Computational Continuum Mechanics

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Kinematics – 1
Lecture - 10-12
Deformation gradient, Polar decomposition, area and volume change

So in today's lecture, we are going to look in to different Strain measures and then little bit of Polar Decomposition theorem we will start.

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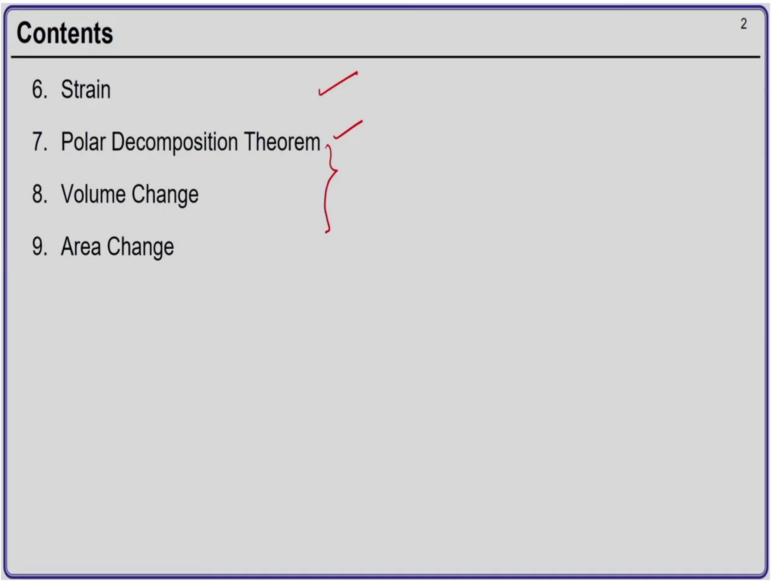

So, in today's lecture we will cover this topic and partially polar decomposition theorem. In next lecture we will cover the or partial polar decomposition theorem and volume change and area change.

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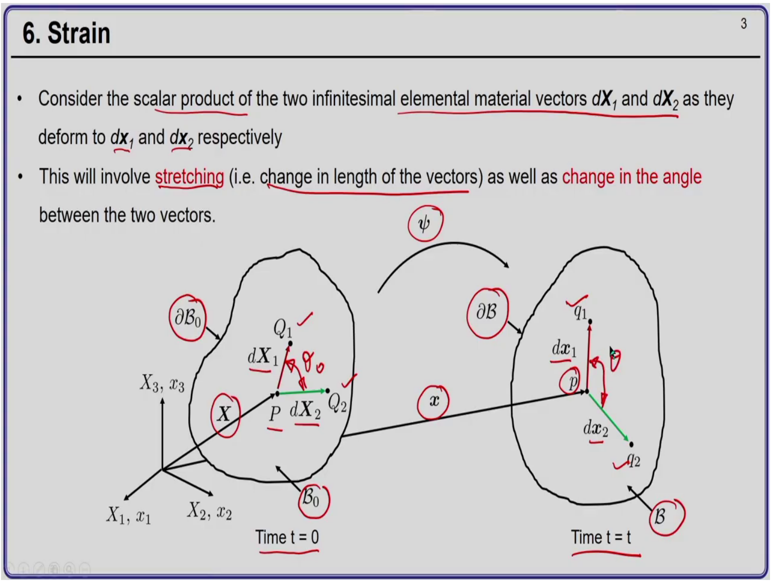

Now, coming to strain, ok; so, to define strain, let us first again look in to this picture, ok. So, at time t equal to 0, our body occupies the initial configuration with volume B 0 bounded by surface del B 0. And then let P be a point whose material vector is X, ok. Let us considered two more points Q 1 and Q 2 and their relative position vector with respect to P is d X 1 and d X 2 respectively, ok.

So, once forces are applied, the body undergoes deformation and then. So, this is the deformation mapping the body, at time t that is in the current configuration has occupied this position where the volume is P and the surface is del B. So, the current position of point p is the small p, the current position vector of point p is small x and the current relative position vectors of point q 1 and q 2 with respect to point p is given by d x 1 and d x 2. So, this is a general setup that we have.

And now, what we do is, we consider the scalar product of this elemental material vectors d X 1 and d X 2 as they deform to spatial vectors d x 1 and d x 2 respectively. So, what would happen? As the material vectors deform, they will undergo what is called stretching; which means there will be change in length of the vectors.

So, P Q 1, P Q 2 these two vectors will stretch; they may elongate, they may compress, but they will be, there may be change in length. And also the angle ok, this angle between the two vectors say theta; theta 0, let us say theta 0. This angle between the two vectors P Q 1 and P Q 2 will also change, ok. So, I have shown here theta 0 changes to theta, ok.

