe the behaviour of the body will remain unchanged which means to describe the behaviour of the body we have many quantities. Like for example, we have green Lagrange strain tensor, we have right Cauchy green tensor there are many such quantities and these quantities will remain unchanged ok.

Now for example, one such quantity is the distance between two points, the another quantity is the state of stress at a point. So, these are the quantities for example, that will not change because, there will be new no new stresses which are generated that the state of stress at a point which and state of stress we defined in the earlier discussions those will these quantity will not change.

So, such quantities are what we refer to as objective quantities ok. So, now, due to this super imposed rigid body motion, the spatial description of these quantities may change.

As you saw in the picture that we had in the previous slide d x the spatial vector dx rotated and became dx tilde so; that means, the spatial description has changed, but what has not changed are some of the intrinsic nature of these quantities ok. So, what is the intrinsic nature of that vector dx? Its the length. So, the length which is given by dx dot dx and taking a square root we get the length.

Now, that quantity has not changed which means if I take dx tilde dot dx tilde and then take the square root I should still get the same length d l ok. Although my vector dx has changed to dx tilde, that is it spatial description has changed, but the intrinsic nature of this quantity which is d x has not changed ok.

So, as I stated earlier study of objectivity is important during the description of deformation, since rigid body motion do not cause stresses to develop ok. So, when the body undergoes rigid body motion there will be no stresses generated inside the body and this we have to ensure that when we are doing our numerical computation and during that numerical computation, if the deformation total deformation of the body is composed of say stretch plus rotation these stresses are computed only because of stretch not because of rotation ok.

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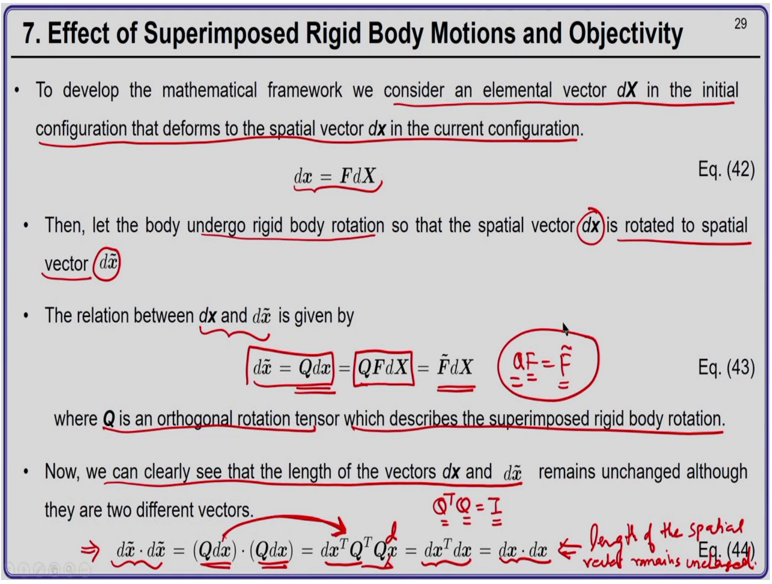

So, how we will do that is we are looking here ok. So, now, to develop the mathematical framework, we consider an elemental vector dX in the initial configuration that deforms to these spatial vector dx in the current configuration ok.

So, we already know these spatial vector dx is the deformation gradient times the material vector d x ok. So, this we already had seen. Now when the body undergoes rigid body rotation for simplicity just take the motion is composed of rigid body rotation, then this spatial vector dx is rotated to another spatial vector dx tilde ok.

So, as I shown in the figure dx rotates to spatial vector dx tilde then the relation between dx and d x tilde is given by dx tilde is Q d x where, Q is a orthogonal rotation tensor which rotates dx to dx tilde ok. Now I can use equation number 42 for spatial vector dx and then I can write Q dx as QF into material vector dX ok. Now I can write QF as F tilde ok. So, what I am doing is QF I am writing as F tilde.

So, this is the deformation gradient tensor which takes me from the initial configuration to the final rotated configuration ok. So, Q here is an orthogonal rotation tensor which describes the super impose rigid body rotation ok. Now as I stated earlier we can clearly see that the length of the vectors d x and d x tilde will remain unchanged.

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Because we are just taking the vector like this pen and we are just rotating it by a certain angle ok.

So, now when you just rotate the length of this pen for example, that vector dx will not change ok. What it means is mathematically dx tilde dot d x tilde should be same as dx dot dx ok. So, dx tilde is nothing, but Q dx ok.

So, there is this Q dx. So, Q d x dot Q d x and that I can write as the first quantity, I can open up the bracket because a dot b is a transpose b and a here is a Q dx. So, I can write the first quantity in the bracket as dx transpose Q transpose Q d x sorry. So, this has to be Q dx ok.

