Computational Continuum Mechanics
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Kinetics - 1
Lecture - 15-17
Cauchy stress tensor, Equilibrium equations, Principle of virtual work

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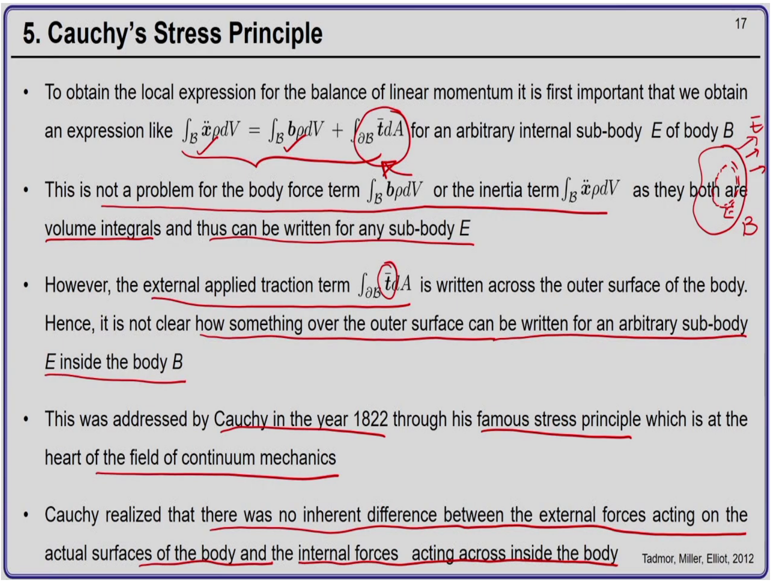

So, the next point we discuss is the Cauchy Stress Principle, ok. So, we have discuss the global form of balance of linear momentum. So, that was in integral form.

So, now we wish to obtain a local expression ok, for balance of linear momentum, ok. And for this it is important that we obtain an expression like this ok, for an arbitrary internal sub body

E of body B, ok. So, you have this body B and then B consider a sub body of body B that is E. And now we wish to obtain a similar expression like this here for the sub body E, ok.

Now, it is not a problem to get a similar expression for these two for the sub body E, because they are volume integrals and then they can be written as well for the sub body E.

However ok, so that is what I have stated here. This is not a problem for body force term or the inertia term as they both are volume integrals and thus they can be written for any sub body E, ok.

However, the problem comes for the traction term that we have; the second term on the right hand side, which is this term over here.

Now, remember this t bar over here is a externally applied traction, ok. So, here it is a externally applied traction. So, this is applied on the physical surface of the body, not on the surface of sub body E, ok.

Now, at this point it is not clear to us how which is an integral, which is written over the outer surface can be written for an arbitrary sub body E, which is inside the body B.

So, this issue that how do we write this integral, surface integral on the physical surface in terms of the surface integral on the surface of sub body E was address by Cauchy in the year 1822, when he gave his famous stress principle which lies at the heart of the field of continuum mechanics.

So, what Cauchy realized was that, there was no inherent difference between the external forces acting on the actual surfaces of the body and the internal forces acting across inside the body, ok.

So, what he realize that, it does not make any difference ok; the way the tractions would act on the outside surface of the body, the traction would act in a similar way on any inside surface of the body ok, sub body E, ok. Then the magnitude of the traction maybe different, but the nature in which they are applied is same.

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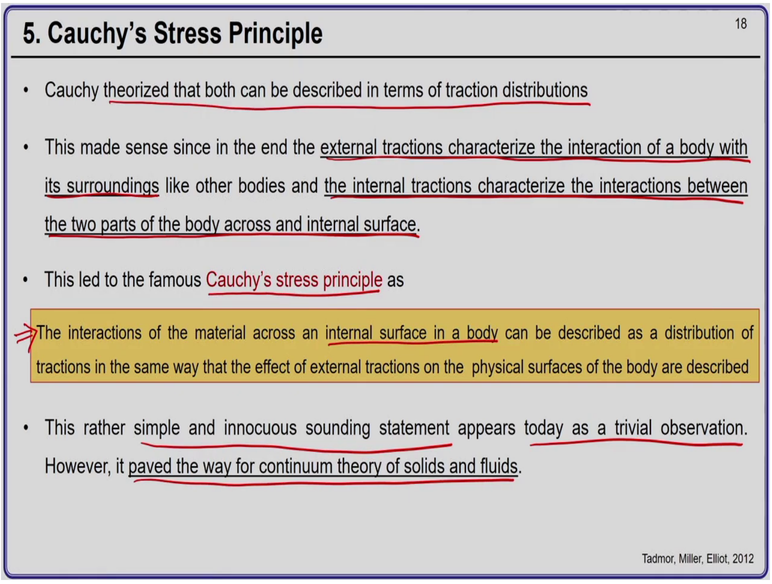

So, then Cauchy theorized that the both can be described in terms of traction distributions, ok. And this would make sense, because in the end external tractions characterize the interaction of a body with its surrounding like other bodies and the internal tractions characterize the interactions between two parts of the same body across the internal surface.