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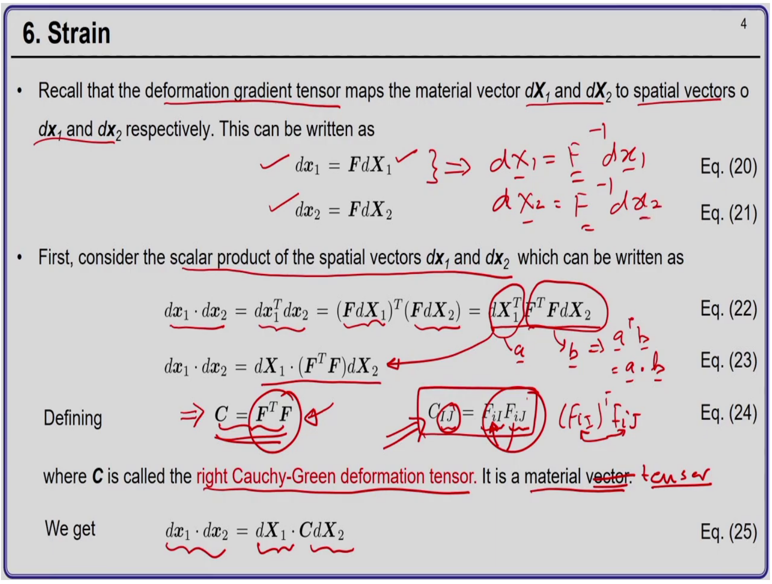

Now, we can recall that the deformation gradient tensor which is a F maps the material vectors d X 1 and d X 2 to spatial vectors small d x 1 and small d x 2, ok. So, mathematically you can write the spatial vectors d x 1 is F times material vector d X 1. Similarly, the spatial vector d x 2 will be polar, I mean deformation gradient tensor F times material vector d X 2. So, now we consider the scalar product of the spatial vectors d x 1 and d x 2 which is shown here, d x 1 dot d x 2.

Now, we want to see, how this scalar product changes and can we express this scalar product in terms of the deformation gradient tensor and the material vectors? So, now, d x 1 dot d x 2 ok, because both are vectors. So, a dot b can also be written as a transpose b. So, we can write d x 1 dot d x 2 as d x 1 transpose d x 2, ok.

Now, using equation 20 and 21; so, from equation 20, I can substitute for d x 1 which is F d X 1 and then from 21, I can substitute d x 2 which is F d X 2. So, now, I have F d X 1 transpose F d X 2. So, I can open up the brackets and I can write d X 1 transpose F transpose F d X 2. Now, this d X 1 is a vector and then F transpose F d X 2 again is a vector ok; because F transpose F will be another second order tensor. So, it operates on vector d X 2 to the give another vector. So, I can write, if I considered this as vector a and this is vector b, so this is like a transpose b. So, I can write this is a dot b. So, I can write this expression as d X 1 dot F transpose F d X 2.

Now, I can define a tensor C. I can define as a tensor C as F transpose F and this tensor C is called right Cauchy-Green deformation tensor, ok. It is a very important measure of deformation. And this is called the right Cauchy-Green deformation tensor. And this tensor is totally a material vector ok, sorry it is a material tensor ok; it is a material tensor; because it totally resides in the material configuration or the initial configuration, I can.

So, this is the direct notation ok; this equation is the direct notation, I can express the same in indicial notation, ok. So, this is the indicial representation of the equation, direct notation for right Cauchy-Green deformation tensor, ok. So, we have C capital I J, ok. So, that is the convention we follow that, indices pertaining to initial configuration are written in upper case and indices pertaining to spatial configuration or the current configuration are written in lower case, ok. So, F transpose is nothing, but F i comma I, ok.

And then F is F i comma J. So, you can see i is common, but the i in the first F is on the left hand side and i in the second also is on the left hand side. So, I can write this as F i I transpose F i J, which will make i in the first F go to the right hand side, and therefore this notation over here is same as this expression over here in the direct notation. So, this is the indicial representation of the direct notation which is given by equation 24. So, right Cauchy Green tensor is a material tensor; you see both the indices are in capital, which means it is resides totally in the material configuration or the initial configuration.

So, if you substitute C is F transpose F in equation number 23, what you will get? d x 1 dot d x 2 that the scalar product of the spatial vectors is equal to material vector d X 1 dot C material vector d X 2.

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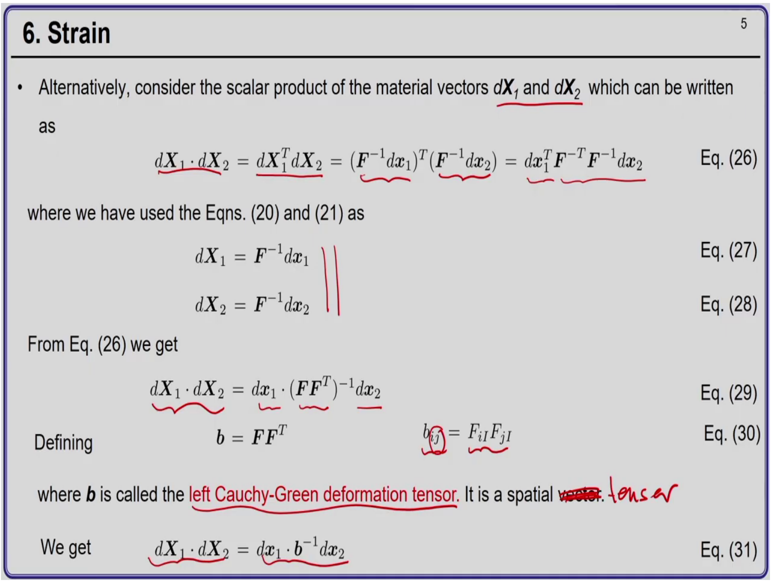

So, alternatively I can go the other way around ok, I can consider the scalar product of the material vectors d X 1 and d X 2, ok. So, I can write d X 1 dot d X 2 and because both are vector, I can write d X 1 transpose d X 2.