Now, because Q is a orthogonal rotation tensor. So, Q Q transpose is identity. Once I have that I can write d x transpose dx which is nothing, but d x dot dx and I can indeed see that the length of the spatial vector remains unchanged ok. So, remember Qf is f tilde this relation we will use ok.

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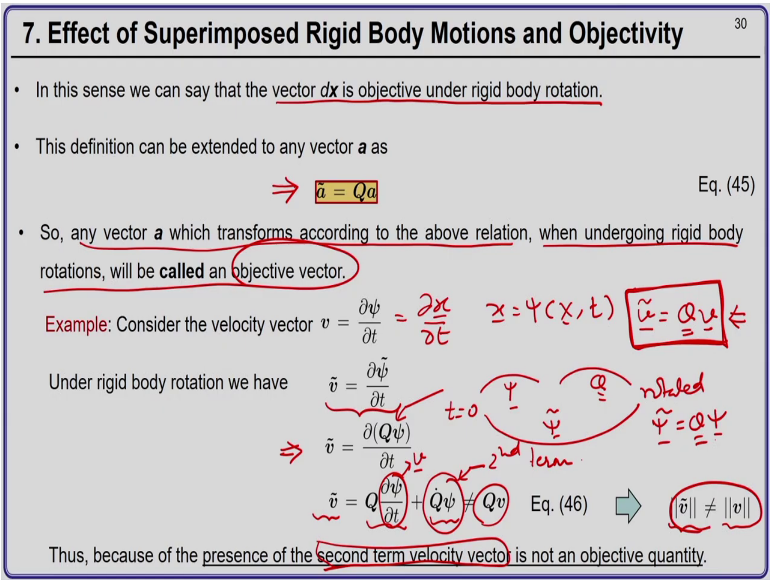

Now therefore, in this particular sense we say that the vector dx is an objective quantity under rigid body rotation ok. So, this intrinsic nature of this vector which is its length remains unchanged ok.

So, this definition we can now generalize and say anything any vector a which transforms according to the relation given by equation number 45 which is a tilde is Q a and a tilde is the vector in the rotated configuration Q is the orthogonal rotation tensor and a is the spatial vector.

Now any vector a which satisfies this condition will be an objective tensor ok. So, any vector a which transforms according to the above relation when undergoing rigid body rotations will be called an objective vector ok. So, whenever we have a relation and we say a certain vector is an objective vector then this relation number 45 will be satisfied ok.

For example, now consider the velocity p which is given by del psi by del t or we can also write del x by del t where x is psi x comma t ok. So, this we have already seen and let us inquire whether velocity vector is an objective quantity is an objective vector. So, if velocity vector is an objective vector then surely it should follow equation number 49 which means v tilde should be equal to Q v.

So, under rigid body motion the velocity vector should transform according to this particular relation. Now let us see whether we get velocity vector as a objective quantity ok.

So, under rigid body rotation v tilde will be nothing, but d psi tilde by d t ok. Now psi tilde is nothing, but Q psi ok. So, this we are already shown between the or not we are already shown, but in the figure if you see on the right hand side there was this relation. So, going from. So, this was psi and this was Q ok. So, this was psi tilde and this was time t equal to 0 and this was the rotator. So, psi tilde will be equal to Q psi now if we substitute psi tilde as Q psi we get v tilde as del by del t of Q psi.

Now, I can open up the brackets and take the derivate and what I get? I get two terms Q del psi by del t plus Q dot psi ok. Now because of the presence of this second term which is shown here this is the second term therefore, we wanted to show that if velocity vector is an objective quantity, it should transform according to this relation v tilde is Qv, but what we are getting?

v tilde is Qv because this is your v Qv plus Q dot psi which is indeed not equal to Qv. So, therefore, we can say that v tilde the norm of v tilde is not same as the norm of v which means that because of the presence of the second term in the velocity vector velocity vector is not an objective quantity therefore, now moving to second order tensors ok.

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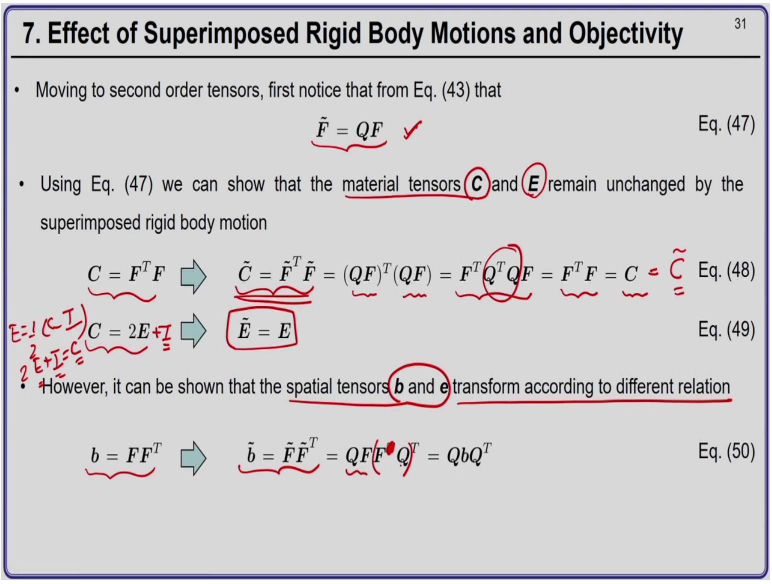

So, before we move that further you notice that we had said the deformation gradient tensor which it takes you from the material configuration to the final rotated configuration, that is F tilde is equal to Q times F ok.

