Computational Continuum Mechanics
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Kinematics - 1
Lecture - 10-12
Deformation gradient, Polar decomposition, area and volume change

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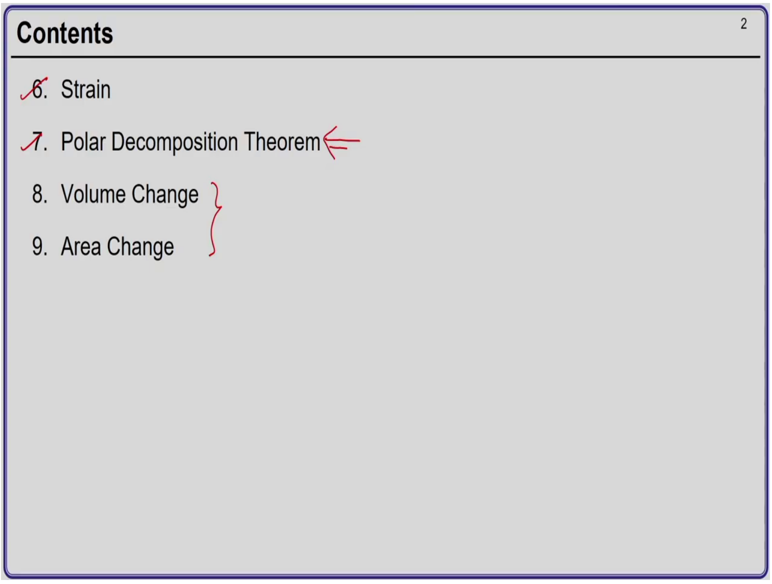

So, welcome back we were discussing following topics. So, we had completed strain and we were midway through Polar decomposition theorem. So, today we will finish the remaining part of the Polar Decomposition theorem and we will see how to relate the volume and area elements in the material and spatial configurations ok. Today we will see how to do left polar decomposition ok.

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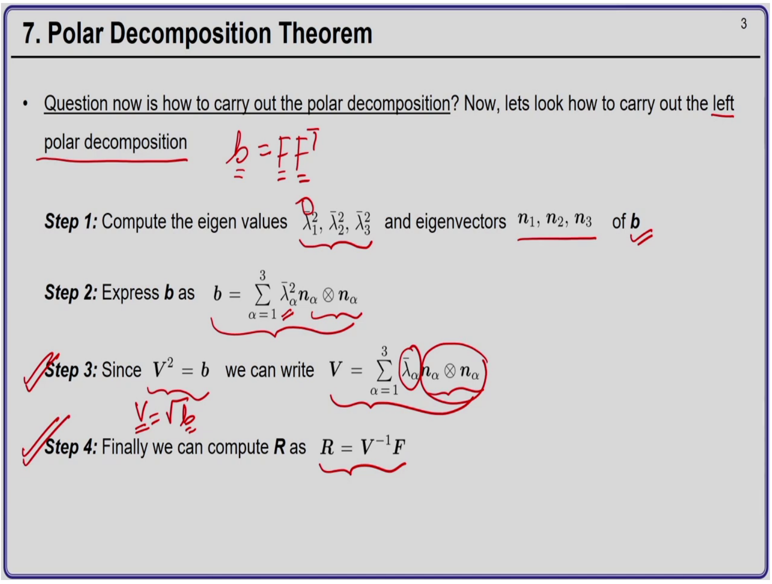

So, that is what we want to do today left polar decomposition. So, we will start with the left Cauchy Green tensor which is given by F F transpose. Now, you have been given the deformation gradient tensor F and now you can compute b. So, the steps to carry out the polar decomposition remains same as discussed previously ok; when we saw how to do polar decomposition using right polar decomposition theorem ok.

Now, the first thing we have to do is; we have to find out the eigenvalues ok. So, make a to make a distinction between the eigenvalues of c and b we put a over bar over the eigenvalues ok, but still we denote the eigenvalues of b as lambda 1 bar square lambda 2 bar square lambda 3 bar square which means; we retain the square term for the obvious reasons that later on we have to take a square root.

So, let these be the eigenvalues and let us small n 1 n 2 n 3 be the eigenvectors; why small? Because b resides in the spatial configuration ok; so by convention all quantity which are in the spatial configuration we denote by lower case letter. Its not always the case, but most of the time spatial configuration quantity that denoted by lower case letters. So, once we have found out the eigenvalues and eigenvectors of B we can use the spectral decomposition theorem and express b in terms of the eigenvalues and the tensor product of the eigenvectors.

So, once we have expressed b in terms of its eigenvalues and eigenvectors and we know that V square equal to b. So, V will be square root of b ok. So, basically what it means we have to take the square root of the tensor b. So, that would be nothing, but V equal to lambda alpha bar n alpha tensor product n alpha ok. So, the eigenvectors have not changed, but the eigenvalues are now square root of the eigenvalues of b.

So, once we have V we can compute the orthogonal tensor R as V inverse F ok. So, once we compute R we have done the polar decomposition ok. We have determine in step 3 the left stretch tensor V and the step 4 we are determine the orthogonal tensor R ok.

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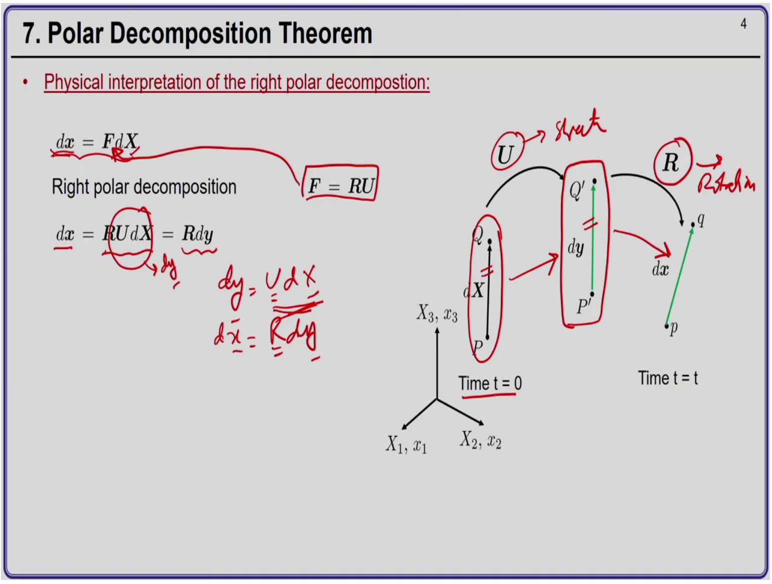

Now, we move to get a physical interpretation of right polar decomposition. So, deformation gradient tensor F maps the material vector d X to the spatial vector d small x that we know. Now, the polar decomposition right polar decomposition gives you F equal to R into U. Therefore, if you substitute for F here if you substitute for F here we will get the spatial vector d x equal to R U times d capital X the material vector d X ok.