Now, I can invert the relation for d x equal to F d x. So, I can write the material vector in terms of the deformation gradient times inverse of deformation gradient times the spatial vector ok, which is given here, ok. So, this you can derive using this relation; you can take the inverse of this, inverse of this will be this F inverse d x 1. Similarly, d X 2 will be F inverse d x 2. So, I can substitute for the material vectors in terms of spatial vectors and I can get d x 1 transpose F inverse transpose F inverse d x 2.

So, we have use equation 20 and 21 and therefore, we get the dot product of the or the scalar product of the material vectors as d x 1 dot F F transpose inverse d x 2, ok. Now I can define another tensor b ok; the symbol b as F F transpose, and this tensor b is called the left Cauchy Green deformation tensor, ok. So, capital C was called the right Cauchy Green deformation tensor and b is called the left Cauchy green deformation tensor, and this tensor is totally a spatial tensor. So, in indicial notation I can write b i j and both the indices are in lower case; which means b resides in the current configuration, b i j is f i I f j I.

So, now, if I substitute equation 30 in equation number 29, I can write the scalar product of the material vectors d x and d X 1 and d X 2 as d x 1 dot b inverse d x 2, ok. So, now, I can express the scalar product of the spatial vectors in terms of the material vectors or I can express the scalar product of the material vectors in terms of the spatial vectors involving F, using F I can do that, ok.

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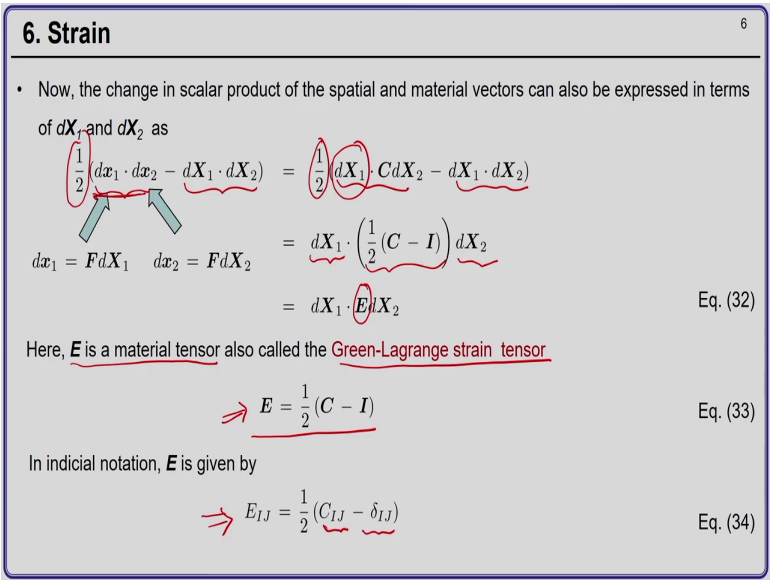

Now, let us consider what happens to the change in the scalar product of spatial and material vectors. Till now we have computed the scalar product of the spatial vectors and the material vectors separately. Now, let us consider what happens to the change.

So, what I will do? We will take the difference of the scalar product of the spatial vectors and the material vectors and we have put 1 by 2, ok. So, the reason for this will be clear when we go to constative relations ok; but right now this consider that, there is a factor 1 by 2 which is multiplied on both the sides, ok. So, it does not change anything, it just have made an factor 1 by 2; there is a reason for this, we will come to it later.

So, d x 1 dot d x 2 was? So, first term was d X 1 dot C d X 2 minus d X 1 dot d X 2. Now, I can take d X 1 out of the bracket from the left hand side that is what happens, and I can take out d X 2 outside the bracket from the right hand side, ok. So, this is what is there. So, what is left in the bracket? 1 by 2 C minus I. So, 1 by 2 C minus I is what remains. And then we write 1 by 2 C minus I as another tensor E capital E,

So, this capital E is a material tensor, it is called the Green Lagrange strain tensor, ok. So, capital E is called the Green Lagrange strain tensor and it is given by this relation, E equal to 1 by 2 C minus I, ok. So, in indicial notation, I can write E I J is 1 by 2 C I J minus delta I J. So, delta I J is a chronical delta in the material configuration.

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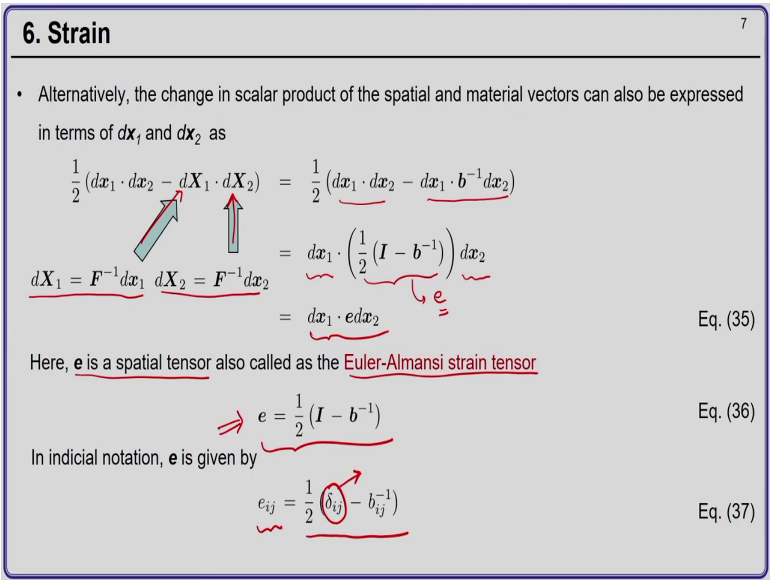

Now, I can go in the other way. So, in the previous slide, we are substituted the spatial vectors in terms of material vectors; I can do the other way around, I can go and substitute the material vectors in terms of spatial vectors. So, d X 1 is F inverse d x 1 and d X 2 is F inverse d x 2; this I can substitute here and then what I can get is after little simplification, what I can get is 1 by 2 d x 1 dot d x 2 minus d x 1 dot b inverse d x 2. So, I can take d x 1 out of the bracket from the left hand side and I can take d x 2 out of the bracket from the right hand side. So, this is what we get d x 1 dot 1 by 2 I minus b inverse into d x 2.

