Computational Continuum Mechanics

Dr. Sachin Singh Gautam
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Kinetics - 1
Lecture - 15-17
Cauchy stress tensor, Equilibrium equations, Principle of virtual work

So, today we are going to start our next topic which is; Kinetics. So, in next 6 lectures we will cover the topic of kinetics and in the first 3 lectures that we will have we will be covering the topics of Cauchy stress tensor, Equilibrium equations and Principle of virtual work ok.

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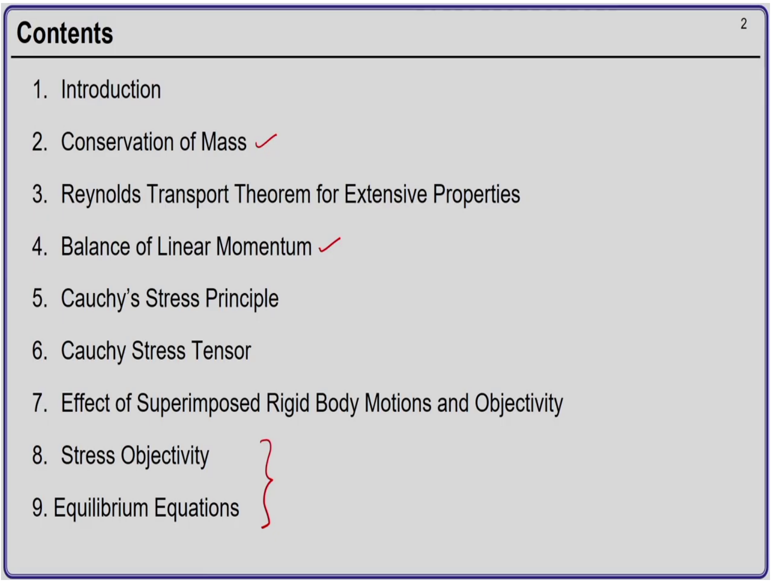

So, following are some of the topics that we are going to cover not necessarily in this lecture, but in the coming lectures. So, we will discuss conservation of mass, then the Reynolds transport theorem that we did towards the end of kinematics we will now be specialized towards extensive quantities ok.

And then, we will discuss balance of linear momentum followed by Cauchy's stress principle and finally, we look into the Cauchy stress tensor ok. We will also look into what is meant by objectivity and what is the effect of rigid body motions and what are it implications on stress objectivity ok. Finally, we will derive equilibrium equation and then we will do principle of virtual work ok.

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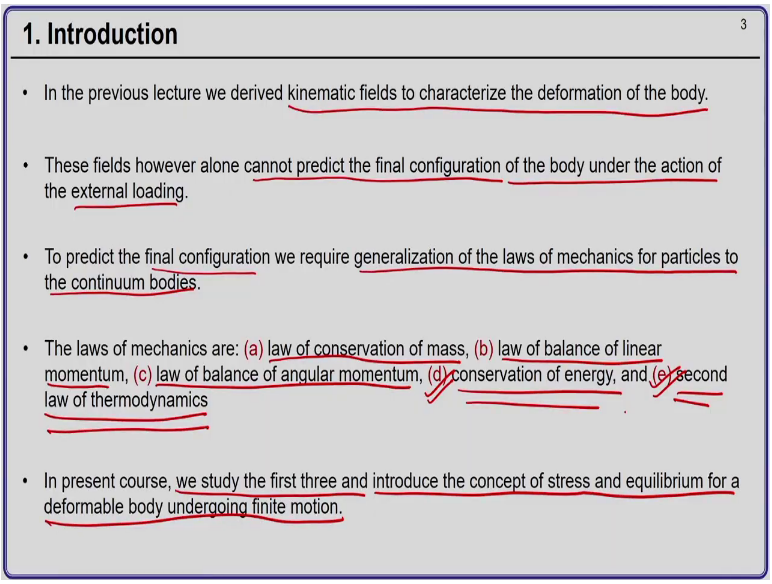

So, in the previous lectures we had derived kinematic fields. So, to characterize the deformation of the body; but these fields alone cannot predict the final configuration or the current configuration of the body under the action of external loading ok. So, just by knowing the kinematic fields it is not possible that you be knowing the final configuration of the body there is something more which is needed ok.

So, to predict the final configuration we require the generalization of the laws of mechanics which are defined for the particles to the continuum systems or the continuum bodies ok. So, we already from our undergraduate and our first year engineering courses we know the laws of mechanics applied two particles or system of particles. Now, we have to generalize these laws of mechanics to continuum bodies body which have infinite number of particles that is continuum particles ok.

So, some of the laws of mechanics which we have to generalize consist of laws of conservation of mass law of balance of linear momentum, law of balance of angular momentum, conservation of energy and finally the second law of thermodynamics ok. So, the last two which is the conservation of energy and second law of thermodynamics these two topics will not be dealt in this course because this is not a course on fully on continuum mechanics. So, we will just look into the first three laws how they can be generalized from system of particles say to continuum bodies ok.

So, we in the present course we just study the first three and introduce the concept of stress and equilibrium for a deformable body undergoing finite motion ok. So, we will not deal with conservation of energy and second law of thermodynamics ok. So, in this present course our material will be hyper elastic material and that would be already satisfying the second law of thermodynamics ok. So, all the constitutive relations that we will derive will always satisfy the second law of thermodynamics ok.