So, U times the material vector d X I can express as d Y and then I can write the spatial vector d x as R d y I can write it as R d y ok. So, how this looks like? Ok. So, first we have computed d y which is nothing but U d x ok. So, initially if you have a time t equal to 0 you have a material vector d X. So, we have not shown the body we are just showing the material vector.

So, what U d x does is; it stretches the material vector d X by certain amount without actually rotating the material vector which means that the vector P Q is parallel to vector P dash Q dash. So, that is the P dash Q dash is not the final configuration its the intermediate just for our conceptual purpose.

So, U d X just stretches the material vector d X by a certain amount and then the final spatial vector is R d y ok. So, R d y means; now the rotation or the orthogonal tensor R rotates the vector d y without actually doing any stretch. So, U causes stretch and R causes the rotation. So, you can see here we have the stretch without any rotation and then we have rotation without having any stretch ok.

So, let me rub yeah. So, now, we can show so we have asserted that the length of the vector d y is same as the length of the vector d x ok. So, this we can actually show.

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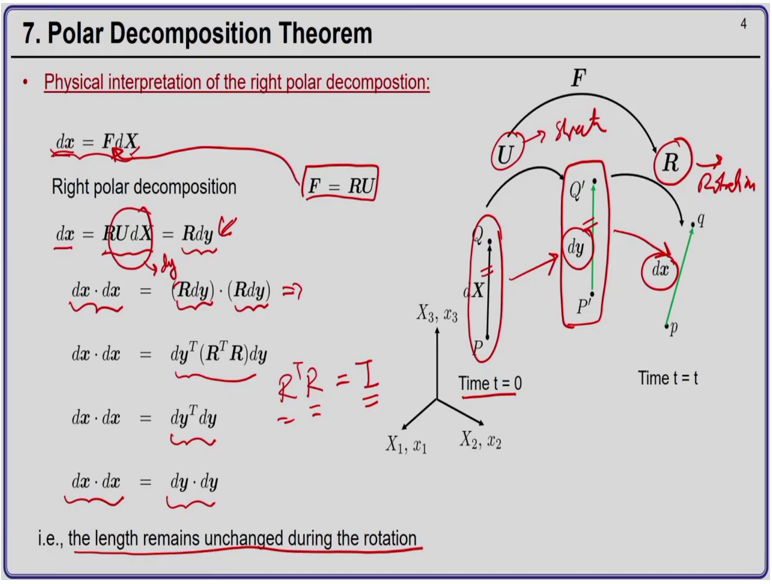

So, we know that d x dot d x that would be nothing, but. So, I mean sorry we want to show that the length of vector d y is same as the length of vector d x ok; because that is where rotation is involved. So, we start with the scalar product of the spatial vector d x dot d x and now d x is nothing but R dy d x is R d y; so substituting that we get R d y dot with R d y ok.

Now, R d y is vector an R, so this is nothing, but dot product of two vectors and I can write this as d y transpose R transpose R d y. Now, we know that R is a orthogonal tensor which means R transpose R will be nothing but identity tensor ok. So, once R transpose R is identity what we are left with is d y transpose d y which is nothing, but d y dot d y ok.

So, we see that these square of the final length d x dot d x is same as square of the stretched length ok; which was obtain by stretched; which means that R does not cause any stretch it only causes rotation. So, the final length remains unchanged during the rotation that is how you can show. Now, coming to the interpretation of the left polar decomposition ok.

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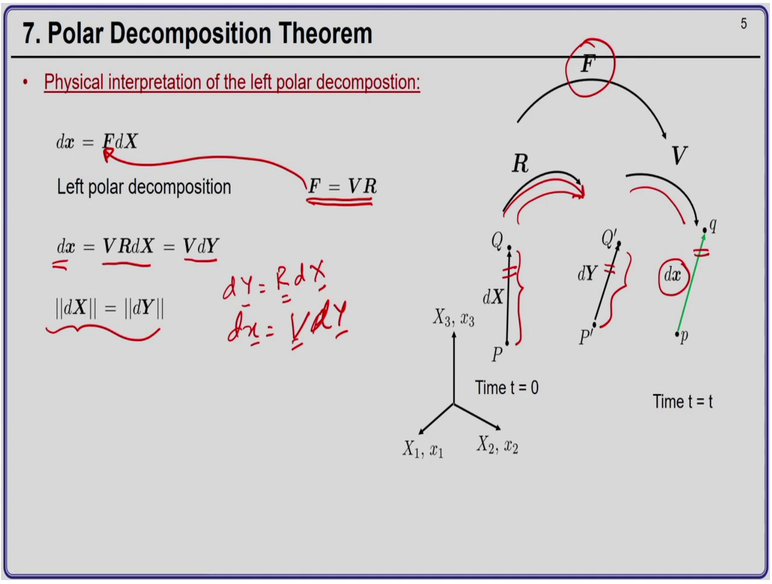

So, in left polar decomposition we have F equal to V R. So, substituting this here so the spatial vector is nothing but V R d X, which is nothing but V d Y d capital Y. So, how can we interpret this? Ok, so first there is only rotation because d Y is sorry this a vector. So, one under bar R d X ok.

So, as in the previous slide we can show that the length of the material vector d X remain same as the length of the vector d Y ok. So, in the first step what we get is rotation ok. So, this material vector gets rotated without change in length which means that the length of vector P Q is same as the length of vector P dash Q dash ok.

Now, next thing is we get the spatial vector V d Y ok. So, this means now we get the stretch. So, the rotated vector which is d Y is now stretched without rotation. So, you see that P dash Q dash is parallel to P Q which is the final position of points P and Q ok. So, the overall deformation F can be split into first rotation and then stretch.