So, next we denote 1 by 2 I minus b inverse as another tensor e. So, lower case e. So, remember Green Lagrange strain tensor was a upper case E, because it was a material tensor, and our convention is all the material tensors will be written in upper case, ok. And all the tensors in the spatial configuration will be written in lower case. So, we have written e, small e here. So, the change in the scalar product of the spatial and material vectors is now expressed as d x 1 dot e d x 2 that is equation 35.

So, here e is a spatial tensor; spatial tensor means, it resides in the current configuration and it is called the Euler Almansi strain tensor, ok. It is called the Euler Almansi strain tensor and it is given by e equal to 1 by 2 I minus b inverse, ok. In indicial notation the same expression can be written as, e i j, e equal to b 1 by 2 delta i j minus b i j inverse, ok. Here delta i j is the chronical delta in the spatial configuration.

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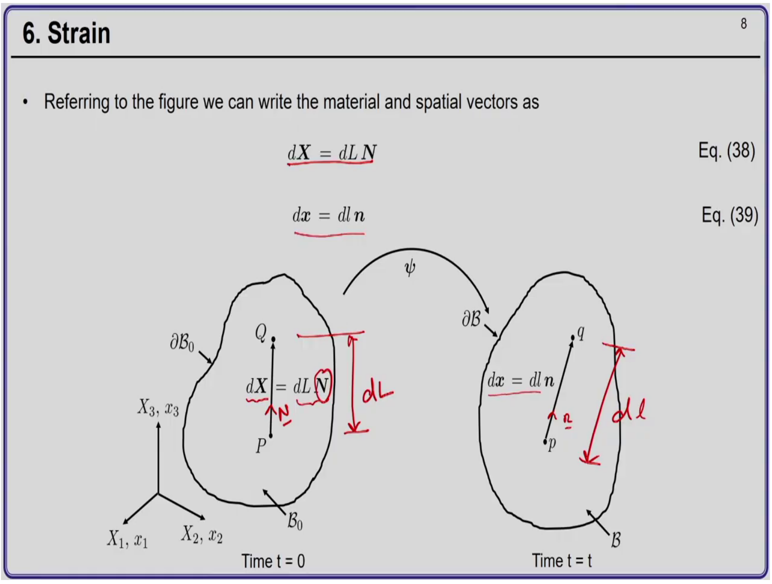

Now, let us consider a special case where d x 1 same as d x 2 is equal to d x, ok. So, in the figure you see we have consider a material vector d X going from P to Q. And let this material vector be written as d L, which is nothing but the length of the material vector times N. What is N? N is the a unit vector along the material vector P Q, ok. So, N is the unit vector along P Q. So, as you know a vector is nothing but its magnitude times the unit vector in that particular direction. So, I can write the material vector d X as d L into N ok, N is a unit vector in that direction.

Similarly, in the current configuration ok; the material vector p q has deformed and has become spatial vector p q, let the length be d l, ok. In that case, the spatial vector d x is written as the length of the vector times the unit vector along p q. So, I can write the spatial vector d x as the length of the vector times the unit vector n.

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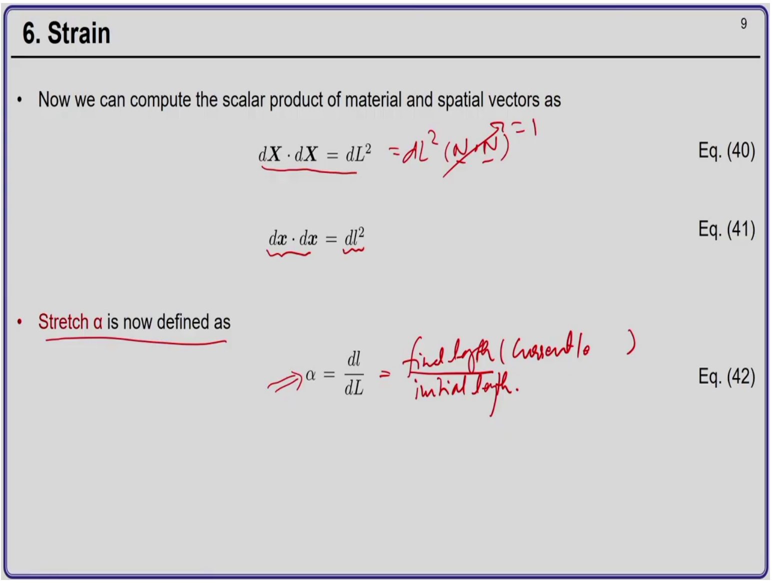

So, once we have this and because I say we can consider d x 1 equal to d x 2 as a special case equal to d x. So, what happens to the scalar product of the material and spatial vectors? So, d x 1 dot d x 2 becomes d X dot d X, ok. And then d X dot d X becomes d L square N dot N. And because N is a unit vector, N dot N is equal to 1. So, what we get? d x dot d x is nothing, but d l square. Similarly, this dot product of the spatial vectors d x dot d x is becomes d small l square,

Now, we can define a quantity called stretch. So, at this point I will introduce this stretch and we will use the symbol alpha. So, stretch is defined as the final length divided by initial length, or final length I can also called current length. So, current length divided by initial length, ok. So, that is the definition of stretch, ok. So, whenever I use the word stretch, you can remember that it is the ratio of current length to initial length.