Now we can show that the material tensors that is right Cauchy green tensor and the green LaGrange strain tensor E they will remain unchanged by the super impose rigid body motion. So, let us start we have we know the relation of the right Cauchy green tensor which is nothing, but F transpose F now C tilde would be F tilde transpose F tilde ok.

So, that is how we proceed to show that any quantity is an objective quantity ok. So, whatever maybe the relation what we do? We write that relation with a tilde over it so, that that becomes an expression in the rotated configuration and then we will try to produce the objectivity statement ok.

Now F tilde from equation 47 we see is Q F. So, we get Q F transpose Q F ok. So, the first transpose A into B transpose is nothing, but B transpose A transpose. So, I get F transpose Q transpose QF and I know that Q transpose Q is an identity tensor because Q is a orthogonal tensor ok.

Because Q is an orthogonal tensor Q transpose Q is I therefore, I what I get is F transpose F and I know F transpose F is nothing, but C. So, what I have shown? The right Cauchy green tensor is nothing, but C is equal to C tilde. So, indeed the material tensor C remains unchanged similarly I know that C ok. So, C is not 2E ok. So, E is 1 by 2 C minus I ok. So, 2E plus I equal to C ok. So, there should be I here ok.

So, C equal to 2 E plus I and I can show because C is equal to C tilde I can show E tilde is E or also the green Lagrange strain tensor is an objective quantity, it does not change with rigid body motion ok; obviously, because these things are defined in the initial configuration. So, they will not have any effect when you rotate the spatial configuration ok. Now however, we can show that the spatial tensors transform according to different relation ok.

We know that b that is the stretch tensor is equal to F F transpose. Now writing this in the rotated configuration we write b tilde is F tilde F transpose what does it make? F tilde is given by equation number 47. So, I can write Q F into F Q transpose and then I take the a b transpose as b transpose a transpose. So, I get F transpose Q transpose ok.

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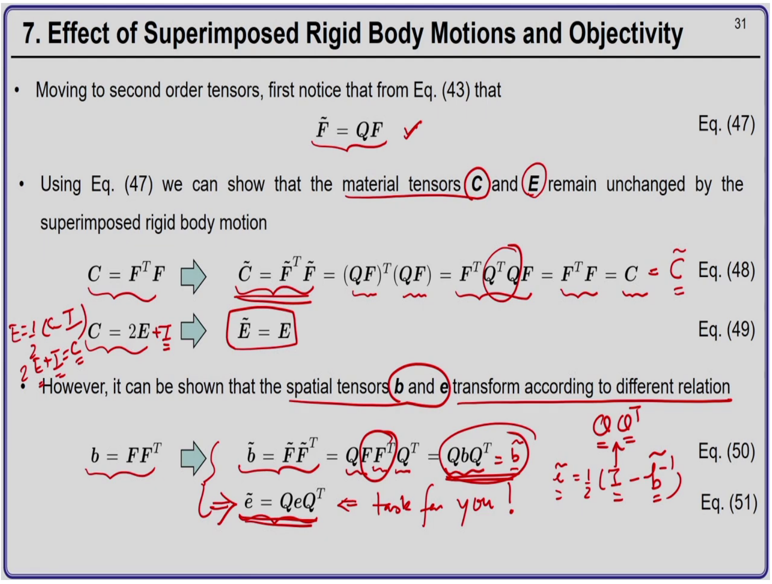

Now this middle term over here FF transpose is nothing, but b. So, I get b tilde as Q b Q transpose similarly I can show that E tilde will be Q e Q transpose.

So, this is to show that indeed e tilde is Q e Q transpose, you know the relation for e, e is nothing, but 1 by 2 I minus b inverse ok. So, e tilde will be I minus b tilde inverse and then b tilde relation you already know from here, you just substitute and you try to show that you get the following relation and this Q I will be Q Q transpose ok.

So, you can take Q from the left hand side Q transpose from the right hand side and you can show that the Euler Almansi strain tensor transforms according to the following relation ok.

Now, it seems there are these tensors b tilde I mean b and e are not objective, but we said the spatial description may change, but their intrinsic value or intrinsic nature of these quantities will not change. As you can see from expression over here b tilde is Q b Q transpose.