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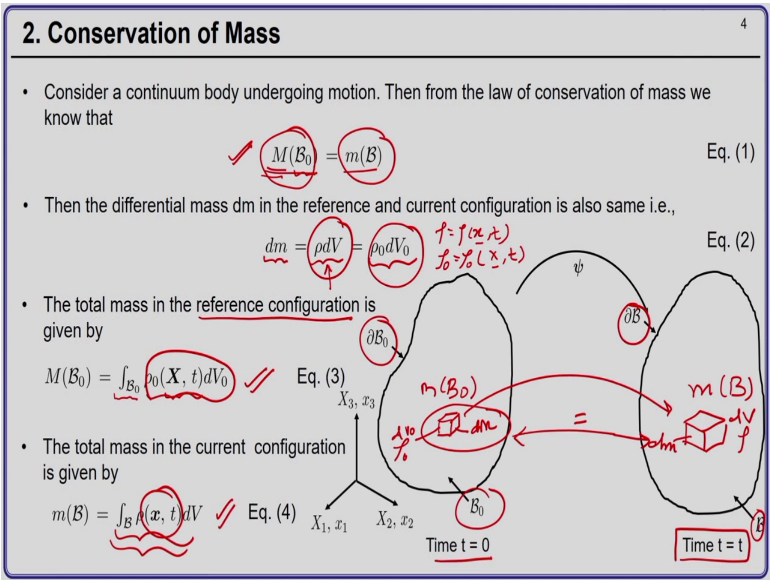

Now, coming to the topic of conservation of mass ok. Now you consider that a body is in the current configuration as usual B 0 is the current volume bounded by surface del B 0 at time 0. And then after deformation at time t it is occupying a configuration B and bounded by surface del B ok.

So, from the law of conservation of mass, we know that the mass of this body before the deformation ok. Let M B the mass and this bracket B 0 means it is a mass of the body in the reference configuration ok. So, the mass of the body in the reference configuration will be same as the mass of the body in the current configuration or the final configuration or in the deformed configuration. So, that is what law of conservation of mass tells us that mass will not be created or destroyed ok.

Now, if that has to be true now let us calculate a mass expression for this mass in the reference configuration as well as in the deformed configuration and let us see what we come up with what kind of equations we are have to deal with ok. So, now if you consider a small say if you consider a small infinitesimal mass d m ok. So, as the body deforms this infinitesimal mass also deforms. Now from law of conservation of mass the mass of this differential element before the deformation should be same as the mass of the differential element in the deformed configuration; which means the mass is nothing but density times the volume ok.

Let d V be the volume in the deformed configuration and let d V 0 be the volume in the reference configuration. So, let density in the reference configuration be rho 0 and the density in the deformed configuration be rho ok. So, the differential mass d m in the deformed configuration will be given by rho times volume d V and in the reference configuration will be rho 0 d V 0 ok. Remember rho is a spatial quantity here; which means it depends on the spatial coordinates and rho 0 is a material quantity it depends on the material coordinates ok.

So, now if I can to get the total mass in the reference configuration what have to do? I have to integrate ok; so I have to integrate this expression over here over the entire volume of the body and this is what I get. So, I have to do this integral over the entire body of the quantity rho 0 d V 0. So, I have explicitly put here in bracket X comma t to show that the density in the reference configuration depends on the reference coordinate ok; rho 0 is in the material description ok.

Similarly, I can get the mass total mass of the body in the deformed configuration ok. And this is obtained by taking integral over this quantity over the current volume integral rho d V ok. So, here you see rho is explicitly a function of spatial coordinates and time t ok. So, now if I use equation 1 equation 3 equation 4 which means; I take equation 3 and 4 and I substitute this in equation 1 what do I get ok?

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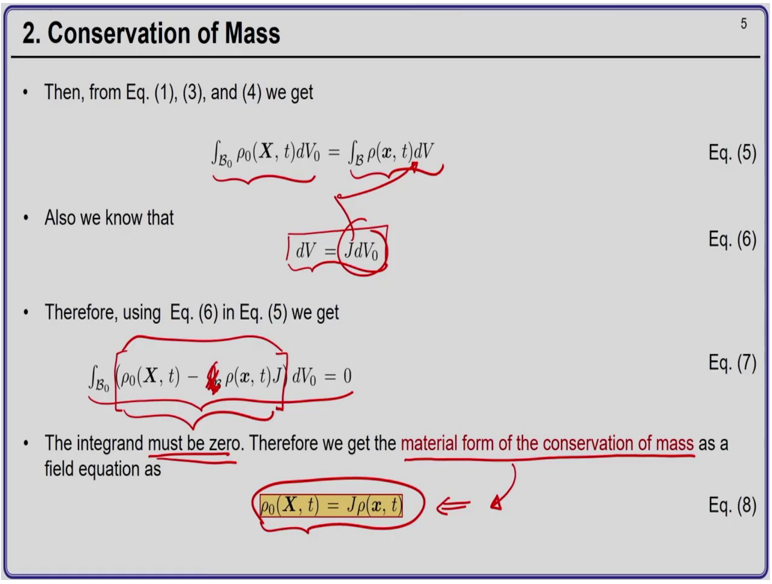

I get the integral ok, the volume integral of rho 0 d V 0 should be equal to the volume integral rho d V ok. Now, I also know from our discussion in the kinematics that the relation between the volume element in the spatial configuration and the volume element in the reference configuration is given by d V equal to J d V 0 ok. So, what I can do now is; I can substitute; I can substitute this over here and then I can write and I can bring the integrals on the left hand side.

So, I can write the integral volume integral or the reference configuration rho 0 minus rho into J d V 0 equal to 0 ok; which means the now because this is applicable for any body; which means this integrant here that you see here this is the integrant this has to be equal to 0. So, what do you get? You get rho 0 equal to J times rho; which means the current density times the Jacobian equal to reference density ok. So, remember this we also derived during kinematics ok.

So, this equation 8 is called the material form of the conservation of mass ok. So, that is a material form which means there has to be I mean there can be something called spatial form ok. So, this is the material form now if I say this is material form definitely there is something called spatial form let us see now we try to get the spatial form of the conservation of mass ok.

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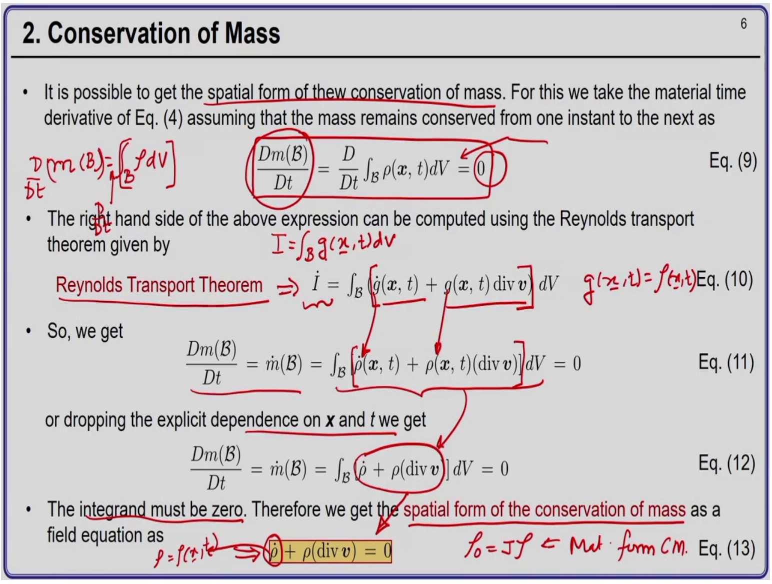

For this what we have to do is we have to take the material time derivative of equation 4. So, equation 4 was m B equal to rho d V ok. Now if I take the material time derivative on both the side D by D t of this ok. So, it will be D by D t of this quantity over here that is what we have written ok.