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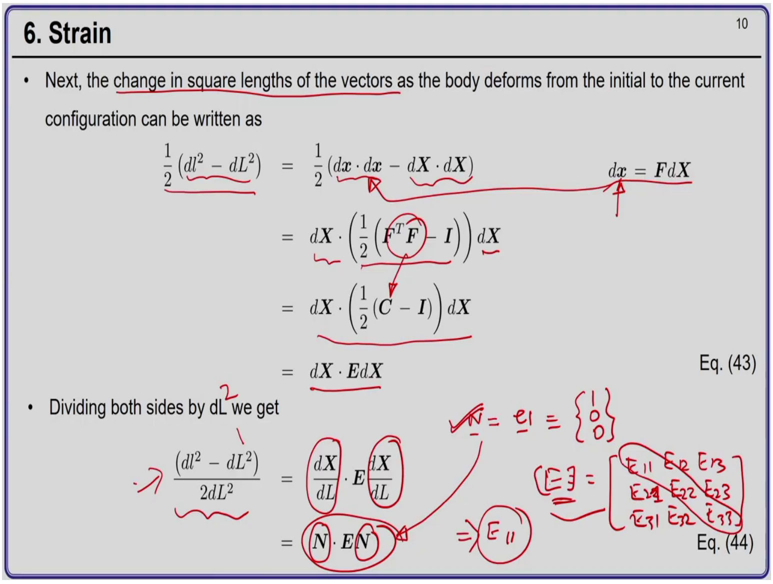

So, now the change in square length of the vectors is what we consider, ok. As body deforms, what happens to the change in the square length, ok. So, we have to compute d l square minus d capital L square and this 1 by 2 is written. So, this is nothing, but 1 by 2. So, d l square is nothing, but d x dot d x; and d capital L square is nothing, but d X dot d X, because d x is F d X now.

So, I can substitute material vector d x as in terms of F d X here ok; I can substitute it here and I can get d X dot 1 by 2 F transpose F minus I into d X, ok. Now, I can recognize this term over here; F transpose F is nothing, but right Cauchy Green tensor C, ok. So, now, I have d X dot 1 by 2 C minus I d X, ok. And 1 by 2 C minus I had define in the previous slides as the Green Lagrange strain tensor E, ok. So, our expression 1 by 2 d l square minus d capital L square reduces to d X dot E d X.

Now, if I divide both sides ok, if I divide both sides by capital d L square ok; capital d L square if I divide on both the sides what I get, d l square minus d capital L square divided by 2 d L square is d X by d L dot E d X by d L. And what is d X by d L? Ok. So, d X is the vector, d L is the length. So, vector divided by its length gives you the unit vector along the direction of the vector N. So, we get N dot E N.

So, now if you remember in one of the previous examples when we were linearizing different strain measures, the third example was Green Lagrange strain ok, 1 D original of Green Lagrange strain. So, now, we can derive that here. So, if I take N has e 1. If I take N as e 1, so N is a unit vector along the x 1 direction e 1 ok; so which in vector notation is 1, 0, 0, it is in the x 1 direction. And if I substitute it here ok, what I what do I get? I get only E, I will get E 1 1, ok. You can do it yourself ok, because E is a second order tensor and if I write that in matrix form; it will be E 1 1, E 1 2, E 1 3, E 2 2, E 2 1, sorry E 2 2, E 2 3, E 3 1, E 3 2, E 3 3.

So, N is 1, 0, 0 and E is this quantity over here. So, if I take the dot, if I substitute everything here; I will get the resultant E 1 1 and that was the expression for Green Lagrange strain in one dimension. So, if you remember this was the expression that we used.

So, you can take different N's and you can get the physical meaning of the different terms of the Green Lagrange strain tensor, ok. So, like if you take N as E 2, you will get d l square by d capital L square by 2 d L square as E 2 2. So, what it means is, like E 1 1 equal to this expression means is; E 1 1 shows, originally if you had a material element vector lying along the E x 1 direction, whose length was d capital L. And now after deformation it becomes d small l, but along the same direction.

Then the change in this square length divided by the original length square will be nothing, but the first component of the Green Lagrange strain tensor which is nothing, but E 1 1, ok. So, E 2 2 will be, you take a material vector along the two direction and after deformation it becomes some other vector. So, the change in square length of the vector after deformation to any before deformation divided by twice of the square of the length in the initial configuration is nothing, but will be E 2 2, ok. Like this we can get the physical meaning of the diagonal components of E. We will come to off diagonal term later.

