Computational Continuum Mechanics
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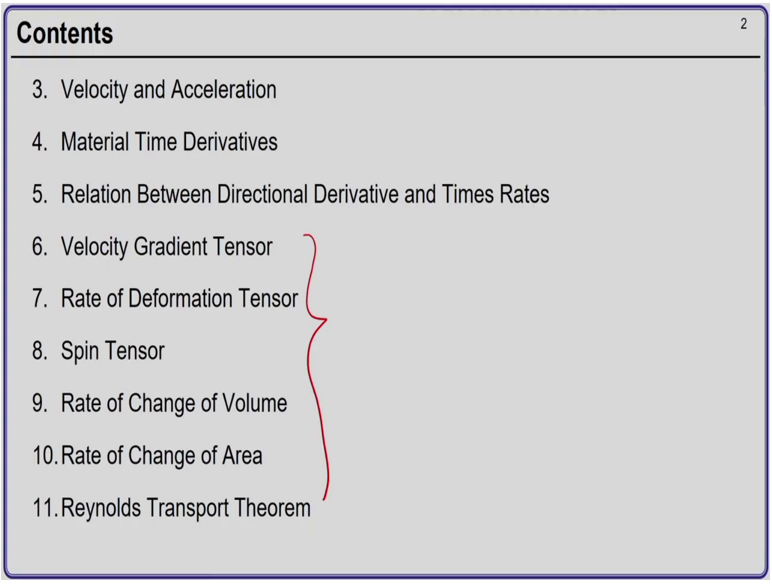
Kinematics – 2
Lecture – 12-14
Linearized kinematics, Material time derivative, rate of deformation and spin tensor



So, today we are going to start the topics of velocity gradient, rate of deformation tensor, spin tensor and then we will calculate the rate of change of volume as well as area. And finally, we will see one of the important theorems called the Reynolds transport theorem ok.

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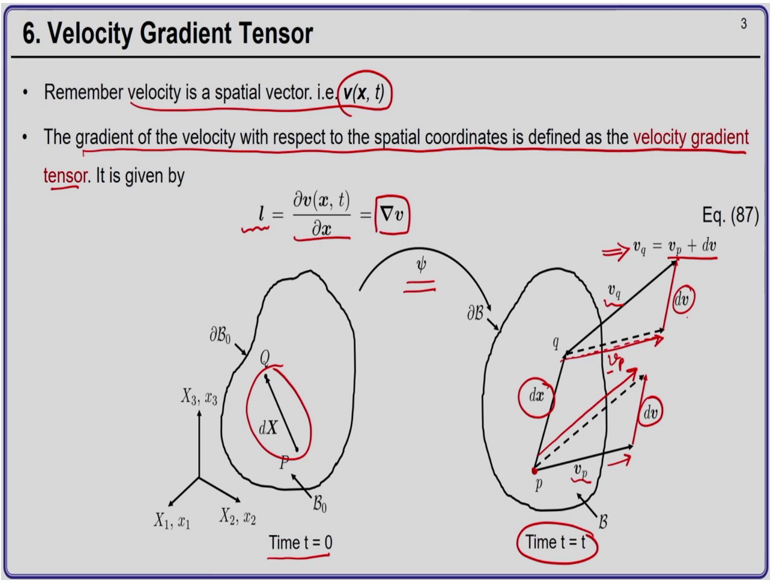

So, remember velocity is a spatial vector we just shown here. So, consider you have a body at time t equal to 0, and then you have a material element d X and as a body deforms and it occupies the current configuration at time t shown here.

So, this material element maps to this spatial element dx. So, now, as the body is moving point p will have a velocity v p and let this arrow show this velocity vector. So, the velocity of point p in the current configuration is v p ok. Similarly, the velocity vector of point q will be v q ok, the direction of velocity at q need not be same as the direction of velocity at p ok.

So, we can draw a vector at q parallel to the velocity vector at p which is v p here shown by dotted line ok. Now, we can other way draw vector parallel to v q at point p which is shown here as dotted line ok. So, the relative velocity vector will be d v ok. So, the relative velocity vector between point p and q will be given by d v ok. So, from the law of vector addition we know that the velocity v p plus d v will give you v q ok. So, the velocity of point q will be velocity of point p plus derivative velocity vector d v between point q and p ok.

So, the gradient of velocity with respect to the spatial coordinates is now defined as the velocity gradient tensor and it is denoted by lower case l and it is given by del v by del x or in short you can write this as gradient of v gradient of velocity ok.

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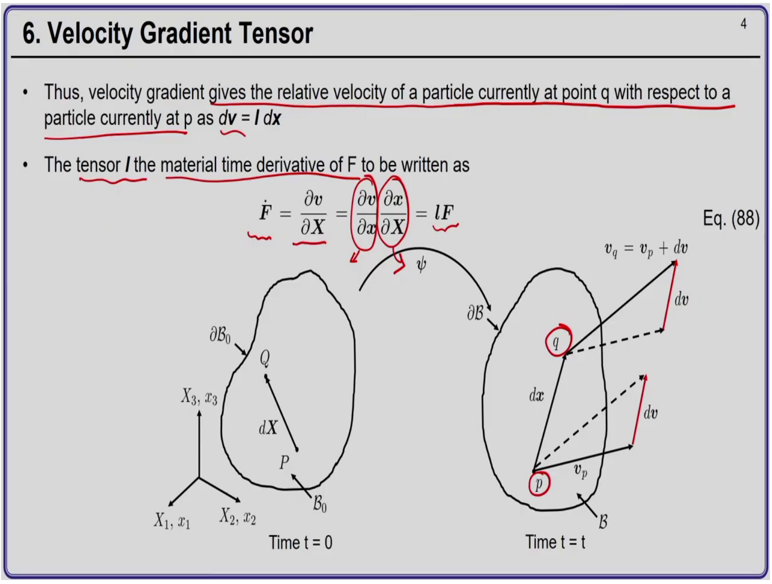

So, what does this velocity gradient tensor actually gives? So, it gives you the relative velocity of a particle; which is currently at point q with respect to a particle currently at point p that is dv equal to l dx ok. So, the velocity gradient tensor gives you the relative velocity of particle at point q relative to point p in the current configuration ok.

Now, the tensor l and can be use to express the material time derivative of F as follows ok. So, F dot is del v by del X, this we already know the material time derivative of the deformation gradient tensor is nothing, but del v by del X ok. So, I can write del v by del X as del v by del x into del x by del capital X.

Now, from equation number 87 on the previous slide we can recognize that del v by del x is nothing, but the velocity gradient tensor l and del x by del X is nothing, but the deformation gradient tensor F. So, I can write F dot as the velocity gradient tensor l times the deformation gradient tensor F ok.

So, using this relation I can relate the material time derivative of the deformation gradient tensor with the velocity gradient tensor and the deformation gradient tensor ok. So, this relation we have to remember because in later slides when we are doing some derivations this will come very often ok.

Now, coming to the next topic of rate of deformation tensor.

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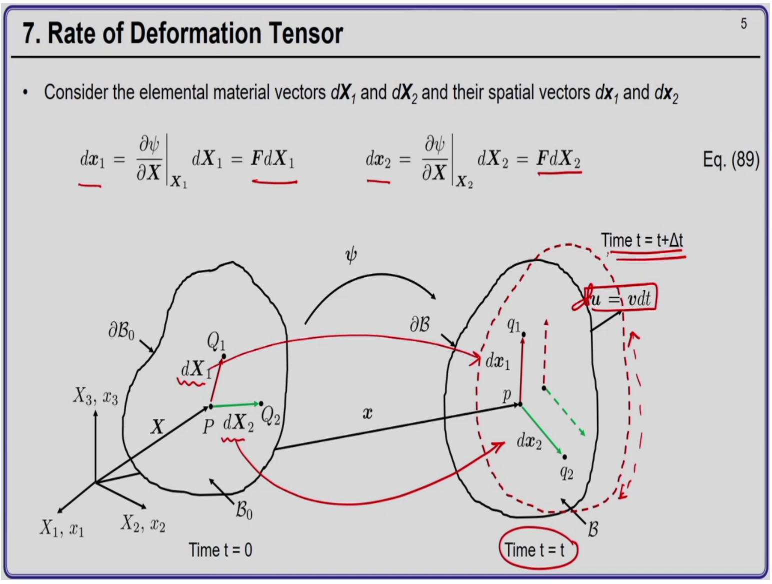

So, you remember we have taken two material vectors d X 1 and d X 2 at point P and we had observed how these material vectors deform to spatial vectors dx 1 and dx 2. So, we know that dx 1 is F d X 1 and dx 2 is F d X 2 that we know ok.