Now, if you remember the Reynolds transport theorem that we discussed towards the end of kinematics and which is reproduced here. Reynolds transport theorem for an integral quantity of a spatial quantity ok. So, if I had remember if I have to take the material time derivative of an integral quantity I which is integral of a spatial quantity g over the current configuration; then Reynolds transport theorem told me that the material time derivative of the integral will be nothing, but g dot plus g divergence of v ok.

Now, in our case here if we see our g here ok. So, if I compare equation 10 with equation 9 I can see g x comma t is nothing, but rho x comma t ok. So, I using this Reynolds transport theorem I can evaluate the right hand side of equation 9. And because the mass does not change with time the material time derivative of the mass is nothing but is equal to 0 ok.

So, now if I use the Reynolds transport theorem what do I get? D m by D t or I can write m dot is equal to integral over the current configuration rho dot plus rho divergence of v. So, this is rho is your g; so all we have done is we placed g by rho in equation number 10 ok.

Now, what happens? This quantity is equal to 0; which means the integrand must be equal to 0 ok. And the integrand has to be equal to 0 which means this quantity over here ok. So, what I have done between these two is, I have just dropped the explicit dependence on the spatial coordinates and time t ok. So, this quantity in the bracket which is the integrand has to be equal to 0 for equation 12 to hold ok. And this equation number 13 is called the spatial form of the conservation of mass ok.

So, the material form was rho 0 equal to J rho that was the material form of the conservation of mass; material form of conservation of mass ok; and the spatial form of conservation of mass is given by rho dot plus rho divergence of v ok.

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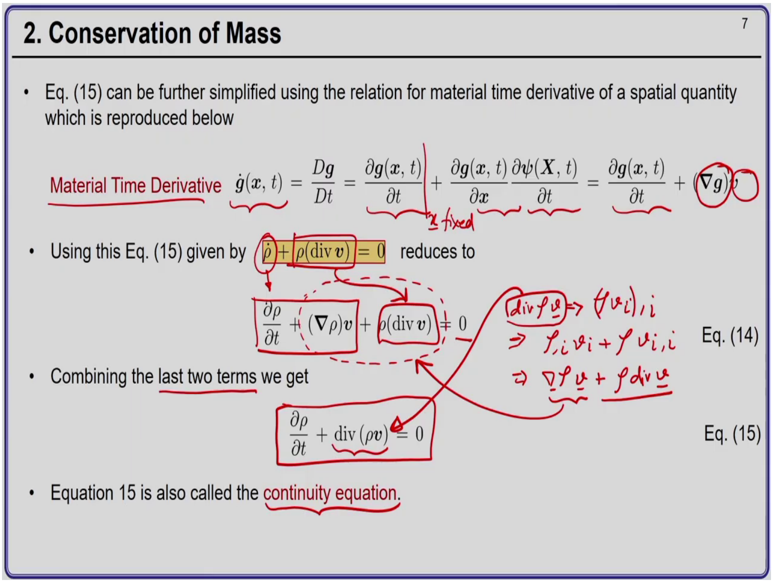

Now, again I can simplify equation 13 further ok. So, to do this what I can do is I recall the concept of material time derivative of a spatial quantity ok. Remember the material time derivative of a spatial quantity g which is written as g dot is nothing, but del g by del t which means I keep x fixed here del g by del t plus the gradient of quantity g times the velocity ok. So, del g by del t plus the gradient of the quantity g plus the velocity ok.

Now, if you see our expression number 13, we have the material time derivative of density sitting over here ok. So, I can express and that density rho ok. So, this density rho here is nothing but a spatial quantity. So, if this is a spatial quantity then using the concept of material time derivative, I can write g dot ok. So, this is the equation. So, I can write our g is rho. So, I can write rho dot as del rho by del t plus gradient of rho times velocity plus rho times divergence of v that is here equal to 0.

Now, I can now concentrate on these two expressions ok. So, this is nothing, but if you do you can show that this is nothing, but divergence of rho v; how? I can show divergence of rho v is in indicial notation I can write rho this is v i bracket comma i and this is nothing but rho comma i v i plus rho v i comma i.

So, rho i is nothing, but gradient of rho times velocity plus rho times divergence of v. Therefore, divergence of rho v is nothing but gradient of rho time velocity plus rho plus rho into divergence of velocity ok. So, that is what we had here therefore, this is what we get ok.

So, combining the last two terms we can get del rho by del t plus divergence of rho v equal to 0 ok. And this equation number 15 is also called the continuity equation its used a lot in fluid mechanics not so much in solid mechanics, but in fluid mechanics ok.