So, what this means is the way the external tractions ok, happened between the body and the surrounding which may be another body; in the same way the traction happen in internal tractions between two different parts of the body interact in the same way, ok.

So, this led to his famous Cauchy stress principle which is stated here. And it states that, the interaction of the material across an internal surface in a body can be described as a distribution of tractions in the same way that the effect of external tractions on the physical surfaces of the body are described, ok. So, you can describe the internal tractions in the same way as you would describe the external tractions on the physical surface of the body, ok.

So, this is rather very simple statement, but and also it may seem a very trivial observation; but back in the day this paved the way for the continuum theory of solids and fluid.

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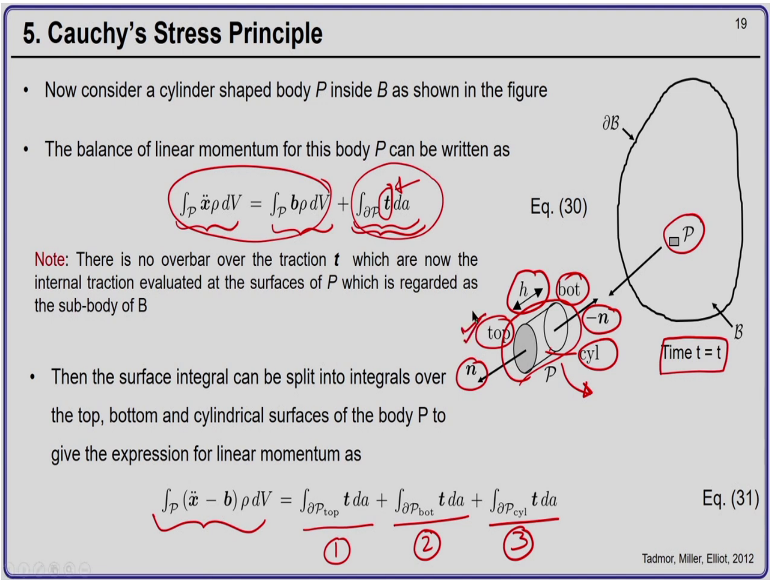

So, now let us consider a cylindrical shape body B as shown in the figure. So, you have a body B at time t and now let us consider a small cylindrical shape body P.

So, this is point P and this is an infinitesimal small cylinder ok, we consider, ok. So, the top surface is given by top, there is a bottom surface and there is a cylindrical surface, ok.

So, the normal to the top surface let us say it is n, and the normal on the bottom surface is minus n and then the height of this cylinder is given by h, ok. With this we can write the balance of linear momentum for this body using the Cauchy stress principle as the inertia contribution, ok. So, the inertia term equal to the total body forces inside the body P plus the tractions on the surface of body P, ok.

Just notice that we have not put any bar on top of traction t, which means that t bar is an external traction and this t is a internal traction, ok.

So, as you take out the cylinder from the body, if you can imagine, if you cut the body P and take it outside; so the internal forces which were there before the body was extracted, will now become the external traction for the cylinder, ok. So, these external tractions on P are actually internal tractions on body B. So, we denote it by symbol t.

Now, we can split this integral on the surface of the, cylinder P into three terms, ok. So, we just take the inertia in the body forces on one side and then this surface integral can be broken into three parts; the first part, the second part, and the third part.

So, the first part is the integral over the, what we call the top surface, ok. So, let me rub this here.