Now, let at time t the body undergo a small displacement u which can be written as v into dt ok. So, sorry this du a small displacement du which is vdt and let at time t plus delta t the red dotted line which is here shows the configuration of the body at time t plus delta t.

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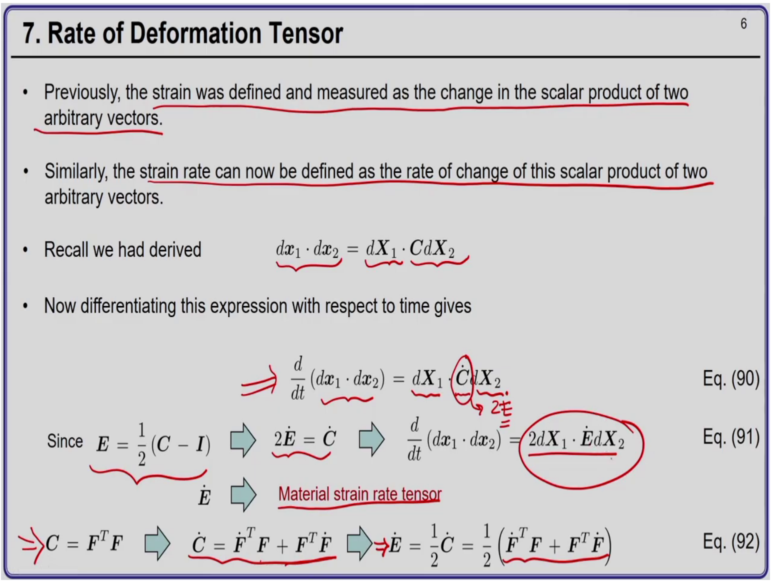

Now, so, we had defined strain earlier and measured it as the change in the scalar product of two arbitrary vectors. So, we took dx 1 and dx 2 in the material configuration and we defined strain by how these material vectors deform how the scalar product of the these two material vectors change and that is how we came up with the concept of strain ok.

So, for strain rate now we have to define the rate of change of the scalar product of the material vectors. So, recall that we had derived as the scalar product of this spatial vectors is nothing, but d X 1 that is the material vector d X 1 dotted with C times d X 2 where C is the right Cauchy – Green tensor and C is F transpose F.

Now, taking the material time derivative on both the sides we take the time derivative on both the sides d by dt of dx 1 dot dx 2 and on the right hand side since d X 1 and d X 2 are the material vectors they are fixed, they are known. Therefore, we have d X 1 dot C dot so, the time derivative of the right Cauchy – Green tensor. Now, we know that the Green-Lagrange strain tensor E is 1 by 2 C minus I ok.

So, which means that if I take the material time derivative of the Green – Lagrange strain tensor so, I can write 2E dot is equal to C dot which means I can replace C dot here with twice of E dot ok. I can replace the time derivative of the right Cauchy – Green tensor with the time derivative of the Green – Lagrange stain tensor and that is what I am going to get ok.

So, where E dot is called the material strain rate tensor because C is F transpose F C dot will be because C is F transpose F C dot will be F dot transpose F plus F transpose F dot which means the rate of Green-Lagrange strain tensor E dot is nothing, but 1 by 2 F dot transpose F plus F transpose F dot ok.

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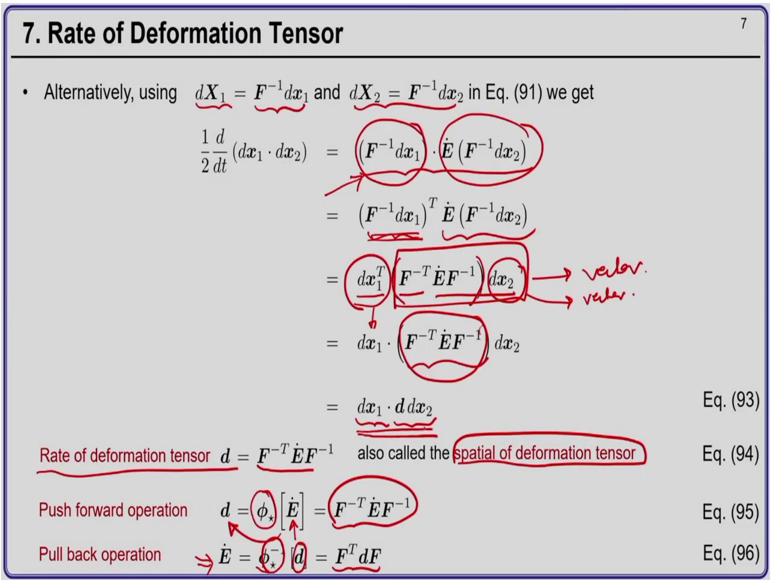

Now, reversing I mean alternatively I can express my material vectors in terms of the inverse of the deformation gradient tensor times the spatial vector d X 1 and d X 2 will be inverse of the deformation gradient tensor times d X 2 spatial vector d X 2. Therefore, if I use this in equation 91, if I use this here dx 1 is F inverse dx 1 and dx 2 is F inverse dx 2.

What do I get? I get F inverse dx 1 dot E dot F inverse dx 2 ok. Now, the first term in this bracket is a vector and this term over here is also a vector and A dot B is A transpose B, where A is this quantity. So, I can write F inverse dx 1 transpose E dot F inverse dx 2 and I can open up the bracket, this bracket over here and I can write dx 1 transpose F inverse transpose E dot F inverse dx 2 ok. So, I can take dx 1 out from the left hand side and dx 2 out from the right hand side.

And, now, this whole expression over here is also a vector quantity because the term in the bracket here ok. So, the term in the bracket here is a second order tensor. So, second order tensor operating on a vector gives you vector therefore, this quantity in the square bracket is also a vector ok. So, a vector transpose vector is nothing, but vector dot vector.

So, I can write this as dx 1 dot F inverse transpose E dot F inverse dx 2. And, now I can denote this quantity in the bracket by symbol small d. So, so, I can write dx 1 dot d dx 2 ok, d is a tensor. I can show that this is the tensor you know how to show something in the second order tensor we have to use the transformation law.

Now, this d is called the rate of deformation tensor d and it is given by F inverse transpose E dot F inverse and this is also called the spatial deformation tensor. Now, d can be treated as the push forward of the rate of Green-Lagrange strain tensor ok. This is the symbol for the push forward operation. So, the push forward of the Green-Lagrange strain tensor that is E dot is nothing, but the rate of deformation tensor and how this push forward is carried out? It is given by this particular expression F inverse transpose E dot F inverse ok.

So, the pull backed ok. So, the pull back of the rate of deformation tensor d is nothing, but the rate of Green-Lagrange strain tensor. So, E dot is pull back. So, this is the pull back sign of rate of deformation tensor and how this pull back is carried out? F transpose d F that is how you carry out the pull back of the rate of deformation tensor.

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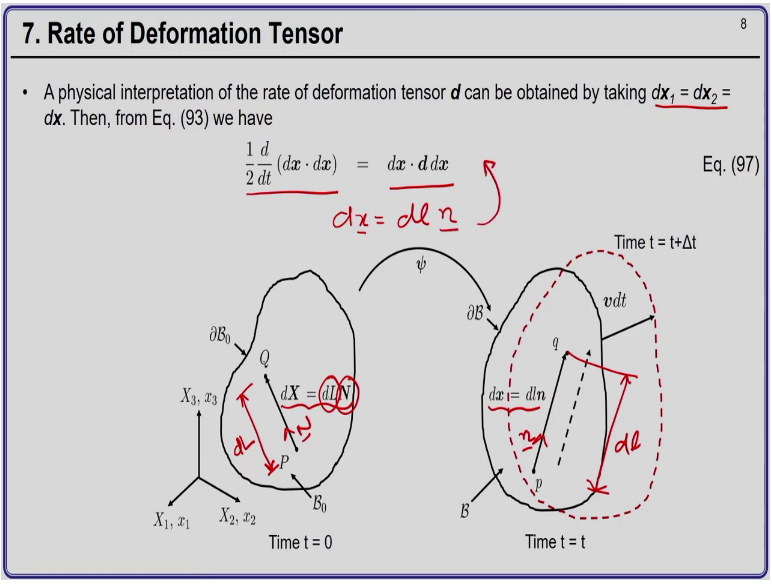

Now, we can have a physical meaning or the interpretation of the rate of deformation tensor if we just consider dx 1 equal to dx 2 equal to dx. So, instead of taking two material vectors we take one material vector and observe what happens to its length ok.