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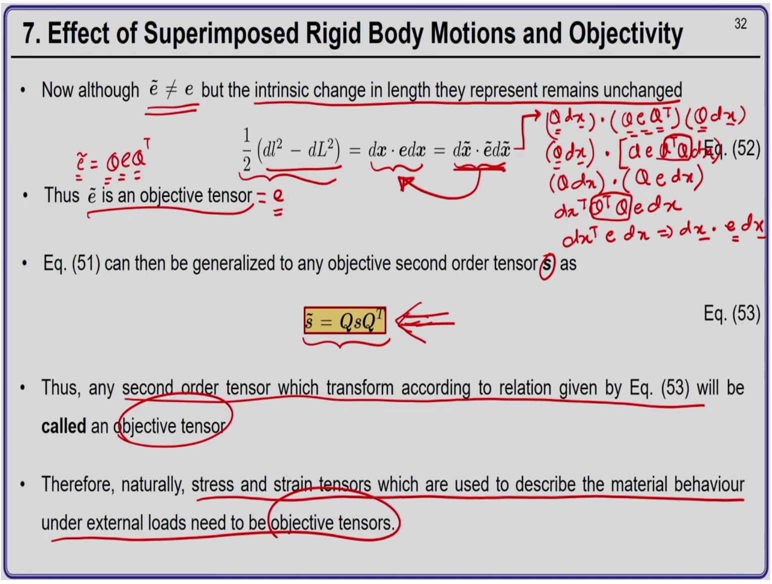

So, the tensor itself has changed, but now let us show that what has not changed ok. Now although say e tilde is not equal to e, but we know that the intrinsic change in length they represent remains unchanged ok.

You will remember from our discussion in kinematics that, the half of the change in the square length of the final element ok. The element in the current configuration minus the square of the length of the element in the reference configuration is given by dx dot e d x and this I can show that it is indeed equal to dx tilde dot e tilde d x tilde ok.

To show that I can start from here and dx tilde is nothing, but Q d x and what is e tilde? e tilde is Q e Q transpose and then dx is Q dx tilde is Q d x ok. So, I am just trying to show that indeed this quantity is equal to this quantity ok.

So, now I can write Q dx dot Q e Q transpose Q d x. Now Q transpose Q is identity. So, I can write Q d x dot Q e d x. Let us not put double under bar its clear from the context. So, now, I can write this as dx transpose Q transpose Q edx Q transpose Q is identity. So, this is d x transpose edx which means this is nothing, but dx dot e d x ok.

So, therefore, although the Euler Almansi strain tensor has changed that is its spatial description has changed, but the intrinsic nature of this tensor which is given by the difference of the lengths square of the lengths has not changed ok.

Therefore, e tilde ok. So, e tilde given by Q e Q transpose will be called in a objective quantity ok. So, therefore, e tilde or rather e is an objective tensor ok. So, this we can generalize to any second order tensor s.

Let us say s is a second order tensor in the spatial configuration therefore, if this tensor transforms under rigid body rotation according to equation number 53 which is s tilde is Q s Q transpose, then this second order tensor s will be called a objective tensor that will be called an objective tensor.

We there will be a lot of second order tensor one of the most important is the Cauchy stress tensor and therefore, we have to later on show that Cauchy stress tensor is indeed an objective tensor that will show in the coming slides ok. So, thus any second order tensor which transforms according to the relation given by equation 53 will be called an objective tensor ok.

So, an objective second order tensor transforms according to this relation under rigid body motion ok. Therefore, naturally stress and strain tensor which are used to describe the material behaviour under external loads need to be objective tensors that is why because the product of stress and strain gives you energy and when the body undergoes rigid body motion.

So, the internal energy which is there as a product of stress and strain should not change ok, but the physical or the spatial description of these tensors which is stress and strain sensors will change. Therefore, if these tensors are objective then the intrinsic nature of these tensors which is nothing, but the internal energy will not change which means there will be no new stresses generated inside the body because of the rotation.

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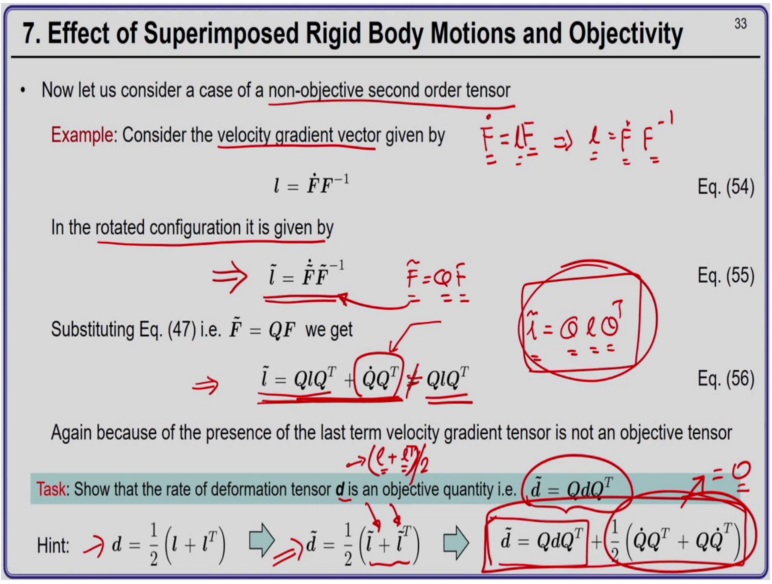