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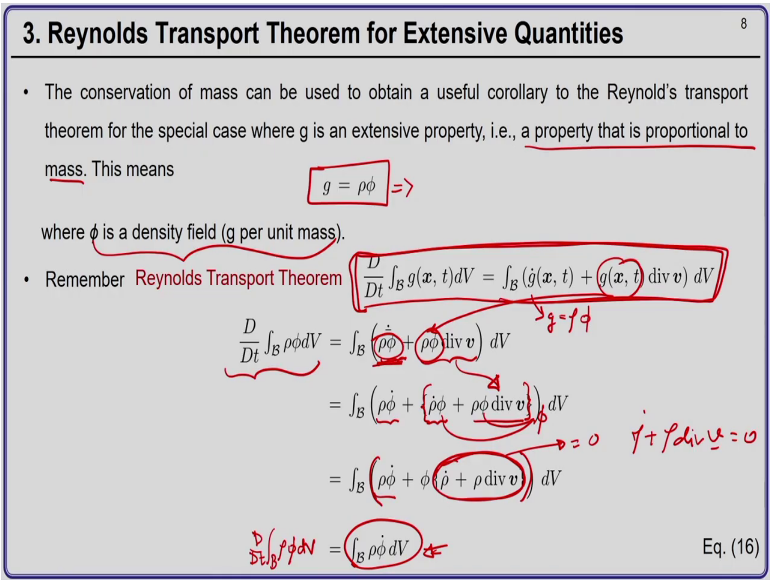

So, now the next topic we have specialization of Reynolds transport theorem for extensive quantities ok. Now what are extensive quantities? Extensive quantity is a quantity which is proportional to mass ok. So, if g is an extensive quantity so it will be proportional to density; it will be equal to density times phi where phi is a density field which means g is per unit mass.

Now, if you have certain quantity like this then your Reynolds transport theorem which is again reproduced here ok. Remember Reynolds transport theorem allowed you to compute the material time derivative of integral of a spatial quantity over the current configuration ok. So, and it was given by following expression.

Now, let us see if we have a special case, where g is equal to rho times phi rho is density and phi is your density field. Then what I can do is, I can substitute g equal to rho phi in this expression ok; if I do this I will get rho phi dot ok. So, this is nothing but the material time derivative of rho phi plus rho phi times divergence of v ok.

So, this is g ok; so which is nothing, but rho phi ok. So, I have just substituted g equal to rho phi. Now I can expand this term ok. So, it is a time derivative of product of two quantities. So, I can expand that and that would become rho phi dot plus rho dot phi plus there will be second term over here which is rho phi plus divergence of v.

So, I can take rho dot phi plus rho phi divergence of v inside the bracket and I can take out. So, because phi is common in both the expression I can take out phi out from the right hand side and then I get rho dot plus rho divergence of v ok. So, I get rho phi dot plus phi times rho dot rho divergence of v integrated over the current configuration ok.

Now, if you remember; the continuity equation or if you basically remembered the spatial form of the conservation of mass. So, this is your same expression the spatial form of the conservation of mass therefore, this is equal to 0 ok. Because we have said rho dot plus rho divergence of v is equal to 0.

If that is equal to 0; we get the material time derivative of an extensive quantity. That is extensive quantity integrated over the current configuration as integration of rho times the material time derivative of that extensive quantity integrated over the spatial configuration ok.

So, D by D t of B rho phi d V is nothing, but integral of B rho phi dot d V ok. So, equation 16 is very helpful and we will see its utility in the coming slides ok. So, this is the specialization of the generalized Reynolds transport theorem for extensive quantity ok.

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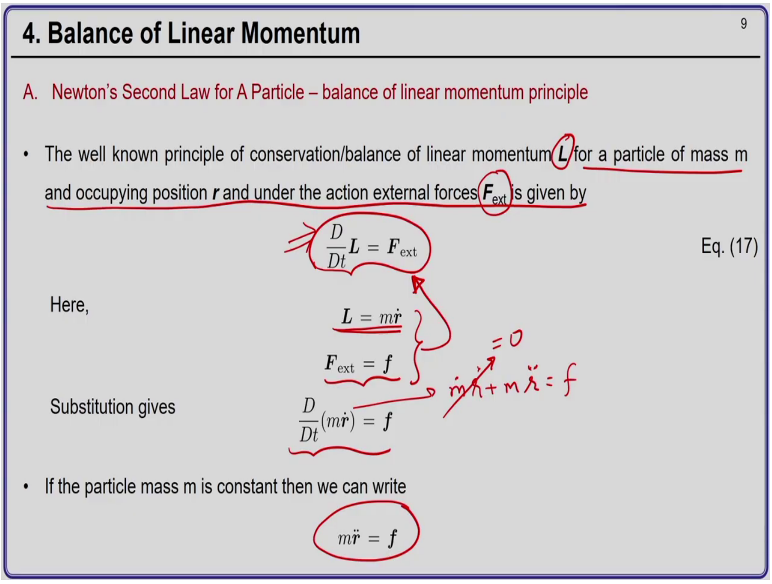

Now, we come to discussion on balance of linear momentum ok. Now before we begin the balance of do balance of linear momentum for continuous body let us start with balance of linear momentum for a single particle ok. So, we know that from the principle of conservation or the balance of linear momentum and the linear momentum is denoted by bold L for a particle of mass m and occupy position r under the action of external forces given by F subscript external.

So, let me write F subscript external is given by the material time derivative of the quantity L is equal to the net external force acting on the body ok. Material because I am concentrating my attention on a particular particle I am not concentrating my attention on a particular position that is why I have to take a material time derivative that is why you have D by D t of L. So, the rate of change of linear momentum is nothing, but the net external force acting on the particle ok.

Now, if the particle mass was m and its position was r then its linear momentum is given by m r dot ok; where r dot denotes the current velocity and F external is equal to small f ok. So, F is the external force acting on the particle therefore, if you substitute these two expression in equation number 17 what we get; D by D t that is the material time derivative of the linear momentum m r dot is equal to external force f ok. And because the mass is constant the mass does not change because this would be equal to m dot r dot plus m r double dot equal to f.

But now, since the mass does not change the first quantity is equal to 0 and then what we get for a constant mass we get mass times acceleration equal to net external force acting on the particle. So, this is nothing, but the Newton's second law for a rigid particle of mass m currently located at position r under the action of forces net external forces f ok.