In right polar decomposition the total deformation can be split into first stretch and then rotation ok; that is the physical interpretation of left and right polar decomposition ok.

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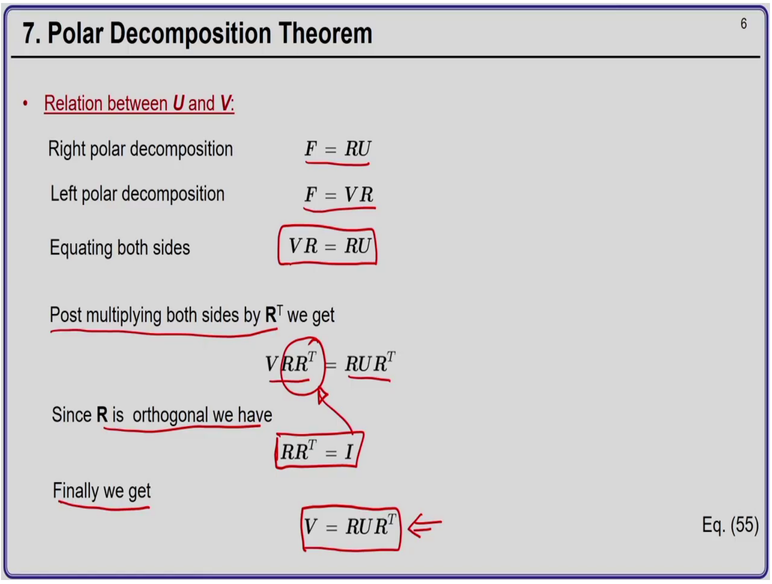

Now, we derive certain relations which will be useful later on. So, first we see; what is the relation between U and V ok; the right stretch tensor and the left stretch tensor. So, the right polar decomposition gives you F equal to R U whereas, the left polar decomposition gives you F equal to V R and because F is same we can write equating both sides V R equal to R U.

Now, if you post multiplying both sides by R transpose we get V R R transpose is R U R transpose

Now, we know that R is orthogonal which means R R transposes identity tensor ok. So, this term over here becomes identity ok. And then finally, we can relate the left stretch tensor V to the right stretch tensor U by this equation V equal to R U R transpose ok.

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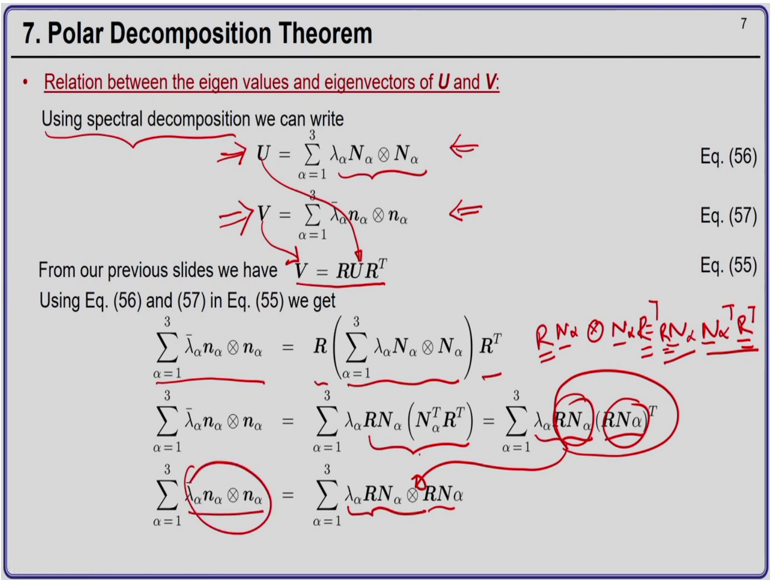

So, now next is we want to have a relationship between the eigenvalues and eigenvectors of the right and the left stretch tensors. So, recall from our previous discussions that using spectral decomposition we can write the right stretch tensor in terms of its eigenvalues and eigenvectors given by equation 56 ok. In similar manner we can write the left stretch tensor in terms of its eigenvalues and eigenvectors which is given the equation 57 ok.

So, now just over a previous slides we had derive this relation V equal to R U R transpose. Now what we do we substitute V and U that is not we are doing substitute V and U from equation 56 and 57 in equation number 55. So, once we do that what do you get? So, left hand side let us keep it same on the right hand side we have R the term in the bracket times R transpose.

So, now I can take R transpose inside the bracket and N alpha tensor N alpha can be written as; N alpha N alpha transpose. And if we multiply pre multiply by R and we post multiply R transpose the right hand side would become R R transpose ok; so that is what we have here we have R alpha and N alpha transpose R transpose.

So, R N alpha is nothing, but a vector and N alpha transpose R transpose again is vector. So, I can write this as lambda alpha R N alpha R N alpha transpose and this because these two quantities are vectors I can replace this relation over here with the tensor product. So, I can write lambda alpha R N alpha tensor product R N alpha ok.

Now, I can relate both sides now I can compare both sides ok. So, I can compare this quantity over here to this quantity over here.

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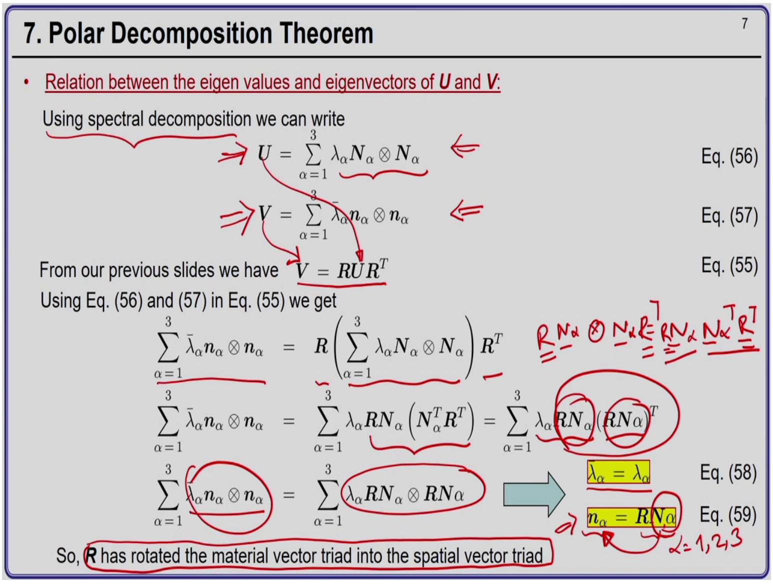

And I can find a relationship between the eigenvalues and eigenvectors of the right stretch and the left stretch strain tensors. So, this is given by.