Now let us consider a case of a non objective second order tensor ok. Let us consider the velocity gradient tensor. Now we know that the material time directive of the deformation gradient tensor was given by lF F dot is lF therefore, l is nothing, but F dot F inverse and let us investigate whether the velocity gradient tensor is an objective quantity, its not an objective quantity and we will show it.

Now, let us write the velocity gradient tensor in the rotated configuration l tilde is F tilde dot F tilde inverse ok. Now we know that F tilde is Q F ok. F tilde is Q F therefore, I can substitute this here and if I do simplifications this I will leave it to you I will get l tilde is Q l Q transpose plus Q dot Q transpose ok.

Now if velocity gradient tensor l was an objective quantity, l tilde should transform according to this relation ok. Well l is a second order tensor it should transform according to this relation, but when we see this expression over here we see that we have this second term which is Q dot Q transpose which in general will not be equal to 0 it will not be equal to 0 tensor.

Therefore, since we are not getting this relation therefore, the velocity gradient tensor is not an objective quantity. So, this is not equal to Q l Q transpose therefore, the velocity gradient tensor is not an objective quantity ok.

We can show that the rate of deformation tensor d which is nothing, but which is the symmetric part of the velocity gradient tenor d is l plus l transpose by 2. We can show that indeed the rate of deformation tensor d is an objective quantity which means d tilde transforms d transforms according to the relation d tilde is Q d Q transpose and how do you show that? You already know that l tilde transforms according to this relation Q l Q transpose plus Q dot Q transpose ok.

So, you know d is the symmetric part you can write d tilde as 1 by 2 l tilde plus l tilde transpose and from equation 56, you can substitute the value of l tilde here and then you can simplify to get the following expression and then you can show that this quantity in the bracket here the second quantity is equal to the 0 tensor how to show that?

I leave it for you as an exercise its very simple to show if you show that this is equal to 0 then d tilde is Q d Q transpose and therefore, the rate of deformation tensor is an objective quantity.

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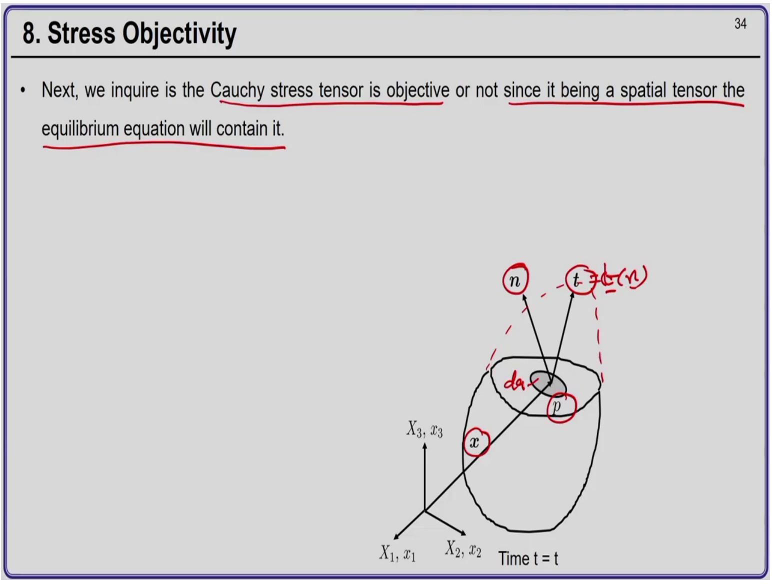

Now, coming to the specific topic of stress objectivity ok. Stress tensor the Cauchy stress tensor is an second order tensor and now let us inquire whether this Cauchy stress tensor is an objective tensor or not ok. Since it is a spatial tensor and it will be used in the equilibrium equation therefore, and this equilibrium equation will see in the coming slides how to how we will get that expression it will contain the Cauchy stress tensor ok. So, now, it is essential that we investigate whether the Cauchy stress tensor is an objective tensor.

So, now you have this body, let us say you had this body and you cut it open ok. So, you had the body, the body was under the action of certain externally upright tractions because of that the body has developed certain stress ok. So, at a spatial location x at point t when you cut it open, let this be area d a which is characterized by normal n and let t be t traction at this point ok.

So, t is the traction at this surface given by the t equal to function of normal n now let us see if we give a rigid body motion.