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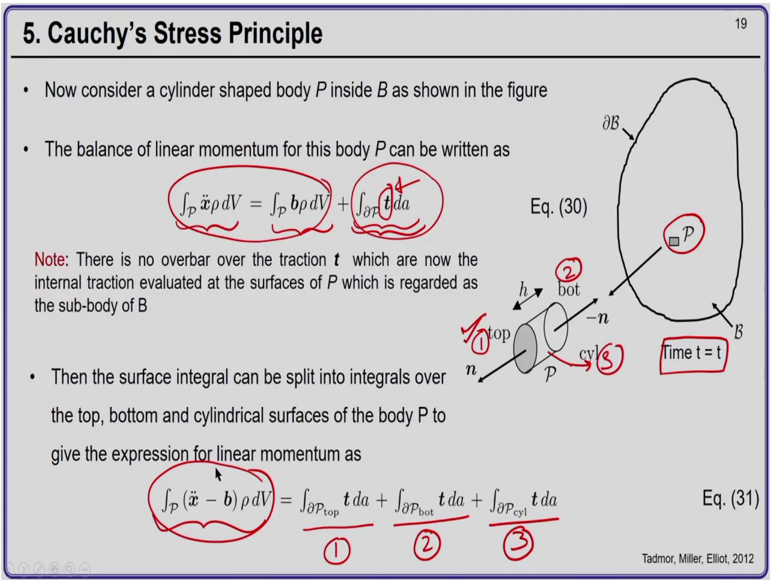

So, the first term is over the top surface, the second term is over the bottom surface and the third integral is over the cylindrical surface, ok. So, this cylinder has three surface; top, bottom and the curved surface. So, I can split the surface integral into three part, ok, which is shown in equation 31, ok.

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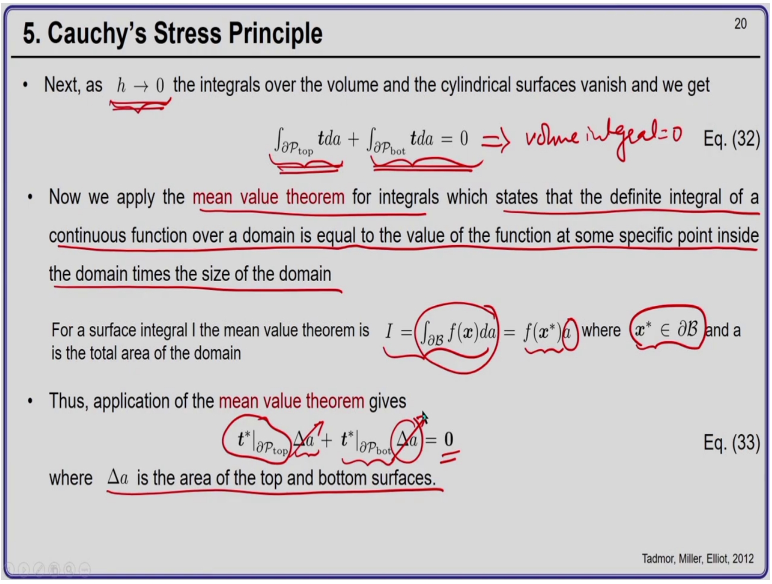

Now, I can make the cylinder go to this height of the cylinder to go to 0; which means I start bringing the top and the bottom surfaces closer to each other. So, what would happen? The cylinder will become at the infinitesimal limit will become a surface and this cylindrical surface will go away, ok.

So, cylinder will become a 2 D surface. So, also this term, the volume integral will go to 0; because as I approach h to 0 volume goes to 0, therefore, all volume integrals will also go to 0, ok. And also the integral over the cylindrical surface; because there once h tends to 0, there is no cylindrical surface left ok, therefore that also goes away.

So, in equation 31 on the right hand side only two terms remain, which is the integral over the top surface and integral over the bottom surface, ok. And that should be equal to 0, because that is the volume integral, ok. So, the volume integral is 0.

Now, we apply the mean value theorem of integrals ok, which states that the definite integral of a continuous function over domain is equal to the value of the function at some specific point inside the domain times the size of the domain, ok.

So, what it means is, if you have an integral over the say, area which is I of a continuous function f x ok; then this integral will be equal to the value of the function at some point x star which belongs to the surface times the total area, ok. So, that is the mean value theorem.

So, this I can apply to our two terms in equation 32, ok. And when I apply this, let us say t star is the traction ok, value of the traction at that point x star, times the total area of the top surface which is let say is delta a plus the value of the traction at the bottom surface at some point x star ok, times the area of the bottom surface which is same as the area of the top surface that is delta a and that should be equal to 0, ok. So, delta a is nothing but the area of the top and bottom surfaces.