So, you can see lambda alpha bar is same as lambda alpha which means that the eigenvalues of U and V are same ok; while the eigenvector of V is equal to the R times eigenvector of U. So, N alpha is R N alpha. So, that is the relation that we come up with.

So, what is R done here? What the rotation tensor has done? The or the orthogonal tensor R; so we also call it orthogonal rotation tensor. So, R has rotated the material vector triad into the spatial vector triad. So, this R has rotated this material vector triad from N alpha to spatial vector triad small n alpha. So, here alpha would be 1, 2, 3.

So, lambda 1 bar is same as lambda 1, lambda 2 bar is same as lambda 2, lambda 3 bar is same as lambda 3. And n 1 is same as n 1 is equal to R capital N 1, n 2 is R capital N 2, n 3 is R capital N 3 ok.

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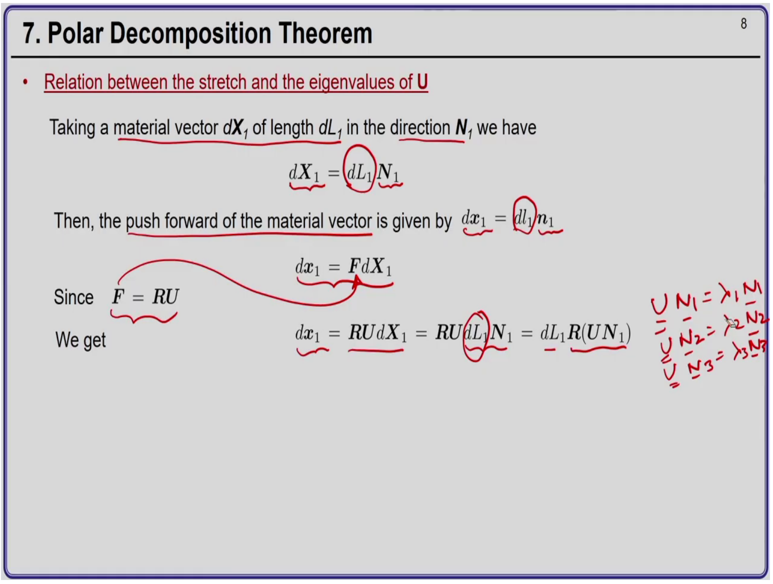

So, once we have this relation let us now move to finding the relation between the stretch and the eigenvalues of the right stretch tensor and from here we will notice why U is called the stretch tensor.

So, now let us take a material vector d X 1 of length d L 1 in the direction of N 1. So, N 1 is your material vector is the eigenvalue is the eigenvector. So, d X 1 will be is a vector so it is will be magnitude of a vector times the unit vector in that particular direction ok. So, d X 1 is d L 1 N 1 ok. So, the lengths of d X 1 is d L 1 ok.

Now, the push forward of the material vector which means d x 1 is given by F d capital X 1 where d x 1 the spatial vector d x 1 is its length d l 1 times n 1 ok; the unit vector in that particular direction.

Now, since from the right polar decomposition F is R U we know that F is R U. So, we can substitute this F here and we can write d x 1 as R U d X 1 ok and now d capital X 1 can be written as d L 1 N 1 its d L1 N 1 ok. So, because d L 1 is a scalar quantity I can take it on the left of this expression. So, d L 1 into R times U N 1 ok.

Now, because N 1, N 2, N 3 are the eigenvalues of U and from the eigen N 1, N 2, N 3 are the eigenvectors of U. So, we know that U N 1 is same as lambda 1 N 1 U N 2 is lambda 2 N 2 and U N 3 is lambda 3 N 3 because lambda 1 and N 1 are the eigenvalues and eigenvectors of U; so, we can have this relation. So, let me rub this.

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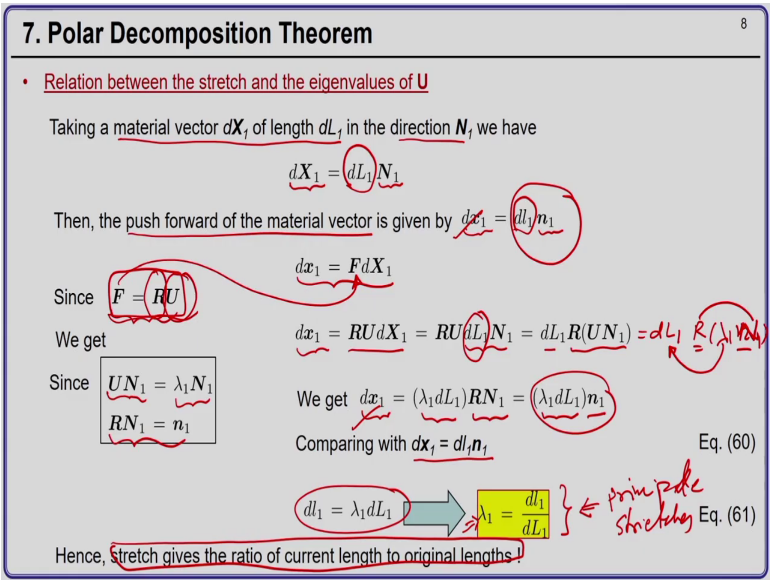

So, we can substitute U N 1 as lambda 1 N 1. So, we can substitute U N 1 as lambda 1 N 1. And now once we have lambda 1 N 1 here we will have d L1 R lambda 1 N 1 capital N because lambda 1 is a scalar I can take on this side. So, we are left with R times N 1. So, R N 1 is nothing, but small n 1 ok. So, using this we will get the spatial vector d x 1 as lambda 1 d L 1 equal to R N 1 which is nothing, but lambda 1 dl 1 equal to lambda 1 d L into n1 ok.