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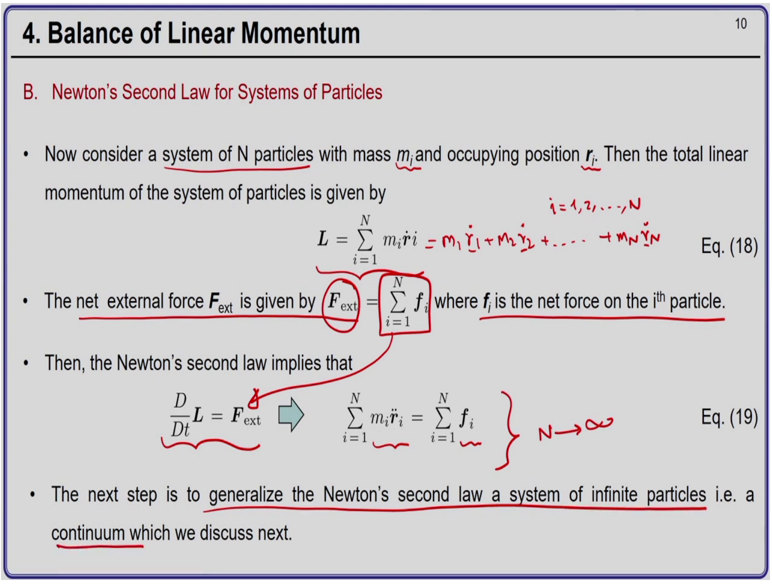

Now, consider a system of N particles and let m i and r i be the mass and position vector of the ith particle and i goes from 1 to N ok. So, then the total linear momentum of the entire system of particles be nothing, but summation over the linear momentum for ith particle ok. So, this is nothing, but m 1 r 1 dot plus m 2 r 2 dot like all the way up to m N r N dot ok. So, this is the total linear momentum for the system of particles ok. Also let the net external force be given by summation over f i; where f i is the net force acting on the ith particle ok.

So, the net force on the ith particle f i if you add for all the particles you get the net external force ok. And now, we know that form Newton's second law that rate of change of linear momentum is nothing, but the net external force acting on the particle. Therefore, when we substitute equation 18 and this equation on the right hand side what we get? We get m i r i double dot equal to f i and summation over i equal to 1 to N. So, this is the equation for Newton's second law for a system of particles ok.

Now what we want to do is let m go to infinity when N goes to infinity we have infinite number of particles and that where we will be able to approximate our continuum ok. So, our continuum is composed of continuum particles infinite number of continuum particles. So, now, our job is to now let N tend to infinity that is where we are going to get our continuum particle. So, you want to generalize the Newton's second law for a system of infinite particle that is a continuum ok.

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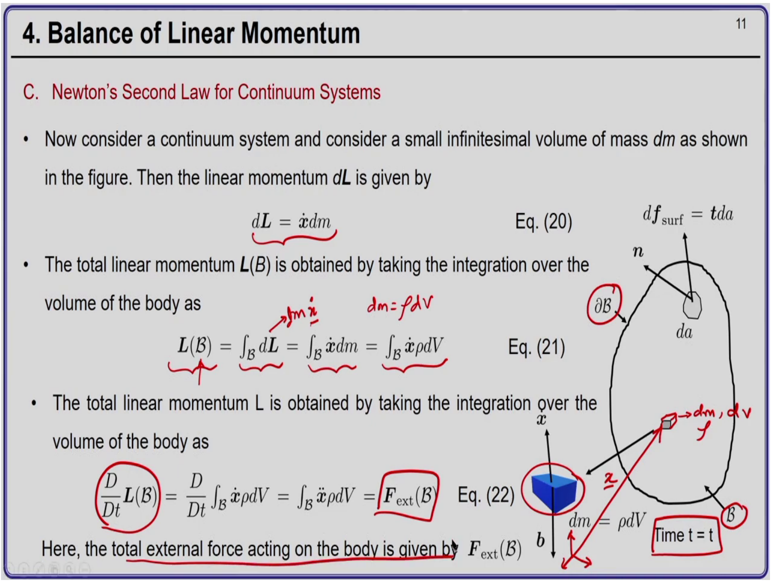

So, in that case now let us consider our body at time t its occupying the configuration B its bounded by del B. Now if you consider a small infinitesimal volume ok. So, the mass of this is d m volume is d V and the density rho ok. Then the linear momentum ok; so this is the zoomed view of that particle.

So, the linear momentum of the particle is x dot into d m ok. So, x dot is the say x dot maybe x is the current position with respect to the coordinate system that is its current position. So, the velocity is x dot and the momentum is mass times velocity mass is d m and velocity is x dot. So, the linear momentum of this infinitesimal mass is in x dot d m ok.

So, integrating over the whole volume we can get the linear momentum of the entire continuum body ok. So, the total linear momentum of the body in the current configuration B is nothing, but integral over the entire volume of d L ok. Now if substitute d L as d m x dot from equation number 20 this is what I get. And then because d m was rho d V I will get equation number 21 for the expression for total linear momentum of the body continuum body which is nothing but integral over the current volume x dot rho d V ok.

Now, the total linear momentum and I can take the material time derivative or the total linear momentum of the body and that should be equal to the external force acting on the body ok.

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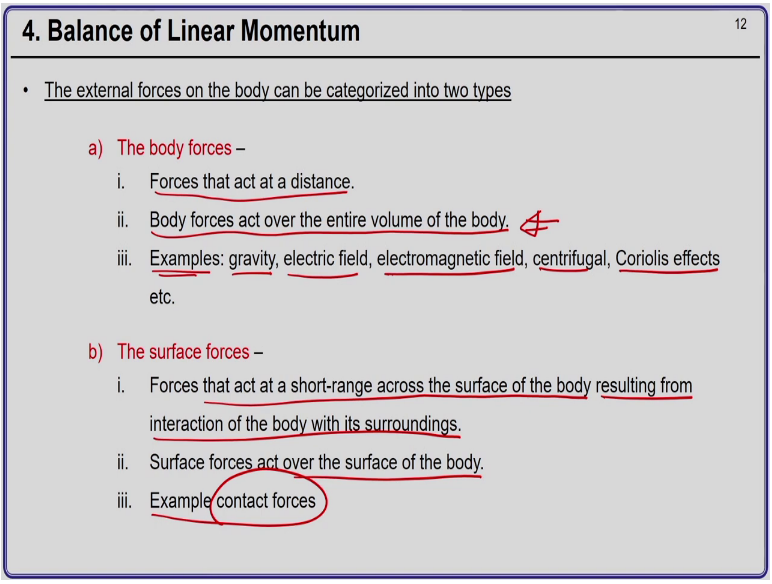

So, here F external is the total external force acting on the body and the total external forces can be categorized into two types ok; one are called the body forces ok. So, the body forces are forces that act at a distance ok. So, the cause of these forces are present at a certain distance from our continuum body they are not nearby the body they are at a certain distance and because of certain effect the body experience certain forces the whole body experience certain forces.