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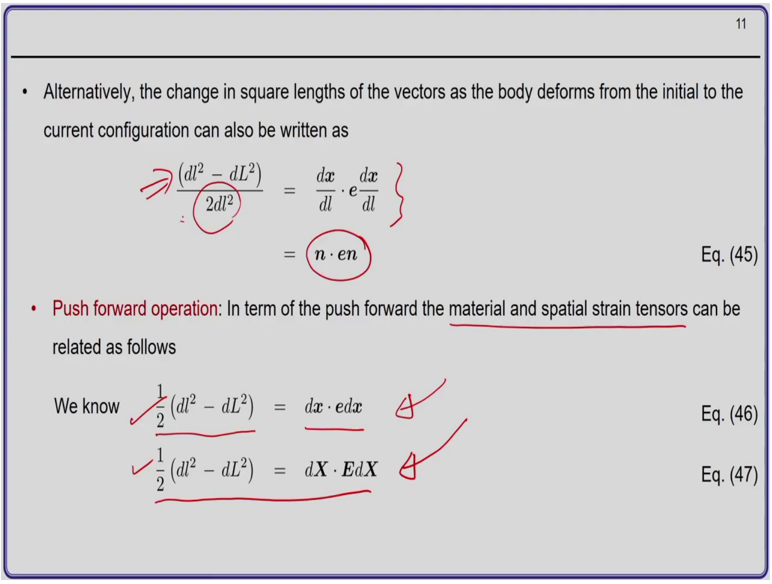

Similarly, instead of in the previous expression where we started, we substituted the material vector, spatial vector in terms of material vector. I can go the other way around, I can replace the material vector in terms of spatial vector and I can derive the relation of change in square length of the vector divided by twice of d l square. So, d l is the current length as nothing, but n dot e n, ok. So, e is the Euler Almansi strain tensor and n is nothing, but the unit vector in this spatial configuration along the vector direction.

So, in terms of push forward operation, now I can do the push forward and pull back operations. So, in terms of push forward operation ok, the material and spatial tensors can be related. How we can do that? So, you notice that 1 by 2 d l square minus d capital L square is nothing, but d x dot e d x ok, this is derived just here. And 1 by 2 d l square minus d capital L square can also be written as d X dot E d X, ok. So, the change in square length either you can express in terms of spatial quantities or you can express in terms of material quantities.

And because the left hand side of both the equations are same; that means, the right hand side of both the quantity should be same.

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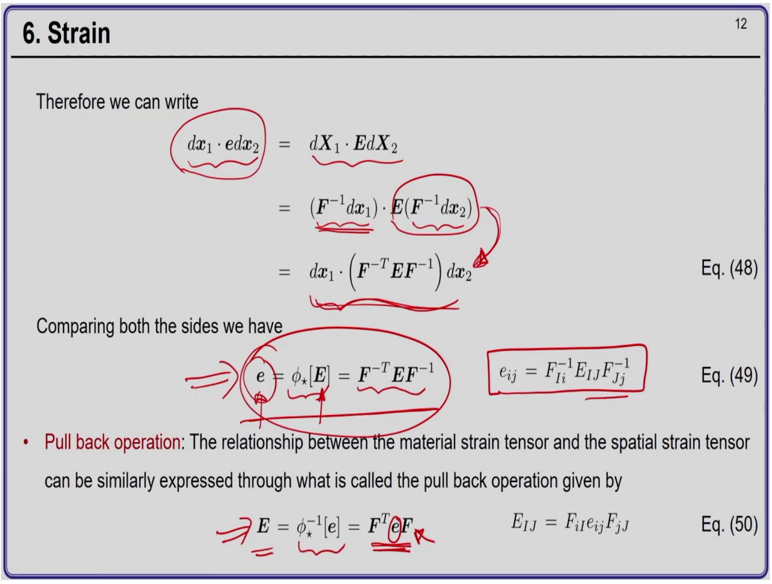

That means I can write d x 1 dot d x 2, ok. Now, for a special case ok, I will just considered two different vectors; instead of same vector d x 1 d x 2 equal to d x, I will consider two vectors. So, instead of d x, I will write the d x 1; so, d x 1 dot e d x 2 will be same as d X 1 dot E d X 2.

And now d x 1 is nothing, but F inverse d x 1 and d x 2 is nothing, but F inverse d x 2. So, I can write this expression as d x 1 dot F inverse transpose E F inverse d x 2. So, going from this step to this step, we requires one additional step that I request you work it out yourself. Noting that in the middle expression, the term on the left hand side of the dot is basically a vector. So, and on the right hand side is also you have a vector. So, a dot b is a transpose b and you can thus simply play between the vectors and you can derive this particular relation, ok.

So, we see that the left hand side we have d x 1 dot e d x 2 and the right hand side we have d x 1 dot F inverse transpose E F inverse d x 2, ok. So, both sides are same. So, I can interpret Euler Almansi strain tensor as a push forward or of the Green Lagrange strain tensor, ok. Remember phi star and square bracket E meant it is the push forward operation of E,

Now, pushing forward e or the Green Lagrange strain tensor gives you the Euler Almansi strain tensor. And the way this push forward is actually done is given by F inverse transpose E F inverse. So, this notation in the direct notation can be written in indicial notation using following expression.

So, e i j is nothing, but F I i inverse E I J F J j inverse, ok. So, that is the indicial representation of the direct notation given here. So, that is the push forward operation. So, the push forward of the Green Lagrange strain tensor gives you the Euler Almansi strain tensor.

Similarly, you can do the pullback operation; instead of d x 1 dot d x 2, I can start the other way around or I can just start from equation number 49. I can start from this expression and I can find Green Lagrange strain tensor in terms of Euler Almansi strain tensor using following expression, ok.