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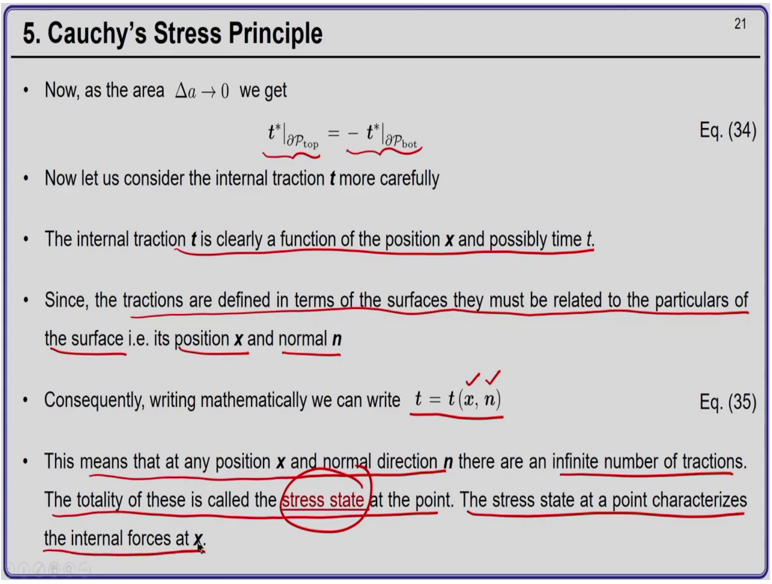

Now, what I do is, I now start making this area go to 0, ok. Now as I start making this area go to 0, what I get? So, here you can realize delta a as it goes to 0 what happens; your traction at the specified point on the top surface will be equal to the negative of the traction at the specified point at the bottom surface.

So, now let us consider this internal traction t more carefully, ok.

So, clearly this internal traction t is a function of the position x and possibly time t, ok. And tractions are specified in terms of the surfaces they must be related to the particulars of the surface; therefore the tractions must be related to the surface on which they act and the quantities that define those surfaces are its position which is x and the normal to the surface which is n.

Therefore, I can write mathematically that the traction is a function of both the position of the surface on which it acts and also the normal.

This means, what this means is that any position, at any position x in the normal n there are infinite number of such tractions, ok. So, at any point x, you can have infinite number of normals; therefore you can have infinite number of tractions.

So, the totality of all these is what is called the stress state at that particular point. So, the totality of all the stress vectors at that point for all possible values of n is called the stress state at that point, ok. So, the stress state at a point characterizes the internal forces at the special location x.

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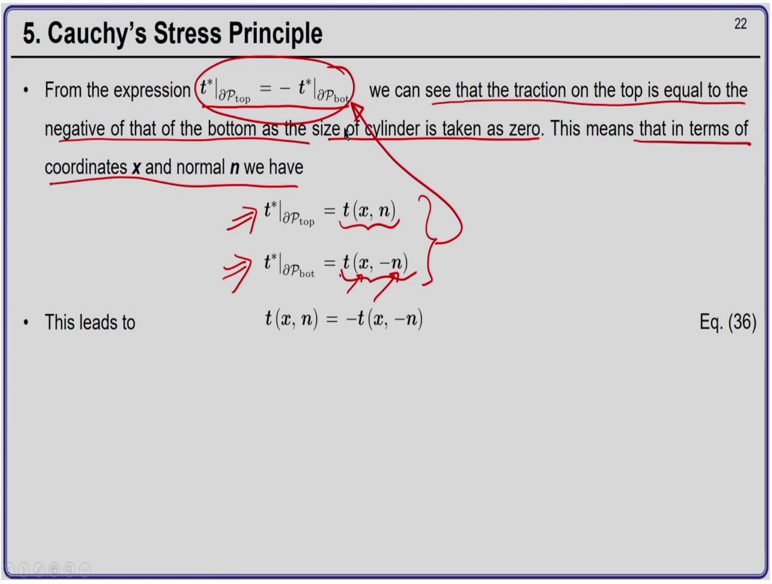

So, now coming back to our expression t star at the top surface is equal to minus of t star at the bottom surface. From this we can see that the traction on the top is equal to the negative of the bottom as the size of the cylinder is taken to 0. Which means that in terms of the coordinate x and normal n, the traction at the top surface will be function of position x and the normal to the top surface, which in our case is n. And then the traction at the bottom surface will be equal to the position x and then the normal to the bottom surface, which in our case is minus n.

And then using these two expression, here in this particular expression, ok. So, if I can substitute these expression here, then what I get.