So, let we consider a material vector d X given by d L into N where N is the unit normal along the vector direction unit vector along the vector direction and d L is the length of the vector. So, this is what is your vector length dL and now, this vector gets mapped to vector spatial vector dx given by dl into n, where n is the unit vector along pq and dl is the length ok. So, d small l is the length of the vector pq ok. So, dx is dl into n.

So, if we substitute this in our previous expression over here equation number 93 if we do that what I will get 1 by 2 d by dt of dx dot dx equal to dx dot d dx. Now, I know that dx is nothing, but dl into n. So, this I can substitute in equation number 97.

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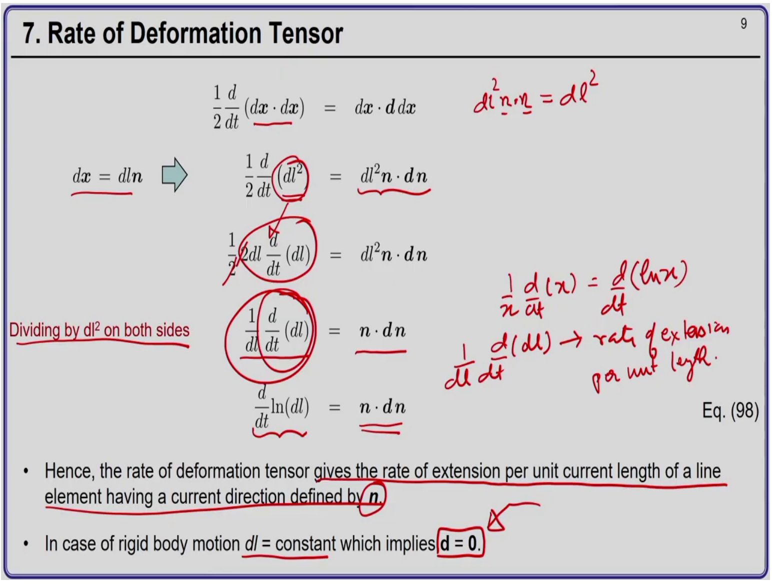

If I do this so, if I do this what do I get dx dot dx is nothing, but dl square into n dot n an n is a unit vector, therefore, n dot n is 1. So, we will get dl square. So, that is what you have it here. Similarly, on the right hand side you will get dl square n dot dn ok.

Now, you can take the derivative of time derivative of dl square it will be 2 dl into d by dt of dl ok. So, we take the time derivative and we get this. So, this 2 cancels out and if we divide by dl square on both the sides if we divide by dl square on both the sides then what we get we get 1 by dl d by dt of dl equal to n dot dn ok. Now, you can recognize that 1 by x d by dt of x is nothing, but d by dt of lnx ok, this is the property ok. So, the derivative of logarithmic is 1 by x d by dt of x. So, this can be written as d by dt of natural log dl equal to n dot dn ok.

So, what does rate of deformation tensor is giving you? It gives you the rate of extension per unit current length of a line element having a current direction defined by n ok. So, this is your. So, d by dt dl this is your rate of extension and 1 by dl will give you rate of extension per unit length ok and this is for a line element whose current direction is along vector n ok. So, along vector n that is what it is giving you. So, sort of it is giving you the rate of stretch of your current line element dx.

Now, what happens in case of rigid body motion? So, you know that in rigid body motion the body does not deform ok. So, the material vector dx will only rotate to dx spatial vector dx, but the length of the vector will not change and from if current position t to position configuration at t plus delta t if there is a rigid body motion then the length of the vector spatial vector dx will not change.

So, dl will not change which means the material time derivative or the time derivative d by dt of dl this quantity over here will be equal to 0 this equal to 0 which means dl is constant. This would imply that the magnitude or the rate of deformation tensor is equal to 0 tensor which means there is no stretching involved. So, this has a physical implication which means that under rigid body motions the body will not generate any strain. If there are no strains generated there will be no stresses generated.

So, this is physically intuitive because if you just rotate a body there will not be any stresses generated inside the body. So, if you have a strain measure for example, strain rate measure for example, d which gives you no change in the strain if there is a rigid body motion then you will get physically no stress. If you derive your stress from d you will get no stress increment in the stress which is physically possible.

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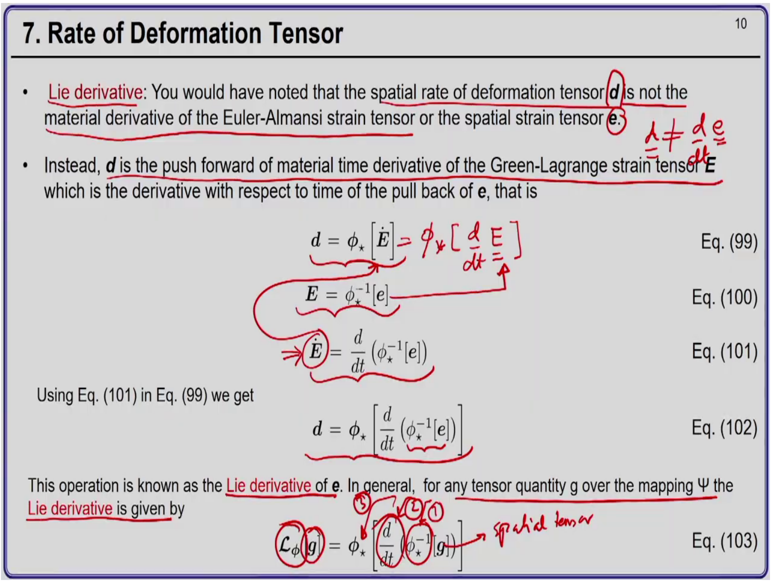

Now, we can also briefly discuss a important concept of the Lie derivative this called the Lie derivative. So, you would have noted that the spatial rate of deformation tensor d is not the material time derivative of the Euler-Almansi stain tensor e ok. So, you would have noted that d is not equal to d by dt of e. See, the Euler-Almansi stain tensor is a spatial strain tensor and d is a spatial rate of deformation tensor, but the rate of e will not be equal to d ok. So, this important concept we have to understand.

So, what is d? d is actually the push forward of the rate of Green-Lagrange strain tensor E and E is nothing, but the pull back of Euler-Almansi strain tensor ok. So, d is nothing, but the push forward of rate of Green-Lagrange strain tensor ok, that we saw in the previous slides. Also the Green-Lagrange strain tensor is nothing, but the pull back of the Euler-Almansi strain tensor E ok. This we saw when we are discussing Green-Lagrange and Euler-Almansi strain tensor.

Now, if I can write this as phi star d by dt of E and what is E? E is this quantity over here. So, I can write E dot as d by dt of pull back of Euler-Almansi strain tensor. I can substitute it here. I can say that d is nothing but the push forward of the material or the time derivative of the pull back of Euler-Almansi strain tensor. So, this operation is called the Lie derivative of e ok. So, in general if you have any tensor quantity g that goes over the mapping psi then the Lie derivative of g is defined as ok.

So, this is the symbol for Lie derivative of the tensor g is nothing, but phi star d by dt of phi star inverse g. So, push forward ok, then material time derivative and the pull back. So, if you want to compute and g is the spatial quantity this is the spatial tensor. So, if you have to take the material time derivative of a spatial tensor and what we have to do? We have to pull it back to the material configuration, take the material time derivative and pull it push it forward to the spatial configuration ok.

So, first is pull back, then the material time derivative and third push forward ok. So, Lie derivative constitutes of three steps – pull back, time derivative, push forward.

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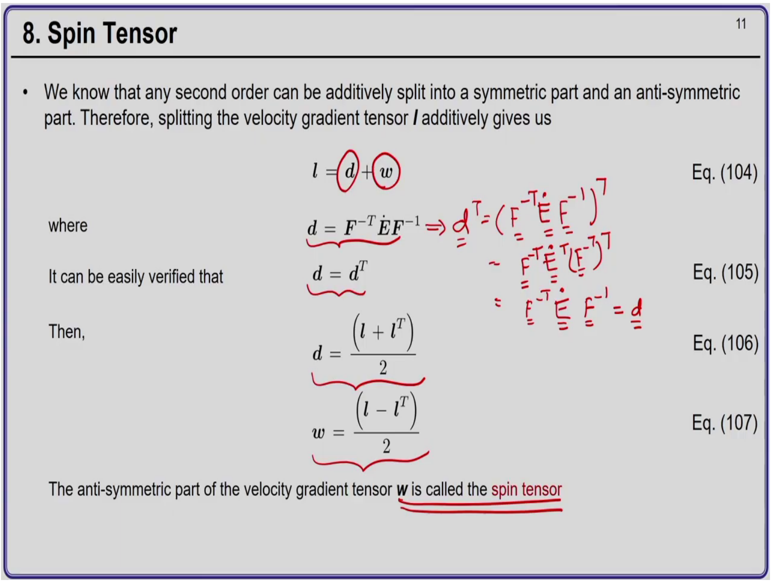

So, next we come to the concept of spin tensor. So, we know that any second order tensor can be split additively into a symmetric part and a anti-symmetric part. So, the velocity gradient tensor l can be split into a symmetric part which is rate of deformation tensor and a anti-symmetric part which is nothing but the spin tensor ok.