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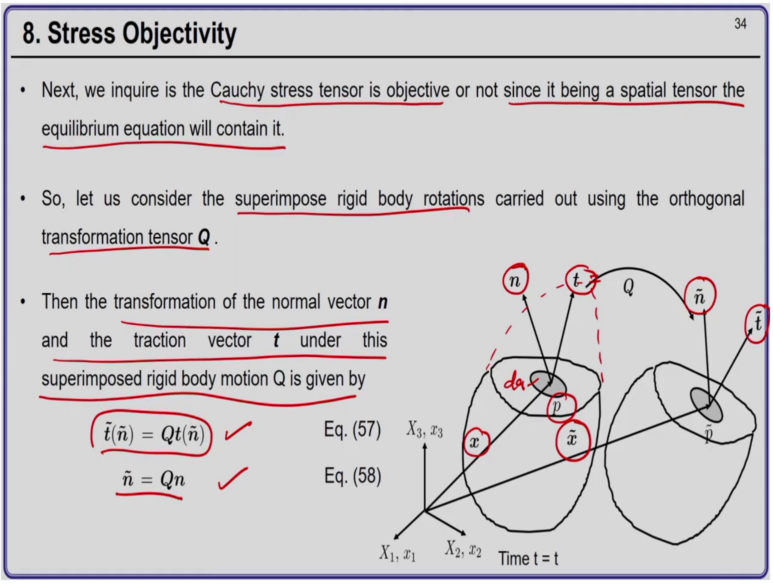

Now, if we give superimpose rigid body rotation using the transformation tensor Q orthogonal transformation tensor Q, your spatial description of the normal and the traction vector will change to n tilde n t tilde and the current position spatial position of the point is x tilde.

So, then as I said here the transformation of the normal vector n and the traction vector t under this super imposed rigid body rotation will be given by t tilde is Q t and n n tilde is Q n.

Now, because n and t are objective vectors therefore, t tilde is Q t and n tilde is Q n ok. This you can show that the length of n tilde will not change and also t will not change therefore, t tilde and t and n are objective quantities and in are given by 57 and 58 ok.

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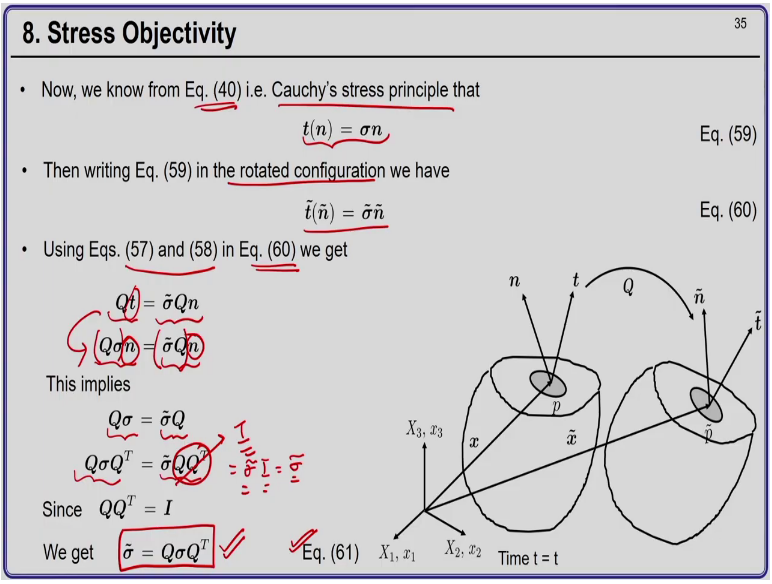

Now we know from Cauchy stress principle that traction at a point traction on a plane whose normal is n is given by sigma times n where sigma is the second order Cauchy stress tensor. This we had derived in equation number 40 in the previous slides.

Therefore now, to investigate whether sigma is an objective quantity let us write equation 59 in the rotated configuration and this is written as t tilde is sigma tilde n tilde ok. So, what we are done? We are just written put it tilde over each quantity in equation number 59.

And now we have been given that traction vector t and the normal n are objective vectors therefore, what we can do is substitute equation number 57 and 58 from the previous slides in equation number 60 and what we get? Q t is sigma tilde Q n this is what we get ok. Now from equation 59 I can see that traction vector t is nothing, but sigma n and that is what I substitute here.

So, I can put equation number 59 here and I get Q sigma Q n is equal to sigma tilde Q n ok. Now if you notice we have n on the right hand side on both the left hand and right hand side and n is present in both the expressions on the left hand and right hand side therefore, this implies that Q sigma is sigma tilde Q.

And now if I just post multiply both sides of the equal to sign by Q transpose I get Q sigma Q transpose a sigma tilde Q Q transpose and because, Q is an orthogonal tensor therefore, Q Q transpose is identity which means this quantity over here becomes identity therefore, sigma tilde I will be equal to sigma tilde ok.