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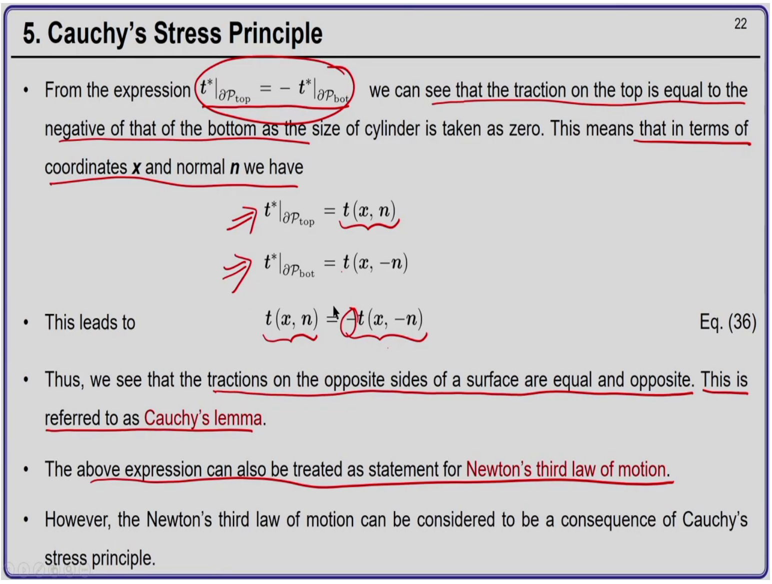

Let me rub this. So, what I get is the traction at the top surface is minus of traction at the bottom surface.

So, the t as a function of x comma n is same as minus of t as a function of x comma minus n, ok.

Thus we see that the tractions on the opposite sides of a surface are equal and opposite and this is referred as Cauchy's lemma, ok.

So, the traction on the same point inside the body, ok. So, the traction on one side will be equal to the negative on the traction on the opposite, 180 degree opposite side.

So, the above expression can also be treated as a statement for Newton's third law of motion ok; that is there is an, when two bodies are in contact, there is an equal and opposite reaction, ok. So, this is equal and opposite reaction. So, the magnitude are same and they are in opposite direction, ok.

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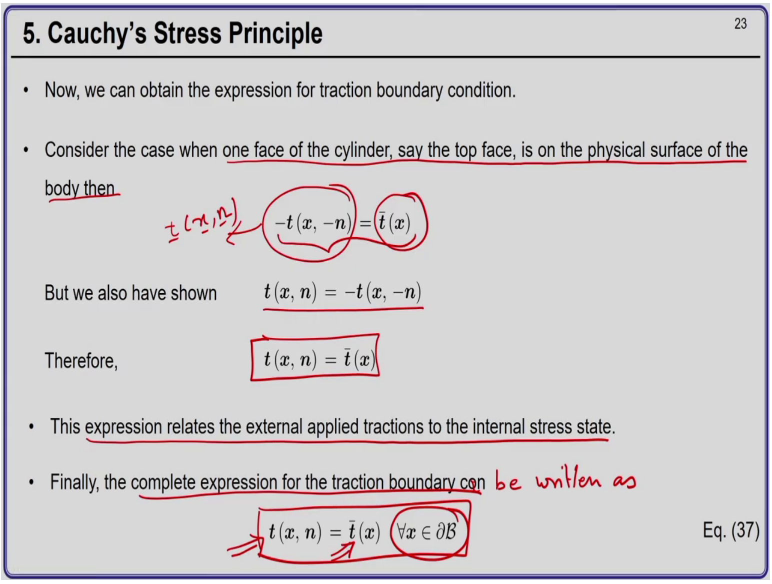

Now, we wish to obtain the expression for the traction boundary condition.

Now, let us consider now that some part of the sub body E now coincides with the external physical surface of the body, ok. Now, what happens ok? In that case, when one face of the cylinder, say the top face, is on the physical surface of the body, ok. So, we had the cylindrical shape body.

Now, let us say the top face of that cylindrical shape body coincides with the physical surface of the body, where you have certain traction. In that case, we have the externally applied traction at that point should be equal to minus of t x comma minus n, ok.

And now, I know from Cauchy's lemma that this quantity over here is also nothing, but t x comma n from the previous slides we know that ok; that is what we already know from Cauchy's lemma.

Therefore the traction t inside the body at position x, whose normal is in the direction n is equal to the externally applied traction at that point. What it means is on the physical surface, say the bottom is your body and the top is the outside of the body. Now the applied traction at the outside will be equal to the traction at that point inside the body, ok. So, that is your traction boundary condition and this expression relates the externally applied tractions to the internal stress state, ok.

So, finally, the complete expression for the traction boundary condition can be written as, can be written as; for any point which is on the surface of the body, the traction inside the body at that surface should be equal to the external traction acting at that point on the body, ok.