Now, d is nothing, but F inverse transpose E dot F inverse. This we have seen in our previous discussions on previous slides. And I can actually show that d is a symmetric tensor which means d transpose is same as d ok. I can just take a d transpose here d transpose will be F inverse transpose E dot F inverse transpose. So, this will be nothing, but F inverse transpose E dot transpose F inverse transpose transpose ok.

So, this is nothing, but F inverse transpose and now, Green-Lagrange strain tensor is a symmetric tensor. Therefore, its time derivative also will be a symmetric tensor which means E dot transpose is same as E dot F inverse transpose transpose is nothing, but F inverse and this is nothing, but the expression for d. Therefore, the rate of deformation tensor d is a symmetric tensor.

So, once we have shown that d is a symmetric tensor therefore, d is nothing, but l plus l transpose by 2 ok. If d is this then w will be l minus l transpose by 2 ok. So, w is called the spin tensor ok. So, this w is called nothing, but the spin tensor. We will come to it why it is called a spin tensor, we will see why this name.

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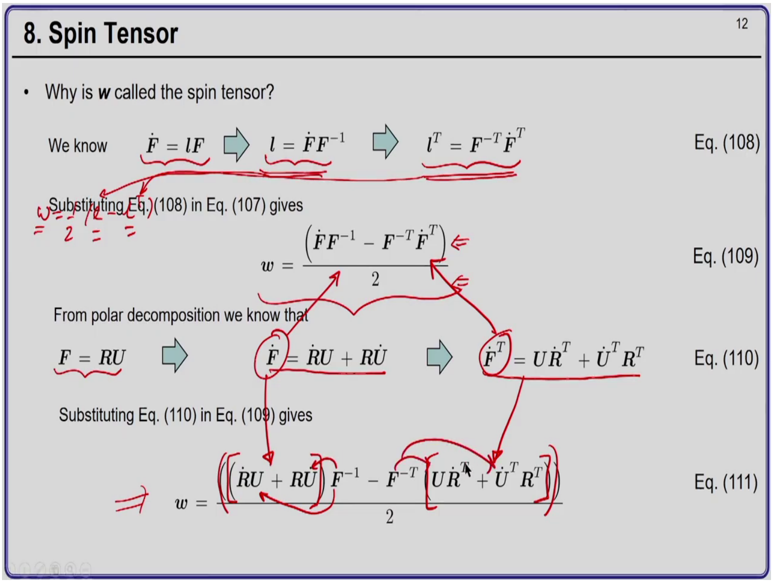

So, why is w called the spin tensor? So, to show that we first notice that the material time derivative of the deformation gradient tensor is the velocity gradient times the deformation gradient tensor ok. So, if F dot is l F; which means the velocity gradient tensor is F dot F inverse which means l transpose is F inverse transpose F dot transpose.

Why we are doing this? I mean we know that w is l minus l transpose by 2. So, this is our l here and this is our l transpose, and because w is 1 by 2 l minus l transpose and I can substitute these two quantities here ok, then what I get w is F dot F inverse minus F inverse transpose F dot transpose by 2 ok.

Now, again what I can do? I can use the polar decomposition theorem I know that for right polar decomposition F equal to R into U, where R is a orthogonal tensor U is a symmetric tensor which is also called the right stretch tensor ok. So, if F equal to RU, then F dot will be R dot U plus RU dot.

So, this I can substitute. I can compute F dot transpose here ok. And, I can substitute both F dot and F dot transpose in terms of R and U in equation 109 I can substitute this here and then I will get following expression ok. I am getting this following expression there is this whole bracket and this is this bracket over here ok. So, this is your F dot transpose and this is F dot. Now, I again I can take F inverse inside, I can open up the bracket I can take F inverse transpose here inside and I can write following expression.

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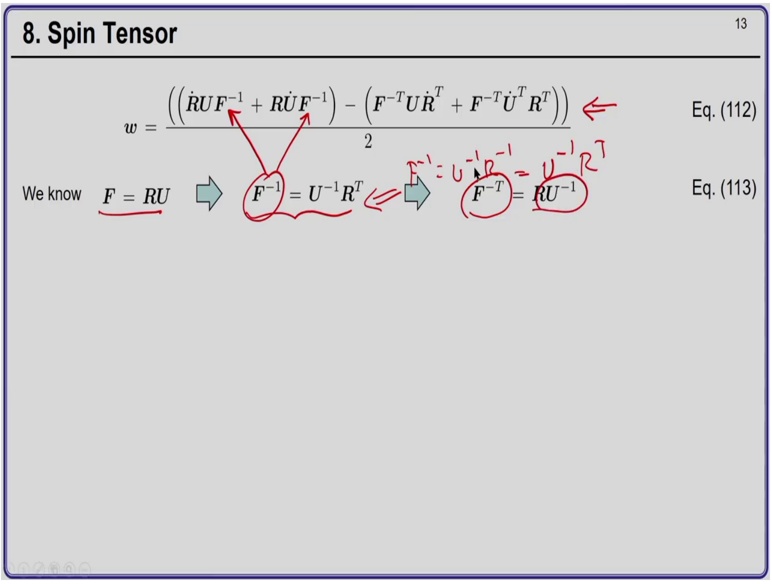

I can write this expression, seems like a very long expression, but we will quickly see how it comes to a very nice small expression. We so, we know that F is RU which means F inverse is U inverse R transpose because R is a orthogonal tensor. So, F inverse is U inverse R inverse and since R is a orthogonal tensor R inverse will be equal to R transpose that is what we have written over here ok. Now, F inverse is U inverse R transpose. So, I can substitute F inverse here. I can substitute F inverse here ok.

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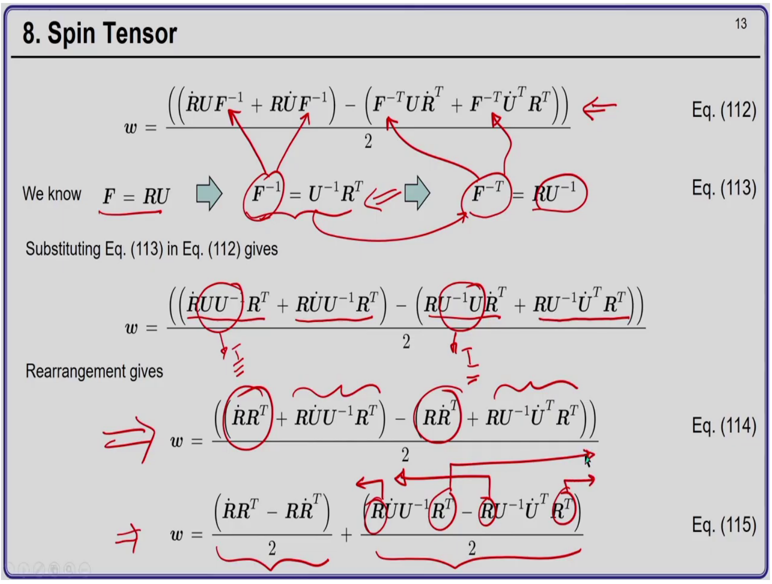

So, F inverse transpose is RU inverse let me just erase this part. So, from here I can compute F inverse transpose and I can substitute F inverse transpose here and here in terms of R and U. If I do this, I will get R dot UU inverse R transpose plus RU dot U inverse R transpose on the as the first term. And RU inverse UR dot transpose plus RU inverse U dot transpose R transpose as the second term. Looks like a very big expression right now, but I can rearrange.

So, what I can do? I can see the other thing you have to notice this is UU inverse is nothing, but identity ok. So, UU inverse here is nothing, but identity and then if I put identity I will get equation 114 ok. Now, what I can do is little rearrangement I can bring this term over here the first and the third term inside one bracket and I can bring the second term and the fourth term inside second bracket ok. If I do this I can write the spin tensor as this first term which does not have any U and the second terms which is contains all the U's or the right stress tensors ok. Up till now we are fine, now what we do?