So, therefore, I get sigma tilde is Q sigma Q transpose and if we match this expression compare this expression with our expression for the objectivity of any second order tensor which is s tilde is qsq transpose we see that equation number 61 matches with the expression that we had in the previous slides ok. Therefore, since this expression is the expression for a objective tensor therefore, sigma or the Cauchy stress strain tensor is indeed a objective tensor ok.

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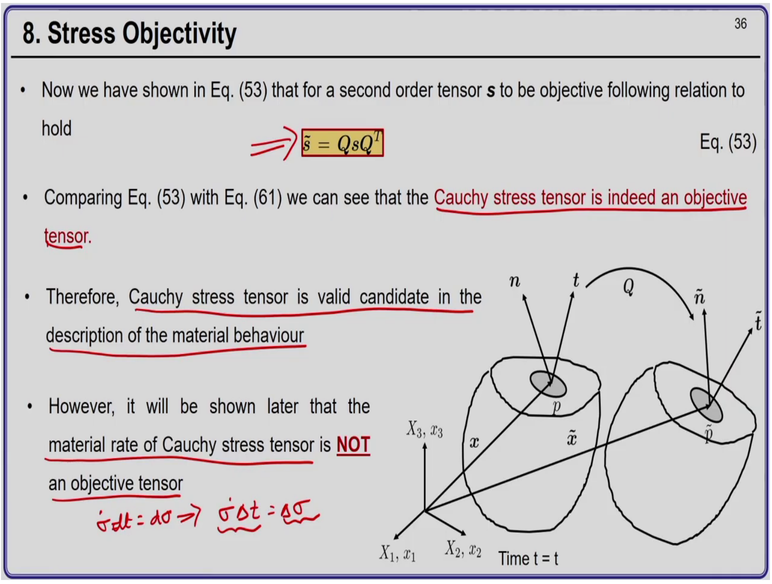

So, which shows that and this if we compare with this expression that is what I said we see that Cauchy stress tensor is indeed an objective tensor that is we have shown ok. And therefore, what this means is Cauchy stress tensor is a valid candidate in the description of the material behaviour ok.

So, we can use this tensor in describing the material behaviour under the action of external loads. However, we should remember and we will see it later that the material rate of the Cauchy stress tensor the rate of change of Cauchy stress tensor which is sigma dot that quantity is not an objective tensor.

So, Cauchy stress sigma is objective, but the rate of Cauchy stress sigma dot is not an objective quantity ok. So, I cannot use sigma dot to compute the increment in the stress. So, sigma dot is nothing, but sigma dot into d t will be equal to d sigma and in case of finite time interval sigma dot delta t will be equal to delta sigma.

See if I have to compute the increment in stress between two time steps, I cannot simply write sigma dot delta d to get the increment in the Cauchy stress because sigma dot is not an objective quantity ok. So, I have to come up with some other objective stress measures that we will see in the next module towards the end of next module where we will discuss a lot of objective stress measures.

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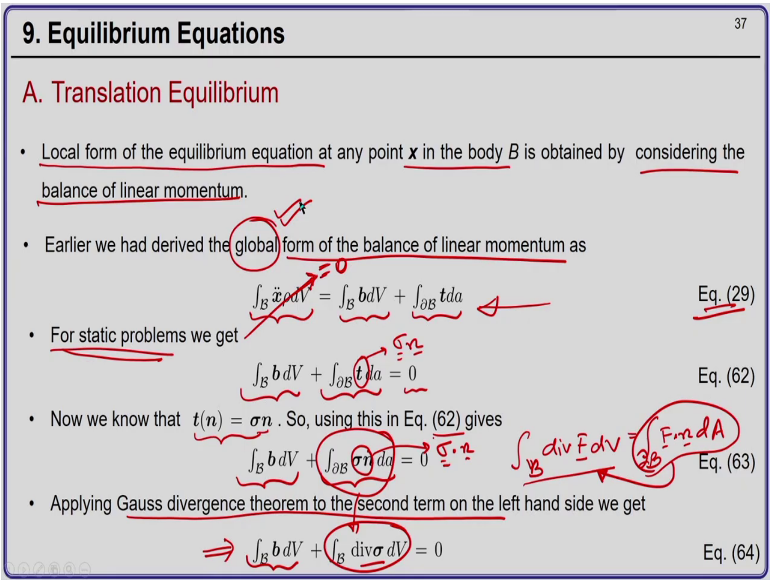

So, let us now move to the topic of equilibrium equation. So, now, we have shown the Cauchy stress principle and also we have shown that the Cauchy stress tensor is an objective quantity ok. So, to derive the equilibrium equation first we consider the translational equilibrium that is the conservation of the linear momentum ok.

So, earlier we have derived what was called the global form of conservation of linear momentum for a deformable body. Now what we want to do is we want to derive the local form of the equilibrium equation at any point x in the body and this we can obtain by considering the balance of linear momentum.