So, what happens? The body forces act over the entire volume of the body. And some of the examples of body forces are; gravity force, electric field, electromagnetic field, centrifugal force, Coriolis effects, etcetera ok. So, these are some of the examples of body forces; remember body forces act over the entire volume of the body ok.

Now, the surface forces ok; the second type of forces are the surface forces. Now surface forces are forces which act in the at a short range across the surface of the body and they result from the interaction of the body with its surrounding ok. So, surface forces act over the surface of the body as the name suggests. And one of the example of the surface forces are the contact forces when you make one body come in contact with the another body then you have surface forces ok.

Now, our next job is to obtain an expression for the net external force of the body when both the body forces and the external surface forces act over the physical surfaces of the body ok; physical surface and volume of the body.

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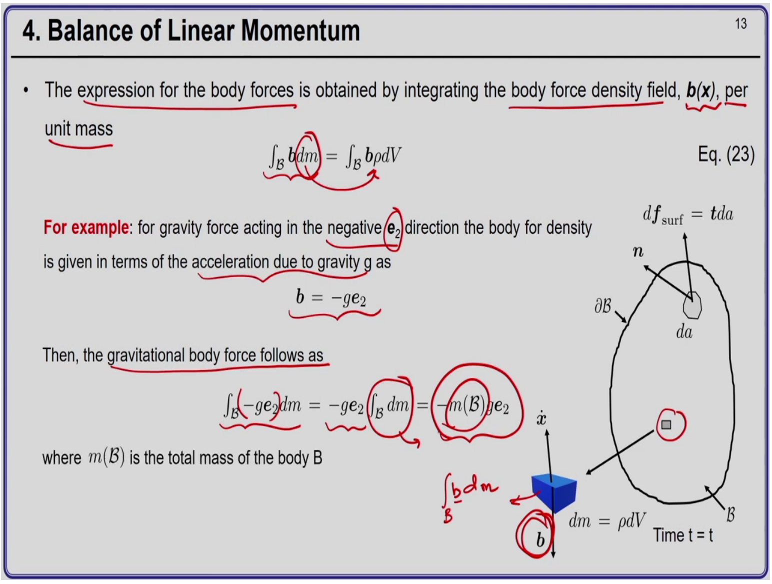

So, now let us see so let us start with the getting the expression for the body forces ok. So, our volume small infinitesimal volume let it be acted over by body force density field b ok;

which maybe depended on the configuration current position of the point and this body force density field is per unit mass ok.

Then the total body force so the total body force will be integral of the body force density field b times the infinitesimal mass ok. So, if b is body force per unit mass then the body force for this small infinitesimal element would be b times d m. So, the total body force will be nothing, but integral over the entire volume sorry integral over the entire volume. And d m is nothing, but rho d V so equation 23 gives you the expression for the body forces ok.

For example; let us take the case of gravity force and let us see if you have a gravity force acting in the negative e 2 direction say; x 1 direction is your horizontal plane and x 2 is a vertical plane so the gravity forces act in the negative x 2 direction or negative e 2 direction ok.

Then the body force are given by minus g times e 2 where g is nothing but the acceleration due to gravity ok. So, the body force field is given by minus g e 2. Therefore, the total gravitational body force in case your body force is composed of gravity is given by integral of minus g e 2 over the entire body ok.

Now, g is a constant 9.81 meter per second square that can be taken outside e 2 is a constant in our case in this course its a constant it can be taken out ok. So, therefore, we have left is integral of mass and this is nothing, but the mass of the body in the current configuration. So, this is minus m times g e 2 is nothing, but your total body forces net body force acting on the body ok.

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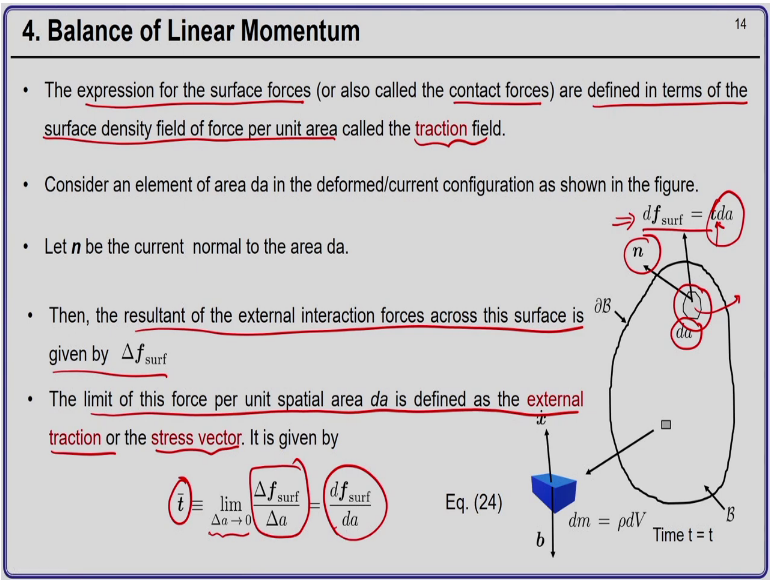

Now, let us see expression for the surface forces they are also called the contact forces and they are defined in terms of the surface density field of force per unit area ok. Remember they are defined in terms of surface density field of force per unit area which is also called as traction field ok. So, the traction is force per unit area ok.

Now, consider you have this small area on the surface of the body the area is d a and the normal to this area is n and the traction is given by t bold t. So, the total because traction is force per unit area that force acting on this infinitesimal area is nothing, but d surf d subscript surf is there to show that this is force on the surface that will be nothing, but force per unit area times the area d a ok.

So, therefore, the resultant external interaction force across this surface is given by delta f surf. So, the limit of this force the limit of this force per unit area d a is defined as the external traction or the stress vector ok. So, it is usually denoted by symbol t bar ok. So, bar over the traction suggests that it is a external specified traction and this is nothing but limit delta a tends to 0 the ratio of delta f divided by delta a and in the limit delta a tends to 0 becomes d f by d a; that is force per unit area ok.