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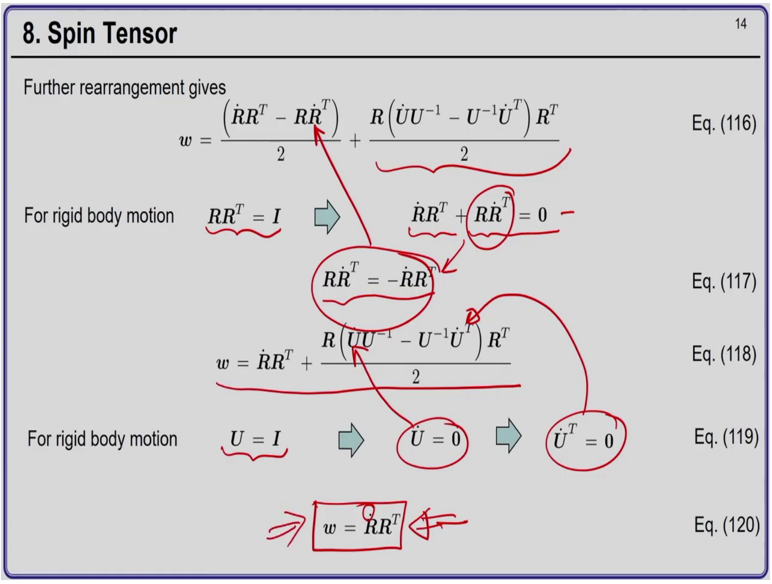

Ok, I can take R outside the bracket from the left hand side and R transpose ok. So, this I can take out from the left hand side and R transpose I can take outside from the left hand side. So, what I am left with is in the second term this quantity R into U dot U inverse minus U inverse U transpose U dot transpose R transpose.

Now, I know that R R transpose is identity because R is a orthogonal tensor. So, RR inverse is a identity R inverse is R transpose therefore, RR transpose is identity. If I take the time derivative I get R dot R transpose plus RR dot transpose equal to 0 tensor ok. So, I can write RR dot transpose as minus of R dot R transpose and this I can use here if I use that here I will get w is R dot R transpose plus the second term.

Now, as a special case if you have rigid body motions in case you have rigid body motions what it means is that the stretch tensor will be equal to the identity tensor ok. So, there will be no stretch which means U equal to I because F is I and then U will become I which means U dot will be equal to 0 ok. So, if U dot is equal to 0, it implies U dot transpose is also equal to 0 tensor. So, this I can use here this I can use here.

So, for rigid body motion I can show that my spin tensor is nothing, but the material time derivative of the rotation tensor times the transpose of the rotation tensor R dot R transpose, there is no stretch here. So, only rotation is involved. So, that is why omega is also called the spin tensor because as a special case I can show it is composed only of spin or the rotation ok. There is a time derivative therefore, the rate of rotation is a spin. So, it is also called the spin tensor ok.

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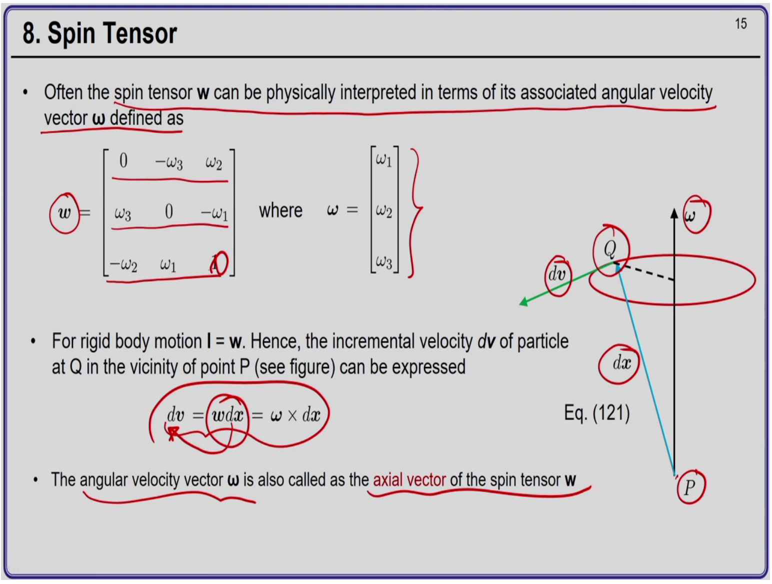

So, I can attach some physical meaning to it. So, often this spin tensor can be physically interpreted in terms of it is associated angular velocity vector omega ok. So, the spin tensor omega is nothing but 0 minus omega 3 omega 2 omega 3 0 minus omega 1 minus omega 2 omega 1 1; oh sorry, this is 0; where omega is the spin vector angular velocity vector this is a component.

So, you can interpret this as if you have a point P and the angular velocity at this point is omega then and you have this point Q whose position vector with respect to point P is dx ok. Then the relative velocity vector dv of Q with respect to P is dv then dv is nothing, but omega cross dx or you can show it also is same as the spin tensor times spatial vector dx. So, the spin tensor maps the relative spatial vector dx to the velocity vector ok.

So, omega maps relative position vector dx to the relative velocity vector dv. So, at the angular velocity vector omega is also called the axial vector corresponding to the spin tensor w.

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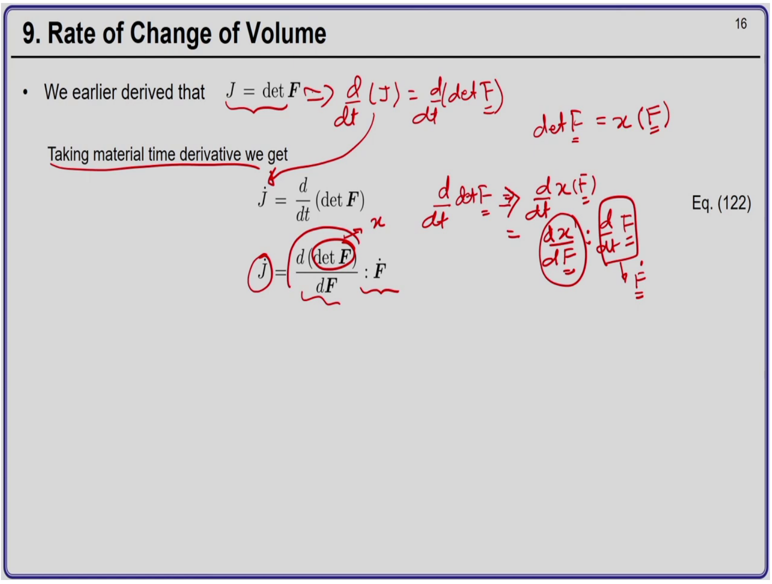

Now, next we come to computing the rate of change of volume and areas first we look into the rate of change of volume. So, we had earlier derived that Jacobian is equal to determinant of F that is determinant of the deformation gradient tensor ok. So, Jacobian characterize the volume change ok.

Now, if we take the material time derivative on both the sides if I take the material time derivative on both the sides which means d by dt of J equal to d by dt of determinant of F and that is what I have shown in equation 122. So, d by dt of J is nothing, but J dot equal to d by dt of determinant of F ok. So, d by dt of determinant of F ok. So, let us say determinant of F is a scalar say x and x is a function of F ok.

So, d by dt of determinant of F equivalently I can write as d by dt of x, where x is a function of F. So, using chain rule I can write dx by d F into now d dx by df is a second order tensor and now our final quantity see on the left hand side we have a scalar. So, dx by d F is a second order tensor to make it a scalar I have to take a double contraction ok. So, I take dx by d F into d by dt of F. So, that is what we have shown here ok.

So, this is our x and d by dt of F is nothing, but F dot ok. So, J dot is nothing, but d by d F of determinant of F double contracted with the rate of change of deformation gradient tensor ok. Now, if you look closely this first term now let me rub this.