So, remember earlier we had derived what is called the global form of balance of linear momentum and that was given by the volume integral of the inertia forces should be equal to the volume integral of the body forces plus the surface integral of the external apply tractions ok.

So, this was already derived in equation number 29 and then for static problem this and in this course we deal with static problem only because if we consider the inertia there is an additional term for time which comes and which needs time integration and because of the number of lectures been fixed we do not drill into the dynamic problems ok.

We only deal with static problem therefore, the left hand side is equal to 0 vector therefore, the volume integral of the body forces, but the surface integral of the traction will be equal to 0. Now from the Cauchy stress principle I know that traction is traction at a point p on a plane whose normal is n is given by sigma into n now if I use this in equation number 62 ok.

In the second term I can put t equal to sigma n. So, what I get? The body the volume integral of the body forces plus the surface integral of sigma n da should be equal to 0. Now remember now I can apply the Gauss divergence theorem on the second term ok. So, Gauss divergence theorem says that divergence of a vector field F integrated over the volume is same as ok.

So, this is B del B F dot n d A ok. So, divergence of the vector field f over the volume can be related to f dot n da integrated over the area. So, I can convert a volume integral to a surface integral therefore, this surface integral can now be converted to a volume integral which is given here ok.

So, we have sigma n which is nothing, but something like sigma dot n its also sigma dot n ok. So, I can write divergence of sigma ok. So, I now I have the volume integral of the body forces plus the volume integral of divergence of Cauchy stress tensor should be equal to 0 ok.

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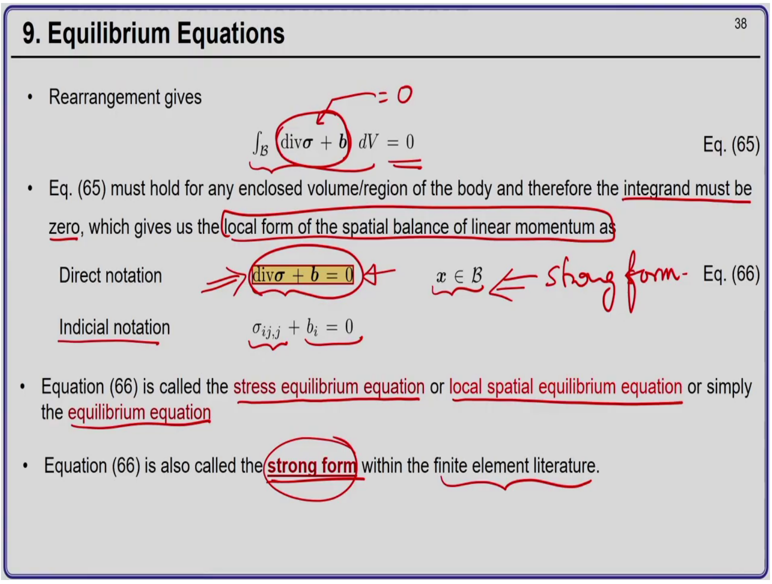

Now I can collect both the terms inside one bracket and rearrange to and get the volume integral of the quantity divergence of sigma plus the body forces per unit volume times the volume equal to 0 ok.

Now this relation 65 is valid for all volume which means that the integrand should be equal to 0. So, this is the integrand and this integrand should be equal to 0 for equation number 65 to hold which means that we just simply put divergence of sigma plus b equal to 0 and this is called the local form of the spatial balance of linear momentum ok.

So, the previous expression that we had here equation number 29 was the global form and here equation number 66 is your local form local because it is valid for any point x which belongs to the current configuration ok. So, that is the direct notation and in the indicial notation I can write sigma i j comma j this is the divergence plus b i equal to 0. So, that is the local formed of the balance of linear momentum in the spatial configuration.

So, equation number 66 is also called the stress equilibrium equation or the local spatial equilibrium equation or simply equilibrium equation. If we had inertia forces also you to do have been called equation of motion ok. Now in the context of finite element literature equation number 66 is also called the strong form ok.

So, this equation over here is also called the or referred to as the strong form ok. Strong form this is because this equation number 66 has to hold for all points x belonging to the current configuration and there are infinite number of such point. So, this equation given by 66 has to hold for all points and also in the context of displacement based finite element method the continuity requirement on the approximation for displacement is higher ok.

We will come to it later that is why it is called the strong form ok. So, the continuity requirement if you use displacement based finite element if you directly want to solve the strong form the continuity requirement will be 2 that is why we need to develop the what is called the weak form and that we will do in the next few slides.

So, next we move to what is called the rotational equilibrium ok. Till now we have seen translational equilibrium now we move to rotational equilibrium and let us see what does rotational equilibrium which means the law of conservation of angular momentum what does it has to imply vis a vis the Cauchy stress tensor.