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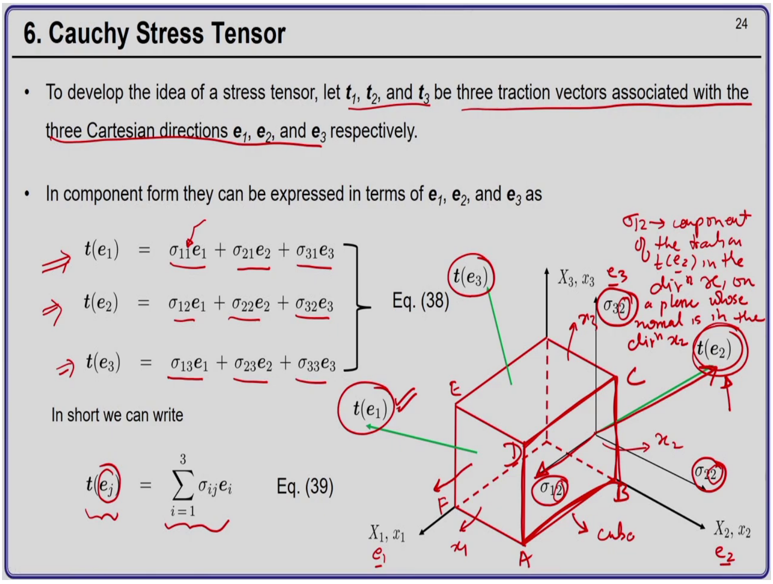

Now, let us come to the most important concept which is the Cauchy stress tensor, ok. Now, to develop the idea of a stress tensor, let say t 1, t 2, t 3 are the three traction vectors which are associated with the three Cartesian directions e 1, e 2, e 3 respectively, ok.

Now, let us consider a cube which is shown here. So, this red shaped object is one cube ok, whose surfaces are along the x 1, x 2 and x 3 directions respectively, ok.

So, we have taken a cube, whose vertical faces are perpendicular to the Cartesian directions e 1, e 2, e 3. So, e 1 is this, e 2 is here and this is your e 3 direction. And let us say we have tractions t e 1, t e 2, t e 3 which act on these planes.

Now, let us say if I take this particular plane. So, this vector, traction vector can be resolved into three directions ok, it can be written in terms of the component. So, this is a vector and this is in a particular direction. So, this vector acts on this vertical surface, ok.

Let us say I will just say A, B, C, D. So, t e 2 acts on A, B, C, D in a particular direction; therefore since this is a vector, I can resolve this vector into one component along the x 1 direction, let us say sigma 1 2, one component along the x 2 direction let us say sigma 2 2 and one component along the x 3 direction which is sigma 3 2, ok.

So, here the second indice 2 is common in all the components, and this signifies the direction of the normal on the plane on which the traction acts. So, the direction of normal here is in the e 2 direction; therefore we have the second subscript as 2. And the first subscript denotes the direction in which the component of the tensor acts, ok.

So, let us say sigma 1 2 would be the direct component of the traction, component of the traction t e 2 in the direction x 1, in the direction x 1 on a plane whose normal is in the direction x 2 that is what sigma 1 2 mean.

So, sigma i j for example, would mean that component of the traction in the direction i on a surface whose normal is in the direction j ok, that is what sigma i j would mean.

Now, coming back to this traction vector, I can write this traction vectors in the component form.

So, the first traction vector can be written as sigma 1 1 e 1 plus sigma 2 1 e 1 plus sigma 3 1 e 1. Notice the second subscript is 1 here, because the normal to this particular surface, let us say E, F. So, the normal to the surface A, D, E, F is in the direction of n 1; therefore the second subscript is 1.

So, the first vector can be resolved into three direction as sigma 1 1 e 1 plus sigma 2 1 e 2 plus sigma 3 1 e 3.

The second vector similarly can be written as sigma 1 2 e 1 plus sigma 2 2 e 2 plus sigma 3 2 e 3.

And the third will be sigma 1 3 e 1 plus sigma 2 3 e 2 plus sigma 3 3 e 3. In short using the concept of indicial notation, I can write the traction on a plane whose normal is in the direction j is summation over i sigma i j e i.

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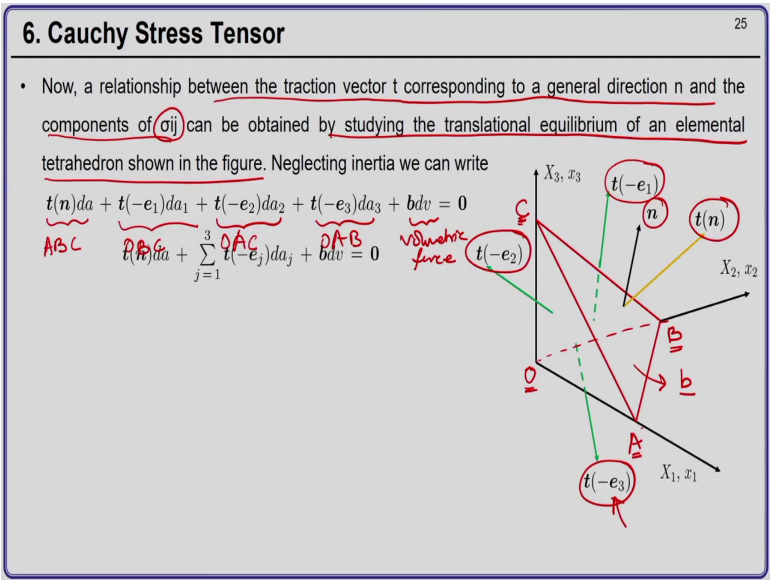