Now, we can compare this relation with this particular relation because we have a scalar times n l in both the expression and the left hand side of both the expressions are same therefore, the scalar multiple of n 1 should be same ok. So, comparing with d x 1 equal to dl 1 n 1 what we get? dl 1 is lambda 1 d L 1 or which gives us lambda 1 as ratio of d l 1 divided by d capital L 1 that is the final length of the vector d x 1 divided by d capital L 1 and this is nothing but the stretch.

If you remember in our previous lecture we have defined stretch as the current length divided by original length and because lambda is the lambda is the eigenvalue of U and this is nothing, but the stretch that is why U is also called the stretch tensor because it has stretch built into it ok.

So, now if you see F equal to R U you want to recognize now that the total deformation the stretch is entirely present in U and the rotation is entirely in R. So, what we have actually achieved by polar decomposition is the deformation is now, split into what is called rotation and stretch.

Why this is important is because we will come to it later that the stresses can develop only when there is certain stretch rotation cannot develop stress ok. So, if you have body and you just rotate it there will be no stresses generated inside that body ok. And I am of course, speaking that you rotate it pretty slowly. So, if you just rotate it pretty slowly there will be no stretch generated. So, rigid body rotations will not generate any stress.

Therefore, once we have to compute increment in the stress because of deformation we have to take out the effect of rotation ok. So, here we have split the total deformation into a rotation part and a stretch part. And then when we are going to find out the increment in the stress during the course of deformation we can know that we have to use U to somehow compute our strains ok.

And if we use U and U only has stretch when we are discussing the physical interpretation of right polar decomposition we saw it only causes stretch and no rotation. Therefore, the strain when computed from U and from that strain when we compute stress you will have stress increment only because of stretching and rotation effects will be taken out of the consideration.

So, this stretch gives the ratio of current length to original length ok. So, now, we know why U is called the stretch tensor because the eigenvalues of U are denote nothing, but the principal stretches. So, these are also called the principal stretches ok.

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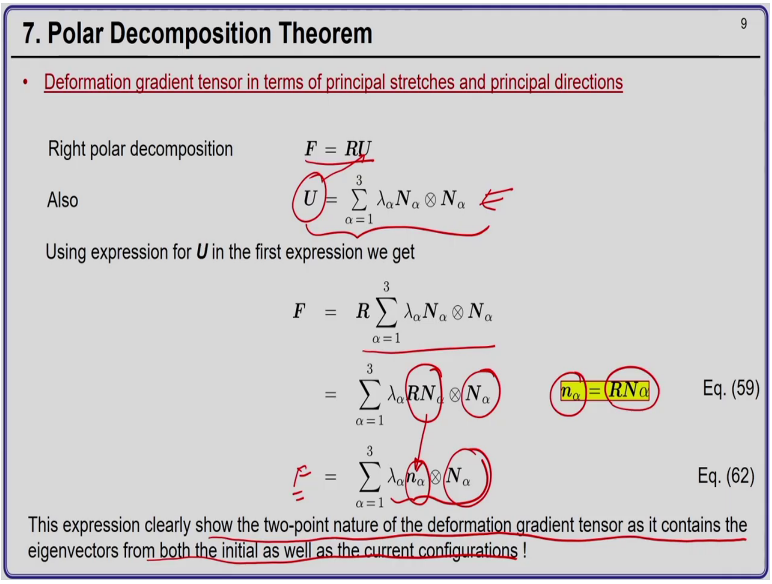

Now, we can write the deformation gradient tensor in terms of the principal stretches and principal directions. Why this is important is; why we are doing this is; it will clearly being route that F is a two point tensor. If you remember in the previous lecture I said F is a 2 point tensor, but its nature why its two point was not clear which will become clear now in this slide ok.

So, right we will start with right polar decomposition F is given by R U and now using spectral decomposition U or the right stretch tensor can be written in terms of its eigenvalues and eigenvectors. So, now if we substitute U from the second expression in the right polar decomposition what we get F is R times this quantity.

And if I take the orthogonal tensor R inside the summation sign we have R N alpha tensor product N alpha and I know that R N alpha is nothing, but n alpha so this we have derived.

So, once we have this relation I can write R N alpha is small n alpha. So, finally, F the deformation gradient tensor is nothing, but lambda alpha N alpha tensor product capital N alpha.

So, now, you can clearly see that F the deformation gradient tensor F is made up of the eigenvectors in the spatial configuration which is given by N alpha and the eigenvectors in the material configuration which is capital N alpha ok.

So, you can clearly see the two point nature of the deformation gradient tensor as it contains the eigenvectors from both the initial as well as the current configurations. So, F has the effect of both the configurations built into it that is why its called a two point tensor ok. So, this expression number 62 will make it pretty clear ok.

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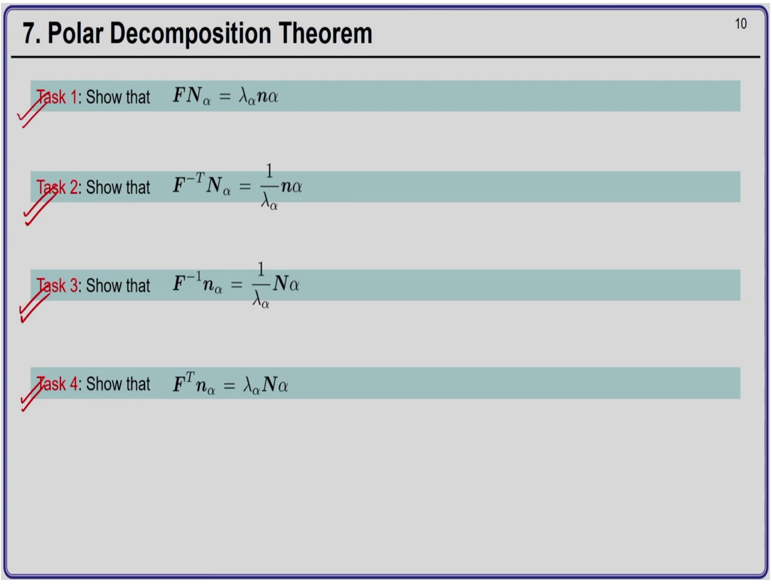

Now, there are certain task for you. So, the first task and these are certain proofs that I will request you try yourself and in case of any problem you can always drop me a message or email and I will be happy to help you out ok. So, these are certain proof that you can try yourself they are very basic ok. And they can be done easily using the concepts that we have discuss till now and this will also strengthen your understanding of the kinematic quantities ok.