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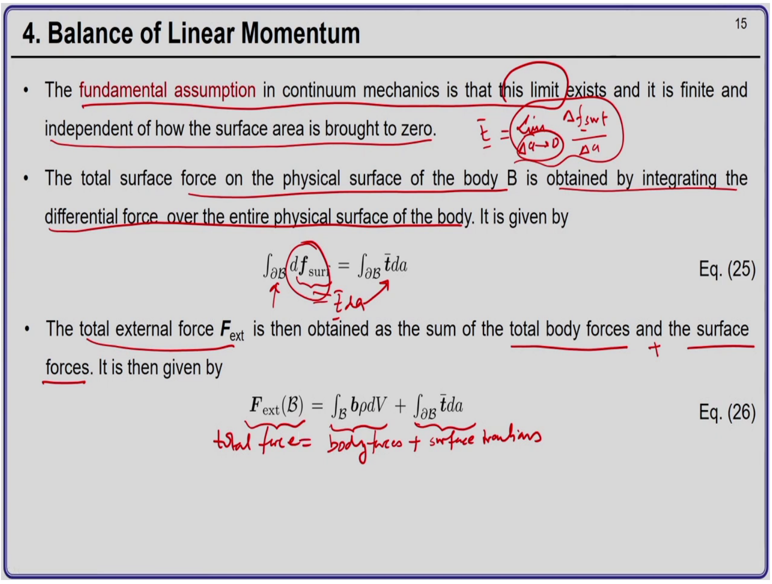

Now, a fundamental assumption in continuum mechanics is that this limit ok; that is in the previous slide we were taking limit delta a tends to 0 delta f surf by delta a; so t bar was defined as this.

So, a fundamental assumption in continuum mechanics is that this limit exists and it is finite and it is independent of how the surface area is brought to zero. So, what your doing is; your bringing this surface area to 0; which means your concentrating more and more at a point your approaching towards a point. So, even if you approach towards a point the limit of this quantity is exists and that is called the externally specified traction ok.

Now, the so total force on the physical surface of the body can then be obtained by integrating the differential force over the entire physical surface of the body ok. So, this is the force on the area infinitesimal area d a ok. So, the total force over the physical surface of the body will be nothing but integration of this quantity over the surface of the body which is; del B ok. And now because d f surf is nothing, but t bar d a I get the total surface force as t bar d a integrated over the current surface of the body ok.

Hence, the total external force will be nothing but the sum of the body forces and the surface forces; so, the sum of total body force plus the sum of total surface forces. So, F external will be sum of the body forces total body forces plus the sum of external traction. So, this is body forces and this is surface tractions; so, total force equal to body force plus surface tractions ok.

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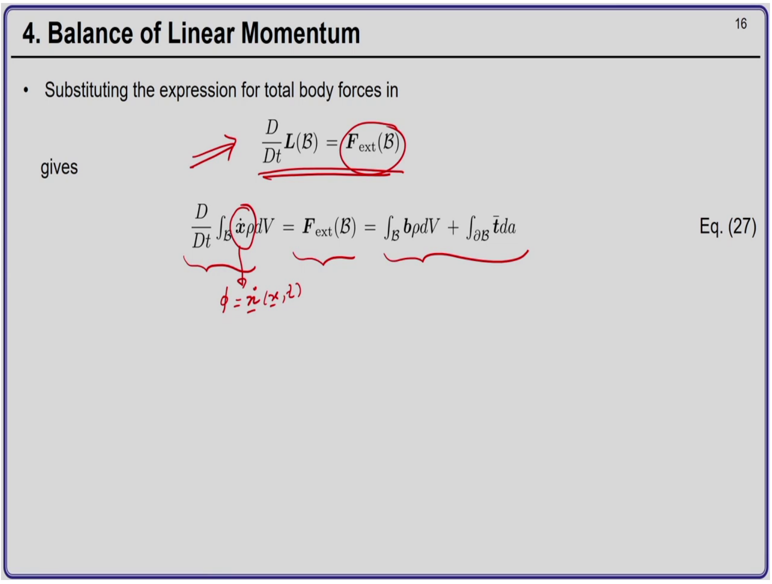

Now, we have obtained the expression for the net external force total net external force acting over the continuum body which is del F external that is F external. So, now, if you substitute the expression for the external forces in the following expression then; we have the material time derivative of the linear momentum total linear momentum is equal to net external force that is which is equal to the sum of the total body forces and the surface forces ok.

Now, comes the application of Reynolds transport theorem for extensive quantity. If you see here our phi here is x dot which is function of x and t x dot is nothing but velocity ok. So, we have a velocity field which is multiplied by density rho. So, from our Reynolds transport theorem for extensive quantity ok; we can just simply take rho times phi dot.

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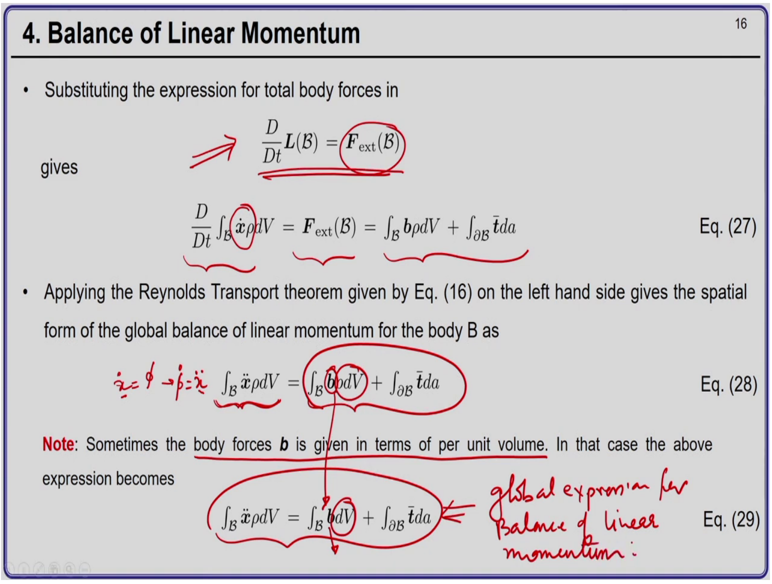

And because our x dot was equal to phi; so, phi dot becomes x double dot. So, we get the integral over the current volume rho x double dot d V is equal to the net external force that is the sum of total body forces and the total surface forces ok.