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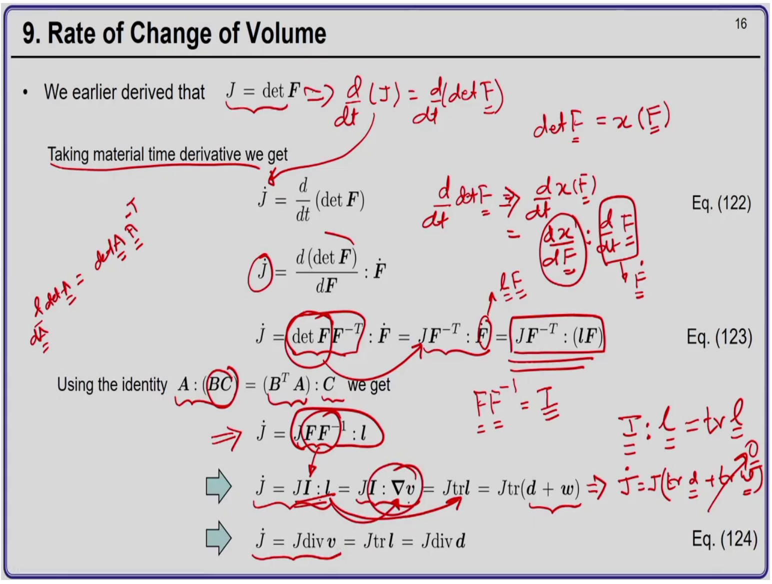

If you look closely this term ok. So, this is nothing, but you have to find out the derivative of determinant of a tensor with respect to it is component ok. And, if you can recall from our discussion when we discussing the mathematical essential for this course that the derivative of the determinant of a second order tensor with respect to the tensor components is nothing but determinant of the second order tensor into F inverse transpose; the inverse transpose of the tensor itself ok.

So, d by d A of determinant of A we had showed is nothing, but determinant of A times A inverse transpose. So, this we had derived ok. So, this I can write and determinant of F is nothing, but Jacobian I can write this as Jacobian times F inverse transpose double contracted with F dot and now, F dot is nothing, but velocity gradient tensor times the deformation gradient tensor F. So, F dot is l F, I can substitute and I can get J F inverse transpose double contracted with l F.

Now, I can use this identity that a second order tensor contracted with a product of two second order tensor B and C is nothing, but B transpose A double contracted with tensor C. So, if I use this here I can show that the rate of change of Jacobian is nothing, but JFF inverse double contracted with the velocity gradient tensor l and what is FF inverse? Ok. So, FF inverse is nothing, but identity tensor ok. So, this is nothing, but identity tensor ok.

So, the rate of change of Jacobean is nothing, but J and I contracted with the velocity gradient tensor l. Now, l is nothing, but gradient of v that we have seen l is nothing, but gradient of v. So, I can also write this expression as JI contracted with gradient of v and we saw that I contraction let me write l ok. So, the double contraction of tensor with respect to identity tensor is nothing, but the trace of the tensor ok.

So, I can write this expression also as J trace of l J times trace of velocity gradient tensor. Now, l is nothing but d plus w. The velocity gradient tensor can be split into the deformation gradient and the spin tensor. Now, I can open up the bracket and I can write J dot as J times trace of d plus trace of omega ok.

So, now you know that d is a symmetric tensor, omega is a anti-symmetric tensor. So, the trace of an anti-symmetric tensor is nothing, but 0 ok. Trace of a anti-symmetric tensor is 0; which means I can write J dot as divergence of U so, this expression over here ok. So, let me rub this. So, I contracted with gradient of a vector quantity is nothing, but equal to the divergence of that quantity.

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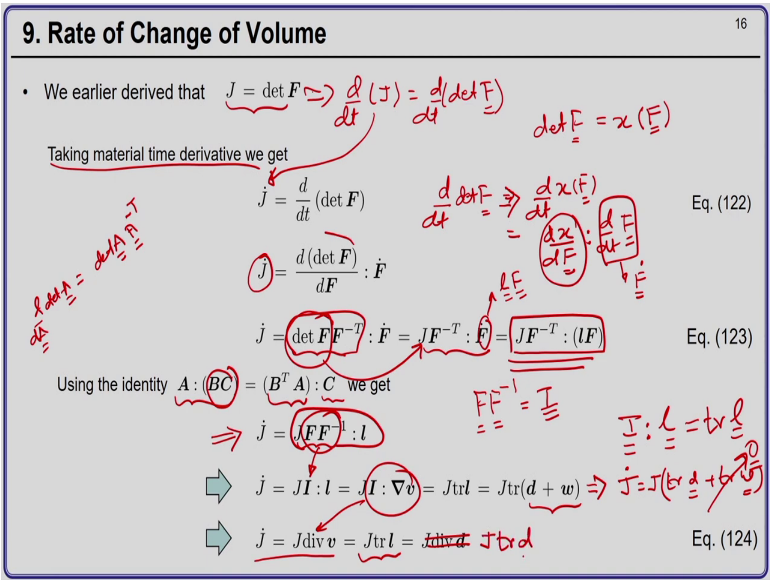

So, J dot is nothing, but J divergence of v which is nothing, but equal to J trace of l is nothing, but equal to sorry J trace of d.

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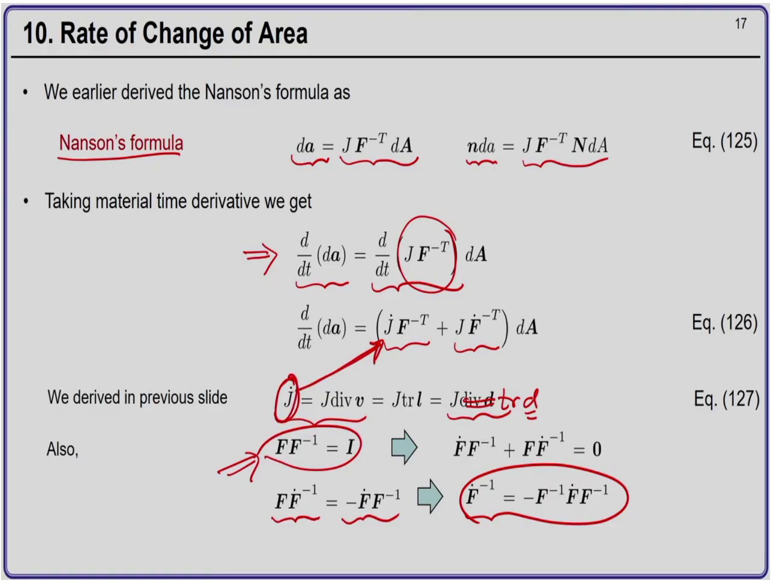

So, with this we come to the next topic which is rate of change of area ok. So, now, we have a infinitesimal spatial area element da and now you want to compute the rate of change of that area ok. So, we know from Nanson's formula that the spatial area vector da is nothing, but JF inverse transpose times the material area element d capital A or I can write nda is equal to JF inverse transpose capital NdA ok.

Now, if I take the material time derivative on both the sides if I take the material time derivative on both the sides I can write d by dt of da is equal to d by dt of JF inverse transpose d capital A. Now, d capital A is a constant quantity it can be taken out ok. So, now, I can open up the brackets here, I can take the time derivative inside the bracket and I can write J dot F inverse transpose plus JF dot inverse transpose.

So, from our previous slides we know that J dot is J divergence of v equal to J trace of l equal to J sorry trace of d ok. So, I can substitute for J dot from here, and then I also know that ff inverse is identity which means I can show that if I take the material time derivative here I can show that FF dot inverse is same as minus of F dot F inverse ok. So, if I use this both of them so, from here F dot inverse is nothing, but this quantity over here and now I can substitute J dot here.

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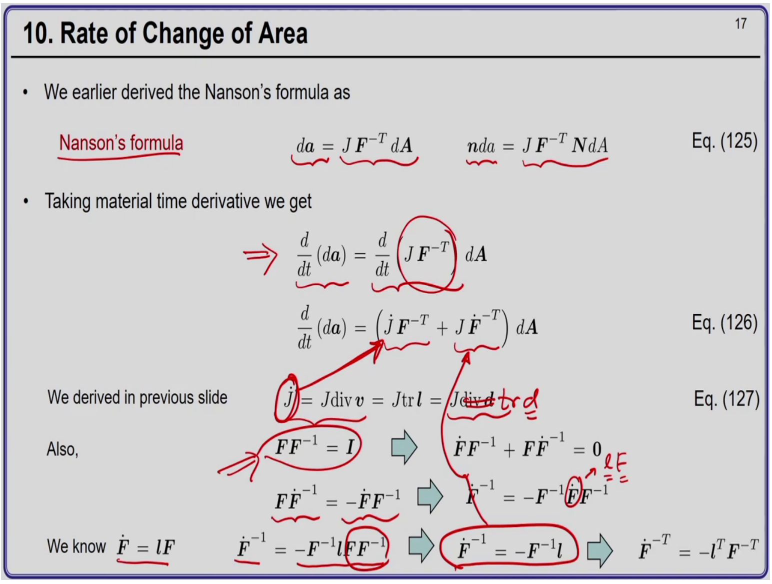

I can further simply F dot inverse. So, we know that F dot is l F. So, F dot inverse ok. So, F dot inverse. So, this F dot is nothing but l F. So, I can have F dot inverse is minus of F inverse l FF inverse. Now, FF inverse is nothing but identity therefore, F dot inverse is nothing, but minus of F inverse l ok. So, I can substitute this expression over here.