So, moving ahead we now come to a final two topics out of which the first one is how to relate the volume change ok. So, as the body deforms from its initial configuration to the current configuration the volume of the body may change.

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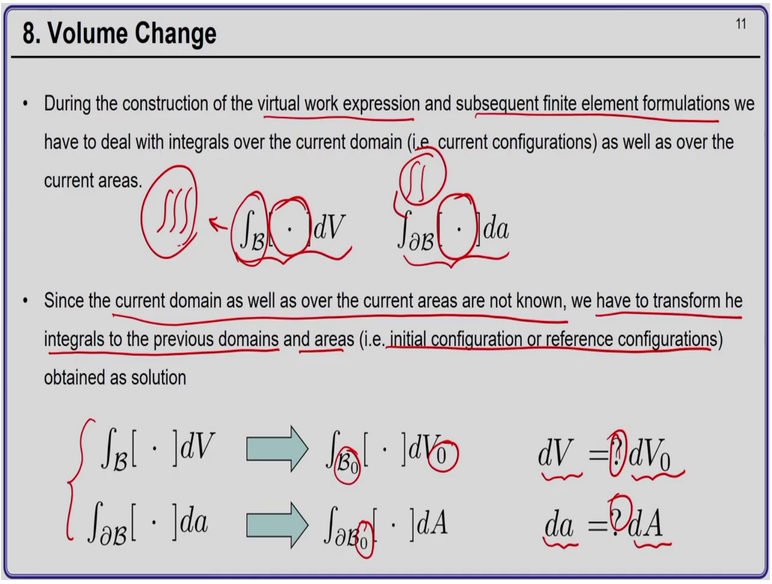

So, how do we relate these volume changes and this is particularly important because when we a writing the virtual work expression and we are dealing with subsequent finite element formulation. We have to deal with integral over the current domain which is the current configuration. So, this is the integral over the current domain as well as over the current areas ok. So, this is the area integral in the current configuration. So, this integral over here is nothing, but is a triple integral remember this is a triple integral and this integral over area is a double integral ok.

So, now you have to and this quantity in the square bracket I have put a dot this can be any expression ok; we will come to it later there will be different expression, but that particular expression over the integrant has to be integrated over the current volume and as well as area ok.

Now, what is a problem with carrying out this integral? Ok. So, the problem is the current domain; as well as the current areas are not known ok; you do not now what is the current domain that is what you want to find out. So, if you do not know the integration limits because you do not know the area you do not know the current configuration you cannot put the integration limits ok.

So, here in the triple and the double integral you cannot put the limits once you cannot put the limits you cannot carry out the integrals. So, what is the solution? And the solution is we have to transform the integrals to the previous domain and areas that is the initial configuration or the reference configurations ok. So, these integrals which are here that is the volume in the area integral have to be transformed to the previous configuration which you had already solved ok.

In the non-linear finite element setting when you are going over certain increments the previous increment which you already know you can transform all your current integrals to that previous configuration ok. So, we will like to transform this integral over the known volume and known areas see we have B 0 here and delta B 0 which means we have to carry out this triple and double integrals over the initial configuration. And we know the initial configuration so it is pretty easy to not pretty easy I mean it is possible to carry out this volume and area integral.

Now, the problem here is how do we relate the volume element d V towards initial volume element d V 0 and the current area element d a to its initial area element d capital A. So, what are these relation? And our objective now is to determine these relation ok.

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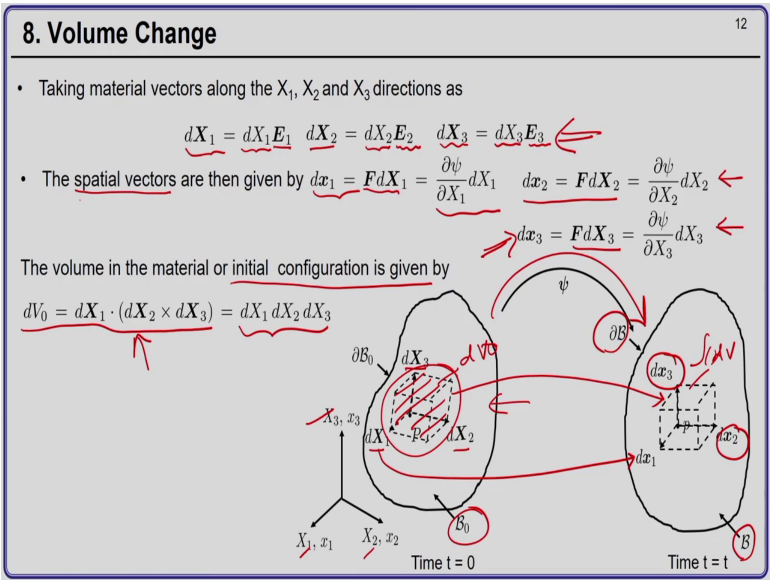

So, first we look into the volume integral. So, the picture here shows; the body at time 0 in that is in the integral configuration and let us say you have this volume. So, at point P you have this infinitesimal volume d V and you have material vectors d X 1, d X 2, d X 3 which are aligned along the X 1 X 2 and X 3 directions respectively. So, in a finite element setting this dotted lines would actually refer to the one particular finite element ok.

So, now as the deformation happens ok; so as the body deform it occupies at time t a certain configuration B bounded by area del B and this volume changes to some other volume. So, the volume changes so let us say this is d V 0 here and it becomes d capital V here. Because equilibrium equations are written in the deformed configuration ok; so we have our integrals here, we have our integrals here and you have to transform all those integrals over this current area a current volume and the areas to integrals over this particular volume that is the initial volumes and areas.

So, how do we do that? So, d X 1 is aligned along the X 1 direction which means it is magnitude of the vector times the unit vector in that direction then this capital E 1. Then d X 2 the second material vector will be the magnitude time the unit vector along the X 2 direction and d X 3 will be magnitude times the unit vector along the X 3 direction ok. We have chosen the Q such that its 3 sides are along the x 1 x 2 x 3 axis.