Now, I can now develop a relationship between the traction vector t corresponding to a general direction n and the components of sigma i j by studying the translational equilibrium of an elemental tetrahedron which is shown in the figure, ok.

Say if I have a tetrahedron let us say O, A, B, C, ok. And this tetrahedron may be obtained by cutting the cube that I showed in the previous slide at an certain angle, so that the normal to the slanted phase is n.

And let us say t n is the traction which is acting on this particular plane, ok.

So, the traction vectors on the three verticals plane, ok. So, the traction vector in the direct on the plane O, B, C which is one which is on the back side is t e 1, t minus e 1; because the normal to that particular plane O, B, C is in the negative x 1 direction, therefore t is function of minus e 1.

So, the traction vector on plane O, A, C will be t function of minus e 2; because the normal to plane O, A, C is in the negative x 2 direction. Similarly, the traction acting on O, A, B is t function of minus e 3 and because the normal to the plane O, A, B is in the negative x 3 direction.

Now, what I can do is and let us say the body forces that act ok, let us say the body forces are b.

So, the body forces per unit volume is b. Now I can say from the translational equilibrium, the vector sum of the forces acting on the slanted phase A B C plus the traction or the forces acting on the plane O B C plus forces on the plane O A C plus the forces on the plane O A B plus the volumetric force is equal to 0, ok.

So, here we for the sake of simplicity, we have just neglected the inertia forces. We are not considering inertia; even if it comes, the result will not change that is why for clarity we have just taken inertia to be equal to 0.

So, the vector sum of forces acting on all the four planes should be equal to 0.

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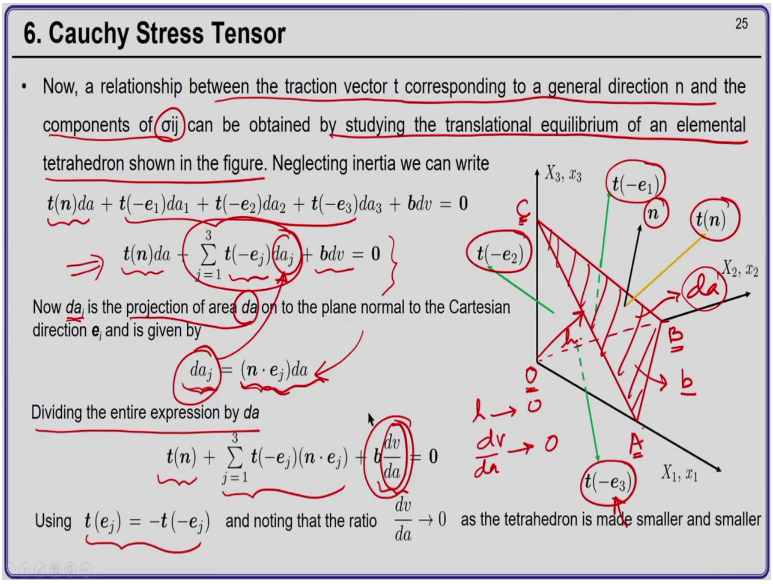

Now I know that, ok. Now, let us say, I will rub this, now I can shorten the above expression; so these middle three I can write as a summation, ok.

So, t n d a plus sigma j equal to 1 to j t of e minus j d a j plus b d v equal to 0, ok.

Now, I know that this area, this ith area is the projection of the area d a, ok. So, area of the slanted phase, let us say is d a; then the projection of this area on the x 1 plane which is O B C will be nothing, but n dot e 1 d a, similarly for the other two direction.

In general I can write d a j is n dot e j d a that is the projection of the area d a, ok. So this is the area d a, which is the slanted phase and if I take the projection of this projected area of phase A B C on the x 1 plane, ok.

If I take the projection on the x 1 plane, I can get the component of d a 1, ok. Similarly I can get for any jth direction d a j is n dot e j d a.