Now, a special mention I should have is that sometimes the body forces b is given in terms of per unit volume instead of being given in terms of per unit mass. If the body forces are given in terms of per unit body force field is given in terms of per unit volume then equation number 28 boils down to this expression.

So, the only difference is here you do not have rho density is absent because b here is force per unit volume. Therefore, you just have to multiply by volume to get force unit of force here b was per unit mass. So, you have to multiply by mass to get the expression for force.

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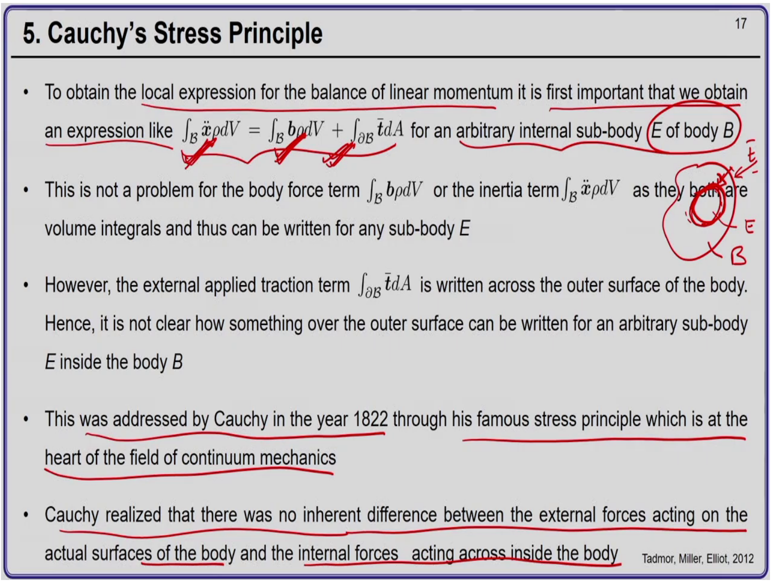

Now, we come to the concept of Cauchy stress principle; now to obtain the local expression for the balance of linear momentum. Now what we have here this is the global expression for balance of linear momentum ok.

Now, we want to get the local expression; why local? Because now we have the case of a deformable body and we wish to determine the local response the response at every location we wish to develop an equation which can give us the deformation at each and every point inside the body. Therefore, we should have some local expression for the balance of linear momentum ok.

For that it is important that we obtain an expression like this here the global expression for the balance of linear momentum for an arbitrary internal sub body E of body B ok. Now if you have a body B ok. So, we already have this expression for this body B. Now if I take a sub body E of this body B ok; then I should be able to have the similar expression for this sub body E ok.

So, now to get this expression for this sub body that is not a problem for these two volume integral terms ok; because you can write these expression because they are volume terms you can just take the volume integral over the volume of the sub body to get the local expression for the left hand side term and the first term on the right hand side. The problem occurs for the second term on the right hand side; because that term is the integration of the applied tractions over the physical external surface of the body. And because this body sub body E is internal; so we do not know the tractions on these body we know the externally applied tractions t bar, but we do not know what is the distribution of traction on the surface of this sub body E ok.

Now, this problem was addressed by Cauchy in the year 1822; where when he gave his famous stress principle which lies at the heart of the field of continuum mechanics ok. What Cauchy gave ok; that is his Cauchy stress principle addressing of this problem where we did not know how the tractions acted on the internal surface of the bodies ok.

So, Cauchy realized that there was no inherent difference between the external forces acting on the actual surfaces of the body and the internal forces acting across inside surfaces of the body ok. So, what he realized that; the way the external tractions act on the external force external surfaces of the body the internal tractions will act in a similar way on the internal surfaces of the body ok.

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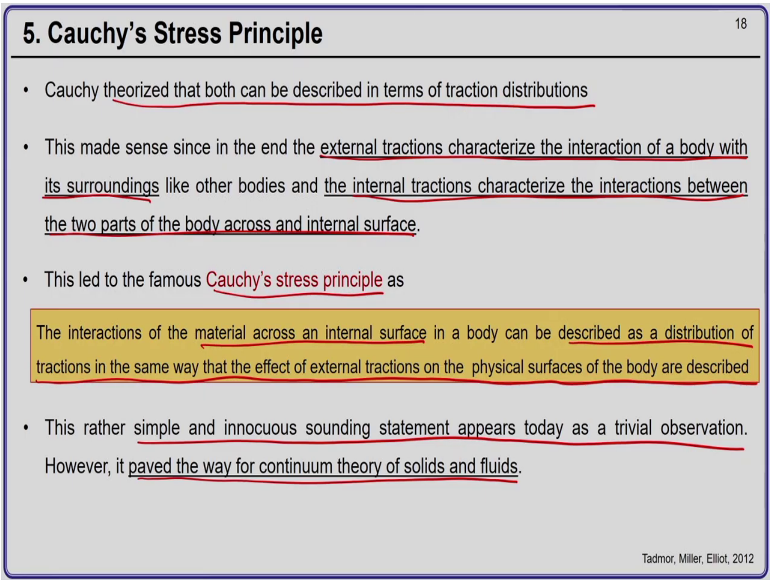

So, once Cauchy realized that he theorized that both of them can be described in terms of tractions distribution ok. And this made sense because in the end external tractions characterize the interaction of a body with its surrounding ok.

So, the external tractions characterize how the body is interacting with the surrounding like the other bodies. And the internal tractions characterize the interaction between the two parts of the same body across an internal interface ok.

So, once Cauchy realized that he gave his famous Cauchy stress principle which is stated here in concise manner. And it says that; 'the interactions of the material across an internal surface in a body can be described as a distribution of tractions in the same way that the effect of external tractions on the physical surfaces of the body are described' ok.

So, this is rather very simple statement and currently it may seem very trivial. But imagine 200 years back this was a very revolutionary idea and this paved the way for the continuum theory of solids and fluids ok.

Now, what will do is; we will come up with the mathematical expression of Cauchy's stress principle. So, what does this mathematically mean?