Computational Continuum Mechanics
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Lecture – 10
Worked Examples – 2

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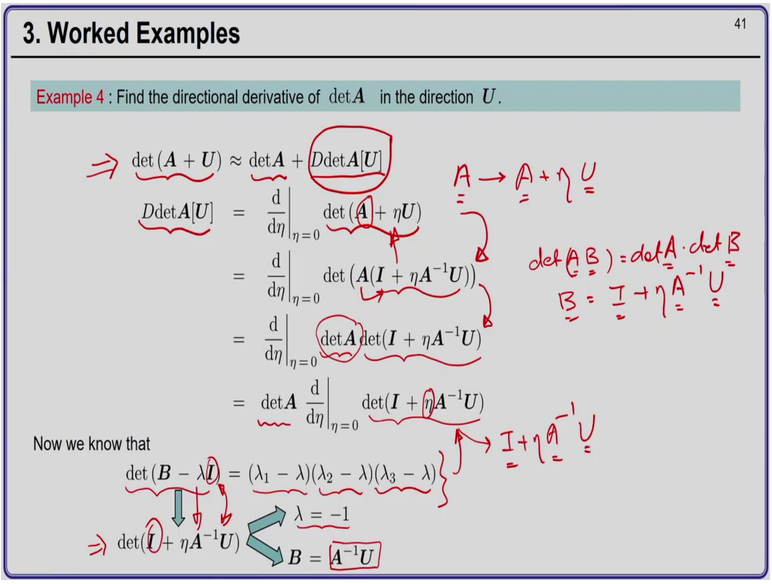

So, the next example that we consider is to find out the directional derivative of the determinant of A where, A is a second order tensor in the direction U which means when the tensor A is incremented by U what is the change in the determinant of A. So, as you can see here we can write the determinant of A plus U as determinant of A plus the directional derivative of A in the direction U ok.

So, this directional derivative is what is the change in the value of the determinant when A is incremented by U. So, we start with our usual definition of finding the directional derivative which is the directional derivative of determinant of A in the direction of U is nothing, but d by d eta of determinant of A plus eta U. So, what we do? We replace A by A plus eta U.

So, now, unlike for scalar quantities finding out the derivative with respect to eta is not that straightforward. So, the way we approach is in the next step what we do is we take out this A outside the bracket ok. So, what it would become is, A times I plus eta A inverse U. You can check for yourself. If you take A inside you will get back this quantity over here.

So, now we will invoke the property that determinant of A into B is nothing, but determinant of A into determinant of B. For us B here is I plus eta A inverse U ok. So, next step when we invoke this relation we can write determinant of A into determinant of I plus eta A inverse U ok.

Now, determinant of A is independent of eta. So, it can be taken outside the bracket, only the second term so, the second determinant is dependent on eta ok. So, we have d by d eta of determinant of I plus eta A inverse U evaluated at eta equal to 0 ok.

Next from the characteristic equation for second order tensor we know that determinant of B minus lambda I, where lambda is the eigenvalue can be written as lambda 1 minus lambda into lambda 2 minus lambda into lambda 3 minus lambda, where lambda 1, lambda 2, lambda 3 are the eigenvalues corresponding to B ok. So, if we compare this relation with determinant of I plus eta A inverse U ok.

So, if we compare these two expressions, we can see that eta is or lambda is minus 1 because the coefficient of I here and the coefficient of I here you can check lambda is equal to minus 1 and B here is A inverse U is same as A inverse U ok. So, let lambda 1, lambda 2, lambda 3 be the eigenvalues of I plus eta A inverse U.

So, we can substitute this expression here with the understanding that lambda 1, lambda 2, lambda 3 are the eigenvalues of I plus eta A inverse U ok.

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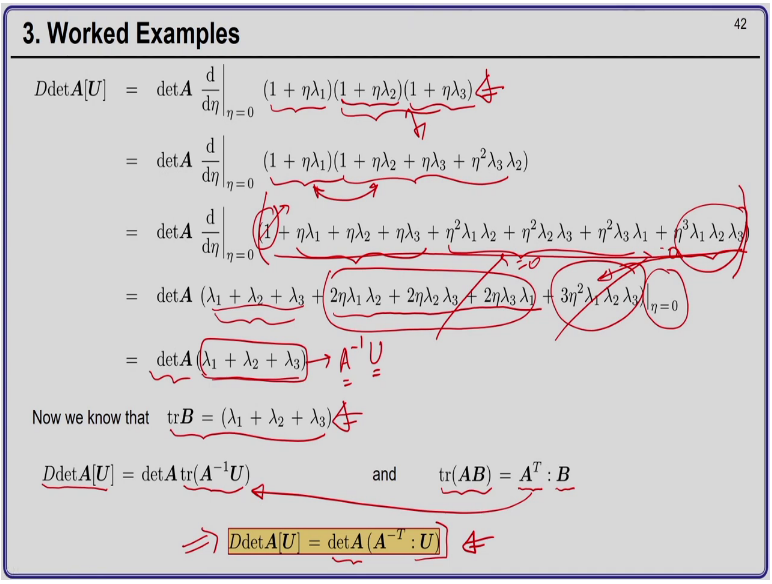

So, this is what we get determinant of A into d by d eta 1 plus eta lambda 1 plus into 1 plus lambda eta 2 into 1 plus eta lambda 3 ok, where eta is the multiplier ok. So, eta was here ok. So, the eigenvalues of a matrix multiplied by a scalar will of a matrix multiplied by a scalar would be the scalar multiplied by the eigenvalues ok. So, the eigenvalues of eta A inverse U will be eta lambda 1 eta lambda 2 eta lambda 3 and that is what we have it here ok.