Now, I can substitute this expression over here, this particular expression. So, d a j in this equation can be replace by following expression, ok.

So, and then I can divide the entire expression, this entire expression by d a; because every term will have d a. So, I can take d a out and I will have the traction on the slanted phase is equal to the sum of tractions on the projected planes plus body forces into the ratio of volume of the tetrahedron divided by area of the slanted phase that should be equal to 0. Now if I start making my tetrahedron go to 0, which means this height h, ok. If the height h start going to 0, then what happens; d v by d a will tend to 0.

So, h tends to 0, h is the height of the tetrahedron from the centre to the this phase A B C; therefore and also t e j from the Cauchy's lemma, we have the traction t of minus e j will be equal to minus of t e j, ok. And as the volume goes to 0, the body force term will go to 0.

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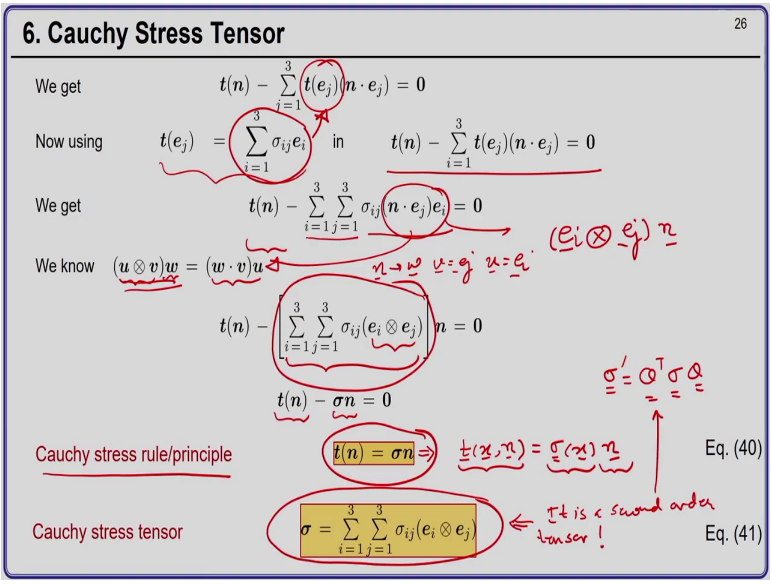

So, what happens? The third term drops away and then I it will have the traction on the slanted phase is equal to the vector sum of the tractions on the projected vertical phases ok, which is given here.

Now, if you remember, we had this particular expression; that traction can be at on any plane e j can be written in terms of the components along the three directions sigma i j e i, that we already had.

I can substitute t e j by this expression over here. If I do this, what I get. So, if I substitute this here, what I get traction on the slanted phase is equal to summation over i and j sigma i j n dot e j e i equal to 0.

Now, recall that a diode of two vectors u tensor product v when operating on a vector w, give you w dot v into u. Now if you see closely here, this is nothing, but like the right hand side of this expression.

So, because n is w and I can see v is e j and my u is e i; therefore this expression can be written as u tensor product v into w, u is i e i, v is e j into w and w is nothing, but n, ok. So, what I get is summation over i and j, sigma i j e i tensor product e j.

And immediately from our definition for a second order tensor, we realize that the expression in the square bracket over here constitutes a second order tensor, ok. And the basis is e i tensor product e j, where i and j go from 1 to 3.

So, I can replace the term in the square bracket with a tensor symbol sigma and I can write the traction on the slanted phase minus sigma times n; n is the normal to the slanted phase is equal to 0.

So, the Cauchy stress principle finally, gives me the traction on a plane passing through position x, having a normal n equal to the stress tensor sigma times the normal at that point. So, more explicitly I can write this as, t x comma n is equal to sigma n n, ok.

So, the traction at position x in the direction n on a plane having direction normal n will be equal to the stress tensor at that point times the normal n, ok. And that is what is called the Cauchy stress principle.

Finally, the Cauchy stress tensor can be identified as sigma equal to summation over i and j sigma i j e i tensor product e j.

So, sigma can be shown that it is a second order tensor, ok. How will you show? In equation 40 you have been given that, t is a vector; which means it is a first order tensor, n is a vector that is it is also a first order tensor.

So, equation 40 you can write in the transform quadrant, which is we can write equation 40 as t dash equal to sigma dash n dash; t dash is Q transpose t, n dash is Q transpose n. Therefore, if you substitute everything and do a little bit of manipulation, you will come to the following relation; that sigma dash will be Q transpose sigma Q, which would mean that with sigma is a second order tensor, ok.

So, this proof the sigma is a second order tensor I leave it to you.