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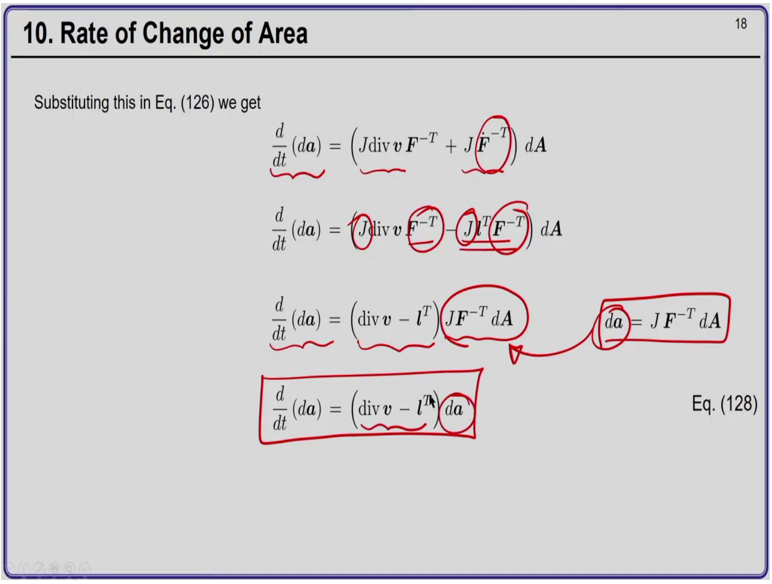

So, I can show then that the rate of change of spatial area element da is nothing, but J divergence of v F inverse transpose plus JF dot inverse transpose ok, where then if I put F dot inverse transpose from our previous slides which we derived here I get following expression ok.

Now, I can take Jacobian outside and F inverse transpose outside the. So, J is common in both the expressions and F inverse transpose is common in both the expressions. So, I can take out J and F inverse transpose outside the bracket and I can write d by dt of da as divergence of v minus l transpose JF inverse transpose da and what is JF inverse transpose?

From Nanson's formula we know that this is nothing, but the spatial area element da ok. So, I can write da here if I write that I can show that the rate of change of spatial area element da is equal to divergence of the velocity vector minus the transpose of the velocity gradient tensor times the spatial area element da ok.

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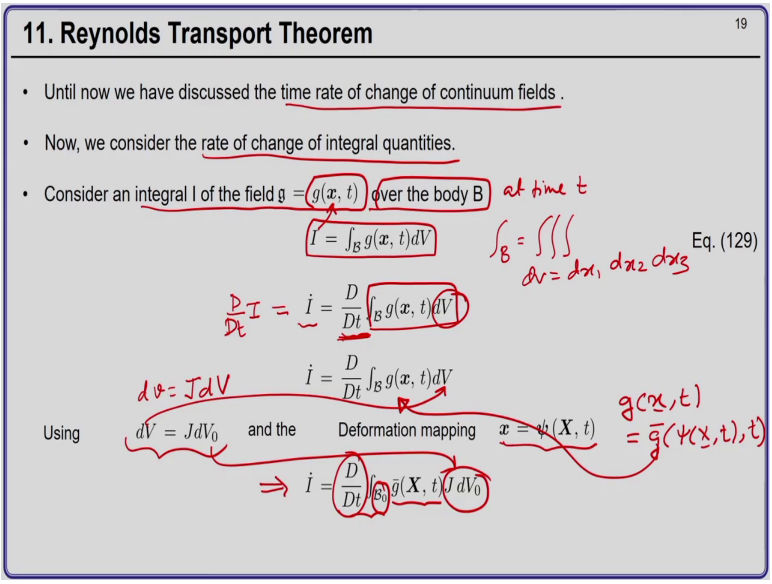

Last we come to the topic of Reynolds transport theorem. So, this is the last topic in kinematics that we discuss and then we will discuss few examples solved examples to complete this topic of kinematics ok. So, Reynolds transport theorem is a very important theorem ok. So, it relates the material time derivative of the spatially integral of spatial quantities ok.

So, till now you would have observed we were dealing with time rate of change of continuum fields ok. Now, what we consider is the rate of change of integral quantities ok, how do you calculate the material time derivative of integral of spatial quantities ok. Now, consider you have a integral I of the field g which is given by g x comma t. So, it is a spatial field, it is a it depends on the spatial coordinates x and over the body B say at time t.

Now, this is the integral I is integral over the volume g of x, t d V ok. So, integral of b is nothing, but the triple integral ok. So, and d V is nothing, but dx 1 dx 2 dx 3. Now, if I take the material time derivative on both the sides if I take D by Dt of I which is nothing, but I dot. So, on the left hand right hand side I will have D by Dt of this integral quantity ok.

Now, the problem is now I cannot take this derivative time derivative inside the integral sign, why? This is because my current volume the current configuration corresponding to the current volume is not known ok. Now, if it is not known, it depends on time then I cannot take this derivative inside. So, how can you evaluate this? So, to evaluate this, what we do is we use this relation ok, the relation between the volume elements in the spatial and the material configuration. So, d V is J d V 0.

Remember, I mean to be clear earlier we were using small dv and capital dV for the material volume element here I am using d V and d V 0 and also using the deformation mapping x psi comma t you can convert g x comma t to another function g bar psi X comma t. So, g and g bar are the same function, but one is expressed in the material coordinates the other we express in the spatial coordinate ok.

So, if I substitute d V here and if I use this here I have transform the integral over the current volume to the integral over the reference volume or the material description or the material volume ok. So, I dot is nothing, but D by DT of integral over the initial volume g bar X comma t and J d V 0. So, this is your J d V 0. Now, I know my material volume, I can now take this differentiation with respect to time inside the integral sign because now I know the configuration material configuration.

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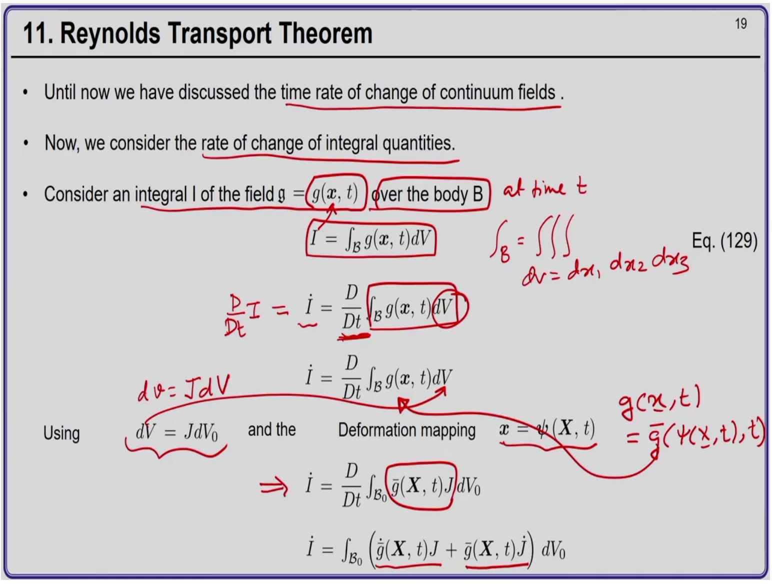

So, I can take the differentiation with respect to time inside the integral and then I can write and I have to take the so, I have to take the time derivative of this integrant g bar J. So, the time derivative of g bar J is nothing, but g bar dot J plus J bar J dot g bar J dot ok.

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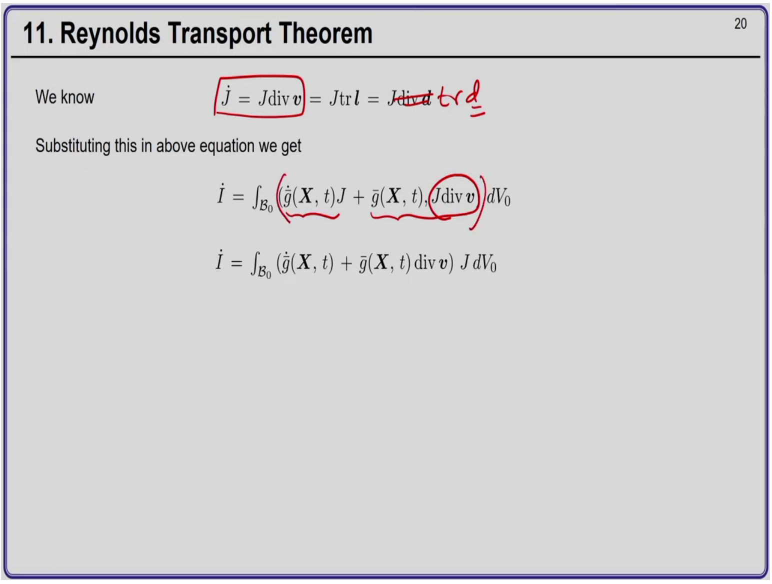

Now, from our previous discussions we know that J dot is nothing, but divergence of v ok, this is trace of d ok. So, J bar is nothing, but J times divergence of v. So, if I use this I can write g bar dot J plus g bar J divergence of v times d V 0.