So, this is phi star inverse e; this means the pullback of the Euler Almansi strain tensor will give you the Green Lagrange strain tensor and this is the way this operation will carried out is F transpose e F, ok. So, if you have Euler Almansi strain tensor and you want to pull it back to the initial configuration, where you will get the Green Lagrange strain tensors; what you have to do? You have to pre multiply Euler Almansi strain tensor by F transpose and post multiply it by F, ok. So, F is present.

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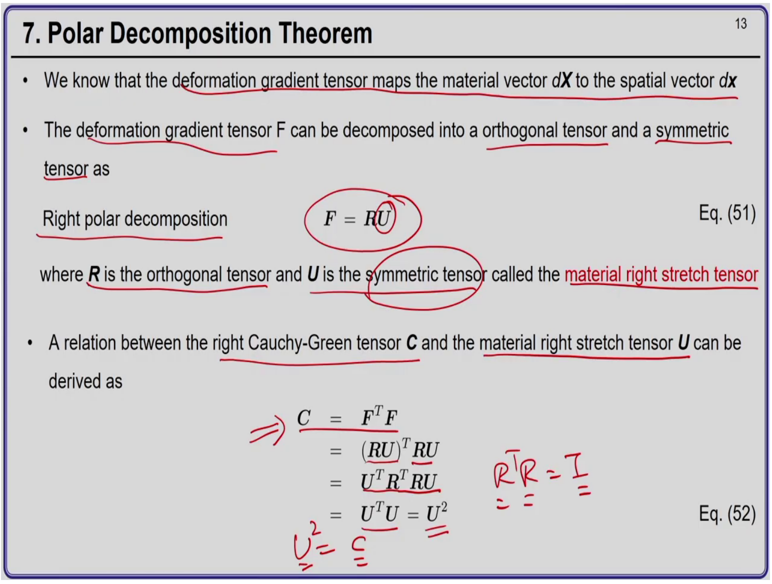

Now, we come to the next topic. So, the physical meaning of all the strain measures will actually see in the next class, where we will do some examples. So, the polar decomposition theorem. So, polar decomposition theorem before we start, you remember that the deformation gradient tensor maps the material vector to the spatial vector d x, ok. So, you had a material vector and then F operated on that material vector to give you the spatial vector d x.

Now, when we were discussing tensors, we discuss that any tensor can be broken into a orthogonal tensor and a symmetric tensor ok, that is what called the polar decomposition theorem. So, in our case the deformation gradient tensor F can be decomposed into an orthogonal tensor and a symmetric tensor as F equal to R into U. And this is called the right polar decomposition; because U which is the symmetric tensor lies on the right hand side of the orthogonal tensor R that is why it is called the right polar decomposition, ok.

So, R is the orthogonal tensor and U is the symmetric tensor and it is also called the material right stretch tensor. So, why the word stretch is here? Because the stretch we have already defined, is it current length divided by original length. So, there is certain relation between stretch and U. So, we will come to it later that is why it is called the material right stretch tensor.

So, we can derive a relation between the right Cauchy Green tensor and the material right stretch tensor U, ok. Remember that F is C is F transpose F; now F from equation 51 is R U. So, if I substitute F is R U, I get R U transpose R U, ok. Now R U transpose is U transpose R transpose R U. Now R transpose R, R is a orthogonal tensor; therefore R transpose R is nothing, but identity, ok. R transpose R R is identity; therefore we have U transpose R transpose R U becomes U transpose U.

And U transpose is same as U, because U is a symmetric tensor; therefore U transpose is U. So, you have U dot U into U which is nothing, but U square. So, the relation that we get is, U square is equal to C or the square of the material right stretch tensor is equal to the right Cauchy Green deformation tensor C ok, that is how we can relate.

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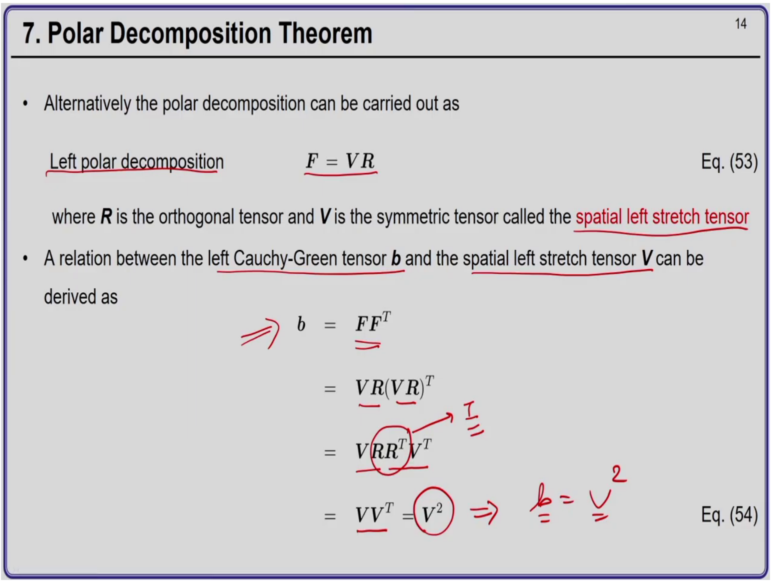

Now, we can alternatively do what is called the left polar decomposition. Here F can be decomposed as a product of a symmetric tensor V times an orthogonal tensor R. And this is called the left polar decomposition; because V, the symmetric tensor lies on the left hand side of R, ok.