Now, the spatial vectors will be the push forward of the material vectors; so, d x 1 so d x so this d X 1 goes to this spatial vector d x 1. So, this is the d x 2 and this is spatial vector d x 3. So, d x 1 is F d X 1 which can be shown as del phi by del X 1 d X 1, d x 2 is F d X 2 which can be shown as del phi by del X 2 d X 2 and d x 3 the spatial vector d X 3 is F d X 3 which is can be shown as del phi by del X 3 d X 3 ok.

Now, the volume element. So, this volume element which is here in the initial configuration will be given by d X 1 dot d X 2 cross d X 3 a dot b cross 3 B cross c. So, this vector triple product scalar triple product will give you the volume and then if you substitute these relations here you can show that the volume d V 0 will be d X 1, d X 2, d X 3 ok.

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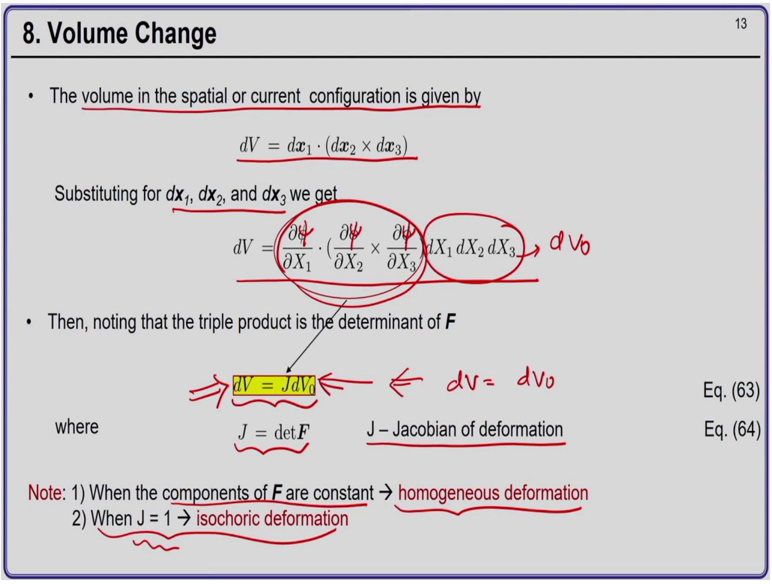

Now, similarly I can write the volume in the spatial on the or the current configuration as d V equal to d x 1 d x 2 cross d x 3. Now, I can substitute for d x 1 d x 2 d x 3 from the previous slides we have those expressions and I can write d V as del sorry this is del psi by del X 1 dot del psi by del X 2 del cross del psi by del X 3 into d X 1 d X 2 d X 3 ok.

Now, this quantity in this circle is identified as the determinant of F and this quantity here is nothing, but your d V 0. So, I can write d V as J times d V 0. So, we are denoting the determinant of the deformation gradient by a symbol J where, J is called the Jacobian of deformation. So, the spatial volume element is related to the material volume element through equation 63 which is d V equal to J d V 0.

So, the final volume is equal to the Jacobian times the initial volume and J is called Jacobian of deformation. Certain points to note when the components are F are constant they do not depend on X 1, X 2, X 3, then this is called the homogenous deformation and when the Jacobian J is equal to 1 it is called isochoric deformation.

Isochoric means if you substitute J equal to 1 in equation number 63 you will get d V equal to d V 0; which means the volume of the in element will not change which means it is a isochoric deformation which is volume preserving deformation.

So, in this course we are not dealing with cases or materials where J will approach 1 ok; that are in compressible materials we are not dealing in this course.

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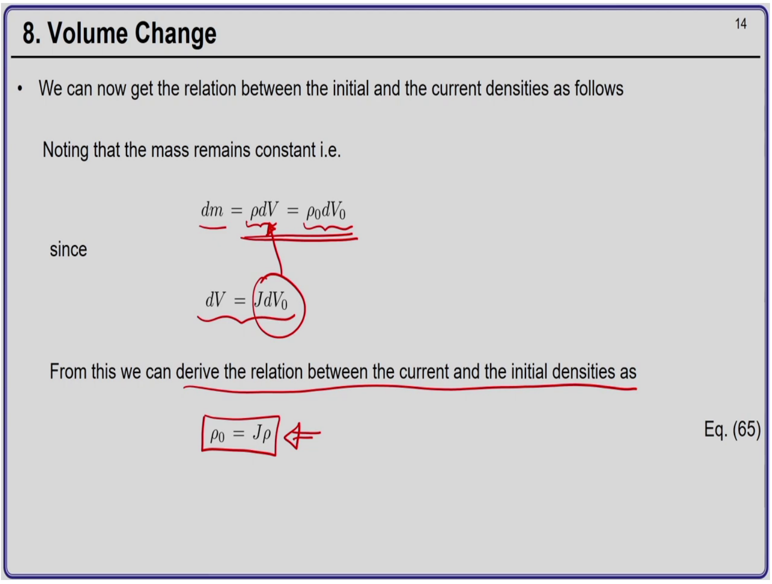

Next we can relate now the densities ok. So, we have related the volumes in the final and the initial configurations d V equal to J d V 0. Similarly, we can derive an expression for the densities in the initial and the final configuration ok. So, how do we do that? The first point we note is that the mass the mass of the element will not change there is no mass loss ok.

In that case we can write the mass of that small q in the final configuration as the density in the final configuration times the volume in the final configuration which is rho d V and because mass is constant. So, d m is also same as d rho 0 d V 0 that is initial density times the initial volume ok.

Now, if we look to these expressions and we also know that d V is J d V 0 ok. So, this I can substitute here this is I can substitute here and then I can derive relation between the current and initial densities as rho 0 equal to J rho ok. So, this is the conservation of mass statement ok. The mass remains conserved which means that the densities are then related by this relation the initial density is equal to J times final density at that particular point. So, this will be again used later.