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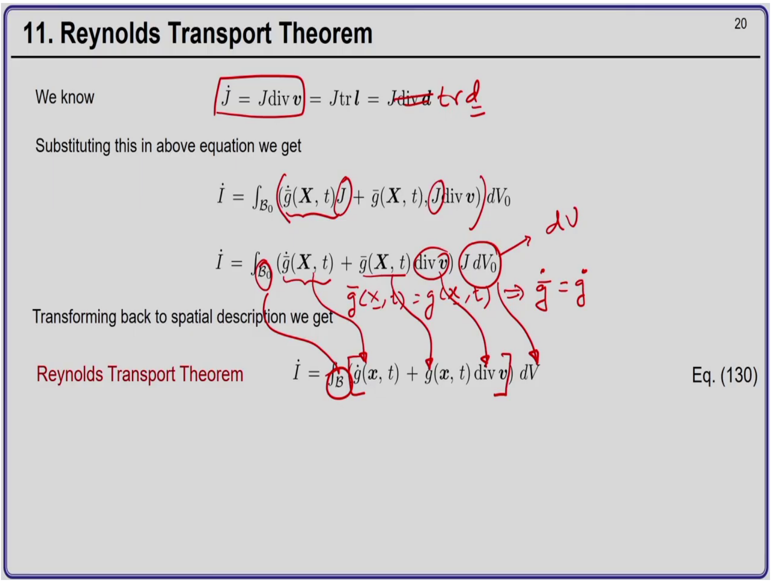

Now, I have see now I have Jacobian ok. Both the terms have Jacobian so, I can take the Jacobian out from the right hand side here and then I can write g bar X comma t as g of X comma t, they are one and the same ok. So, using this I can derive ok. So, g bar is g therefore, g bar dot is nothing, but g dot and this J d V 0 is nothing, but d V ok. So, I can write the first term here becomes g dot and this becomes g and then you have divergence of v and J d V 0 becomes d V and this integral over the reference volume becomes integral over the current volume ok.

So, now, I have the expression for the material time derivative of the integral of a spatial quantity over the current volume and this is given by expression shown in equation 130.

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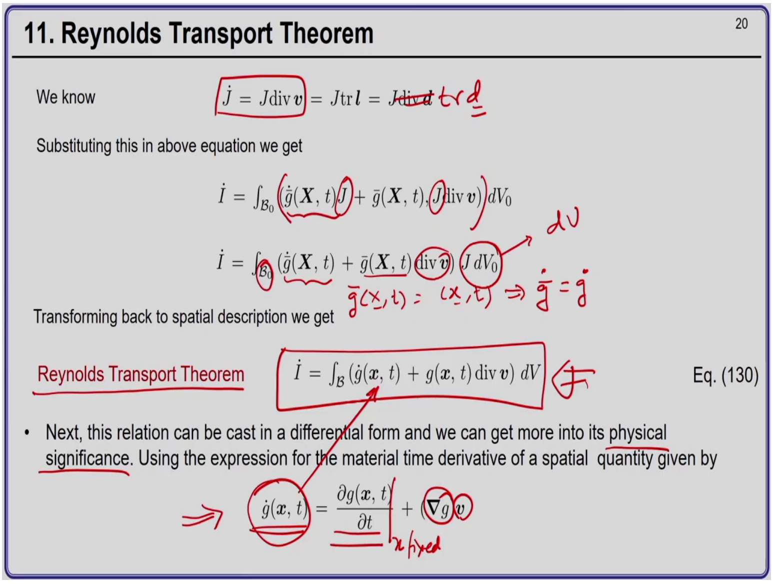

And, this is called the Reynolds transport theorem ok. This is a very important theorem ok. It helps you to carry out the material time derivative of an integral of a spatial quantity over the current volume ok.

Now, I can relate the I I can get more physical interpretation of this expression 130. So, what from our previous discussion on how to compute the material time derivative of a spatial quantity, we know that the material time derivative of the spatial quantity g will be nothing, but del g by del t keeping x fixed plus the gradient of g times the velocity vector.

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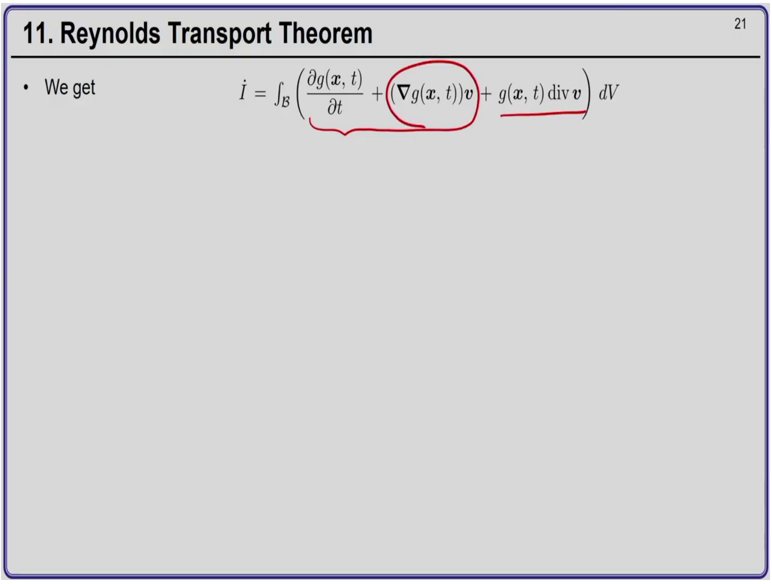

Now, if I use this here if I use this here what do I get? I get del g by del t plus the gradient of g times velocity plus g times divergence of u ok.

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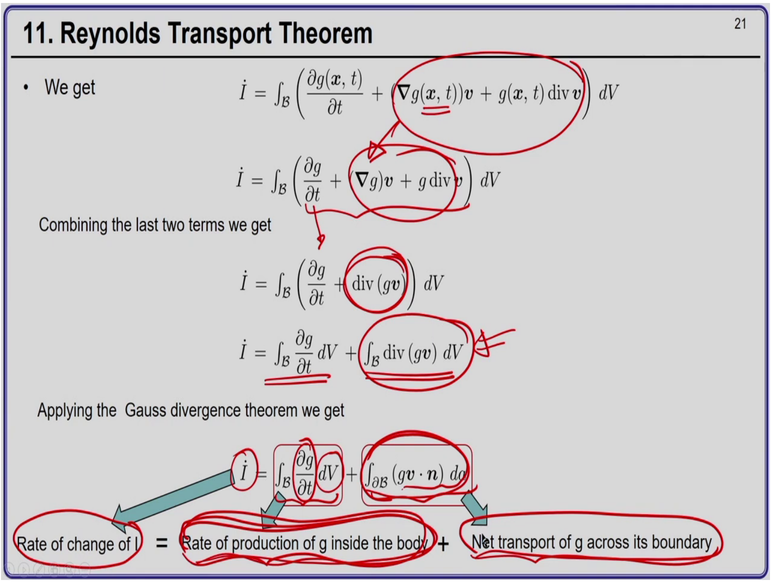

Now, these if I look into this term over here this term is nothing, but if I drop x comma t, I can neatly write like this. It is the same expression, but just x comma t has been removed and this second expression is nothing, but divergence of g into v and the first term remains there itself.

Now, in the second term over here I can split into two, one term plus the integral of a divergence term. And, now I can apply Gauss divergence theorem, I can convert this volume integral over here to an integral over the surface area divergence of g v over d V will be nothing, but the integral over the surface gv dot n da. Therefore, I split I dot into 2, one over the volume and the other over the surface. So, the total rate of change of I is nothing, but the rate of production of g inside the body ok.

So, del g by del t is the production of g at a point current spatial position x and then when you take the integral over the volume you get the total rate of production of g inside the body plus you have the second term which shows that this is nothing, but the net transport of the quantity g across its boundary ok. So, I dot is nothing, but so, the Reynold transport theorem gives you nothing, but the rate of production of g inside the body plus the transport of g across its boundary ok.

So, with this we have covered all the essentials of kinematics which is required for this course. So, next we will do few examples.