And V is also called the spatial left stretch tensor. So, the comprehends of both the material right stretch tensor and the spatial left stretch tensor has some physical meaning. We will come to it, but right now this consider that the word, it has the word stretch into it.

As we had done in the previous slide, we can find a relation between the left Cauchy Green tensor b and the left stretch tensor V, ok. So, we know that b is F F transpose; an F from equation 53 is V R. So, if you substitute F as V R here, we write V R into V R transpose, ok.

So, V R transpose is nothing, but R transpose V transpose and we have V R. And R R transpose is identity; because R is a orthogonal tensor, ok. So, we get V V transpose and V is a symmetric tensor; therefore V transpose is V and therefore we can write V square, therefore finally we derive b is equal to V square, ok. So, the left Cauchy Green tensor is square of the spatial stretch tensor b, left stretch tensor V.

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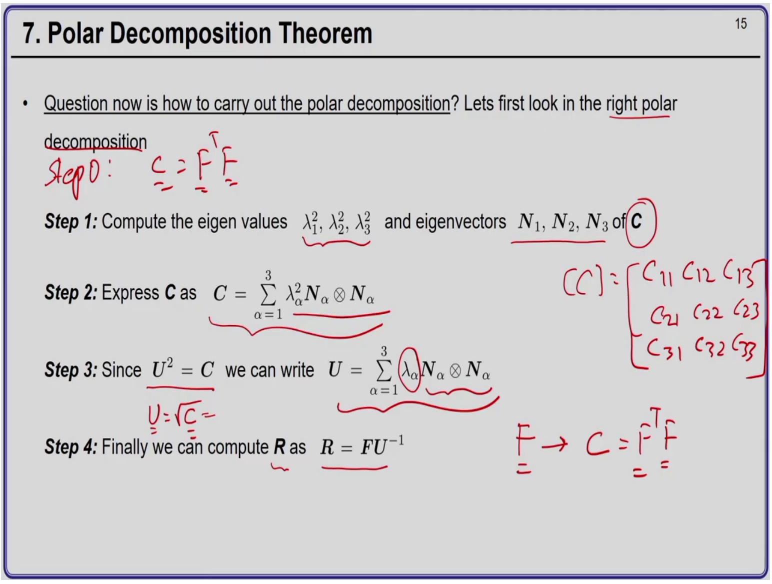

Now, the question comes; when we are doing a finite element computation, you need to carry out this polar decomposition of the deformation gradient tensor, ok. So, you need to determine what is U; because U contains stretch. So, how do we do this polar decomposition? So, first we look into how we carry out the right polar decomposition.

So, the first step is let us say, we solve the eigenvalue problem ok, where we find the eigenvalues and eigenvectors of the right Cauchy-Green tensor C. So, let lambda 1 square, lambda 2 square, lambda 3 square be the 3 eigenvalues of C, ok. Remember we have not written lambda 1, lambda 2, lambda 3; but rather we are writing lambda 1 square, lambda 2 square, lambda 3 square, there is a square which is present.

And let N 1, N 2, N 3 be the eigenvectors of C, ok. So, in the first step we find the an eigenvectors of C. So, that is done by writing C in matrix notation; the C is C 1 1, C 1 2, C 1 3, C 2 1, C 2 2, C 2 3, C 3 1, C 3 2, C 3 3. So, once you have this 3 by 3 matrix, you can solve for the eigenvalues as a eigenvectors of this matrix which is nothing, but the right Cauchy-Green tensor,

So, once you have found out the eigenvalues eigenvectors using spectral decomposition theorem; I can express C as the linear combination of the eigenvalues times the tensor product of the eigenvectors, which is summation over alpha, alpha goes from 1 to 3 lambda alpha square N alpha tensor product N alpha, ok. This is nothing, but lambda 1 square N 1 tensor product N 1 plus lambda 2 square N 2 tensor product N 2 plus lambda 3 square tensor product N 3 tensor product N 3, ok.

So, now I have determined the eigenvalues eigenvectors. Now, I know the relation U square is C. So, U will be nothing, but square root of C. So, from your linear algebra, you would recall that to find the square root of a matrix; what we have to do? We have to find the eigenvalues and eigenvectors and then we take the square root of the eigenvectors and use these spectral decomposition; because eigenvectors do not change.

So, eigenvalues we take the square root and use the spectral decomposition to reconstruct the square root of the matrix. So, here we have to take the square root of C; which means I take the square root of the eigenvalue. So, eigenvalue of C was lambda square, so it becomes lambda, ok. So, I have lambda and then the eigenvectors remain same. So, I have N alpha tensor product N alpha. So, U will be nothing, but lambda alpha N alpha tensor product N alpha ok, summation over 1 to 3, ok.

So, this will give you, step 3 will give you the right stretch tensor U, ok. So, once you have U and you already have F; F is given to you, that is why you are doing the polar decomposition, ok. So, from F what we do, we first calculate C, ok. So, C is F transpose F. So, here I should have written one more step, say step 0 we can write; imagine C as F transpose F. My F is already there with U; you are having step 3, you have computed U.

Now, the orthogonal tensor R will be nothing, but R equal to F U inverse, ok. So, take you take the inverse of U and then U, pre multiply U inverse with F to get R ok. So, R can then be obtain, ok. Why we the need to do this kind of polar decomposition, will be clear later.