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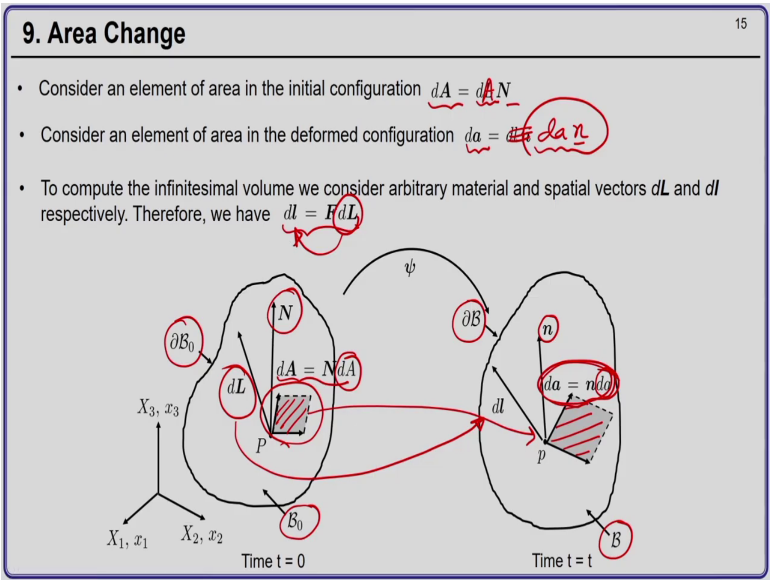

Coming to the final point: So, the final point now is how do we relate the areas in the initial and the final configuration? So, consider again this body the initial configuration is B 0 bounded by surface del B 0 which after deformation we have in the final configuration B bounded by surface del B.

And then we consider at point P a small infinitesimal area d A ok. So, there is small area d A shown by this dash lines dash I mean dash lines that is the area and the shaded portion is that particular area which after deformation occupies this I mean this region ok.

So, let N be the normal to this area ok. So, capital N is the normal to the area in the initial configuration and small n is the normal to the area in the current configuration. So, the area element d capital A will be nothing, but N d A and the area element in the final configuration will be n d a where the scalar capital D a is the area magnitude of the area and this small d a scalar d a is the magnitude of the current area ok. So, d A will be d capital A n and sorry this is d a n ok.

So, the area vector is magnitude times the normal N is the normal to the area and d a ok, the current area element will be the magnitude times the normal n ok. So, once we have this we can compute the volume by considering a small arbitrary material vector d capital L ok.

And how much volume this area infinitesimal area d a and this material element d l occupy. And we see that this material vector is map to the spatial vector dl. So, we know that d l is F d capital L ok. So, the deformation gradient maps this vector d capital L to vector d small l ok.

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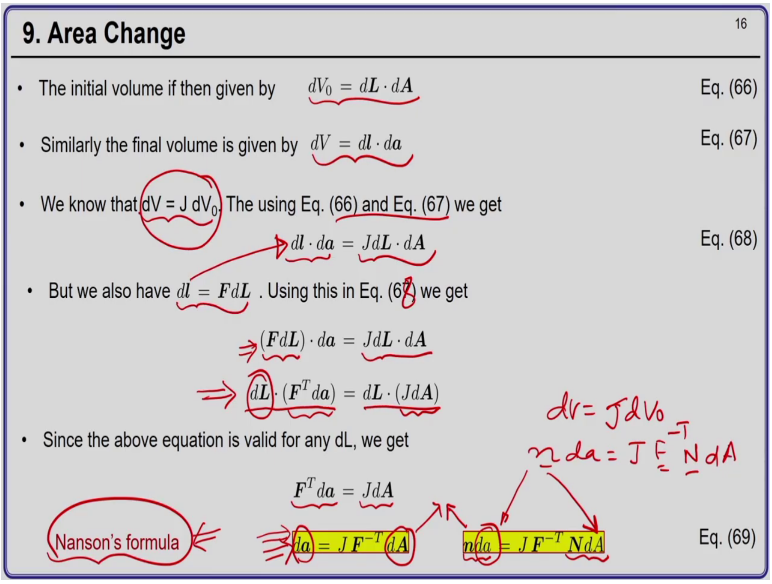

Now, what will be the volume? The volume bounded by the area vector d A and the arbitrary vector d L it will be say d V 0 is d L dot d A that is the volume that is bounded by these two vectors. Similarly, the final volume will be dl dot d A. Now, we know that d V is J d V 0 ok. So, what we can do now is; from equation number 66 and 67 we can substitute in this relation the value of the current volume and the initial volume and we will get d l dot d a is J d L dot d capital A ok.

Now, we know that d l is F d capital L ok. So, using this if you use this in this relation ok; if you use this in this relation over here ok; so this will be equation 68. So, what we get? F d L dot d a is J d L dot d A ok. So, we can rearrange and we can write this expression as d L dot F transpose d a is equal to d L dot J d capital A. And now since d L was an arbitrary material vector therefore, these terms inside the brackets must be same. So, if they are same then F transpose d a is same as J d capital A ok.

So, now I can take multiply both sides by F inverse transpose. So, if I multiply both sides by F inverse transpose I can now relate the area element in the deformed configuration to the area element in the undefromed configuration. So, d a is J F inverse transpose d capital A ok.

Now, remember that the area integral ok; so when you take the area integral d a is a scalar and here d a is a vector, but I can write d a vector as n d a and d capital A as N d A ok. So, now, I have this relation; I have this relation of the initial of the final area element area and relation with the current or the initial area d a so n d a is J F inverse transpose N d A and this relation is known as Nansons formula.

So, when you have to transform the area integral from the current configuration to the area element in the reference of the initial configuration; then you have to use what is called the Nansons formula ok. So, either of these relation is the Nansons formula both are same so this is the Nansons formula that you will be using.

So, now we have derived the relation between the volume elements in the initial and the final configuration and also we have derive the relation between the area elements in the initial and the final configuration.

So, later on, when we go and write integrals when we are writing the equilibrium equation or the virtual work expression in the current configuration then we have to transform those integrals to the initial configuration for the computations because the current configuration is not known.

I similarly transform my relations integrals in the initial configuration and then I can compute. In the initial configuration I know my integration limits and then I can compute and to do that we use these relations d V equal to J d V 0 and n d a is J F inverse transpose N d capital A.

So, with this we come to end of this lecture. So, we have covered a certain part of kinematics ok. And what is remaining now is to solve one or two numerical problems ok. So, in the next lecture we will start with and few numerical examples and then we will discuss about the linearize kinematic ok. We will discuss what is meant by material time derivative ok. We next we will meet in next lecture ok.

Thank you.