So, now, we can open up the bracket, you open up the first two brackets and this is what you are going to get ok. So, you open up first two brackets, you get this quantity and then you multiply this bracket with this bracket you end up with this expression 1 plus. So, you have a constant term, you have a linear term in eta, you have a quadratic term in eta and you have a cubic term in eta ok.

So, once you have this the next step is you take the derivative of this quantity in the bracket with respect to eta and this is what you get the constant term does not have any eta. So, it goes away ok.

The constant term goes away, the linear term all the etas go away. So, you are left with lambda 1 plus lambda 2 plus lambda 3; for the quadratic terms you are left with 2 eta lambda 1 plus 2 eta lambda 2 lambda 3 plus 2 eta lambda 3 lambda 1 and for the last term you have 3 eta square lambda 1 lambda 2 lambda 3 and this you have to evaluate at eta equal to 0.

And, you substitute eta equal to 0 this term goes to 0 and this term goes to 0. So, these two last two terms drop off and you only ok. So, last two you know I mean the last four terms drop off and only the first three terms remain which is nothing, but lambda 1 plus lambda 2 plus lambda 3.

So, the directional derivative of determinant of A in the direction U is nothing, but determinant of A multiplied by lambda 1 plus lambda 2 plus lambda 3; remember, lambda 1 lambda 2 lambda 3 are nothing, but the eigenvalues of the matrix A inverse U.

So, we can simplify it bit little bit more. So, you would remember from our previous discussions that trace of a matrix can be written as sum of its eigenvalue. So, now, we also have the sum of eigenvalues here and these are eigenvalues corresponding to the matrix A inverse U ok. So, using this property we can write the term the sum of the eigenvalues that particular term as trace of the tensor A inverse U ok.

So, now we have reduced the directional derivative of A in the direction U as determinant of A into trace of A inverse U. So, this can be simplified further if you remember the property of trace which is the trace of A into B where and B are two tensors is nothing, but A transpose double contraction with B ok, this is the property ok.

So, once you remember this property we can use it here. We can use it here to further reduce this expression the right hand side to determinant of A into A inverse transpose contracted with U ok.

So, finally, this is what we have derived that the determinant of A the change in the value of determinant of A when A gets incremented by U is nothing, but determinant of A into A inverse transpose double contracted with U ok. So, this is a very important derivation that we have done and this will be used later on for our constitutive relations ok. One of the examples that we are going to discuss next also has this derivation this result used ok.

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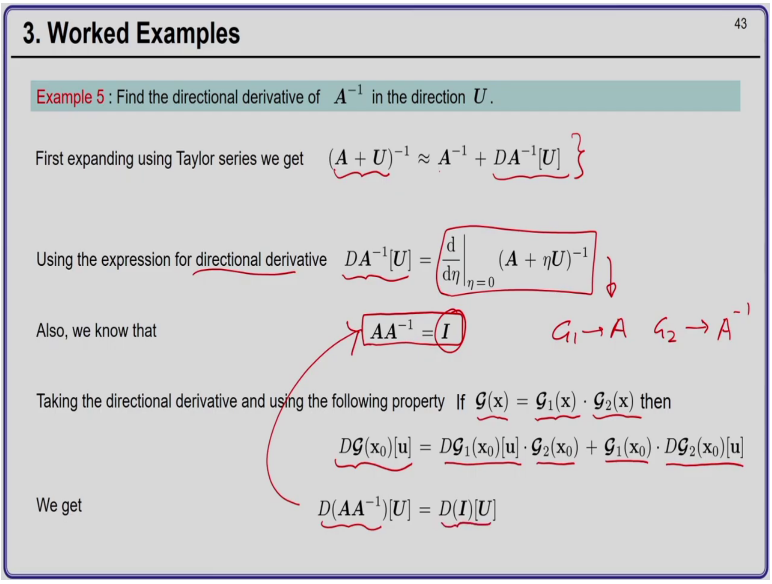

So, once we have done this we move to our next example, where we want to find out the directional derivative of inverse of a tensor in the direction of U ok. So, what we want to say is if A gets incremented by U, what is the change in the value of A inverse how much change is there. Again, first you can do the Taylor series expansion. So, A plus U inverse will be nothing, but A inverse plus the directional derivative of A inverse in the direction of U.

So, this second term is what is the change ok. Now, this directional derivative is what we now need to find out ok. So, again we can use the expression of directional derivative. So, the directional derivative of inverse of the tensor in the direction U is nothing, but d by d eta of A plus eta U the whole inverse evaluated at eta equal to 0.

Now, again this is a bit complicated to do. So, rather than proceeding directly from this point what we do is we first notice that for a invertible tensor; invertible means for that tensors for which A inverse exists we have this property A A inverse is identity tensor second order identity tensor ok. So, this is true and also the identity tensor is a constant tensor ok.

So, if you remember the property of directional derivative the second property that we discussed that if G is a function non-linear function which is a function of product of two functions G 1 G 2, then the directional derivative of G at point x 0 in the direction U was directional derivative of G 1 at x 0 in the direction U multiplied by G 2 plus G 1 multiplied by directional derivative of G 2 evaluated at x 0 in the direction U ok.

So, this property we are going to use here our G 1 is A and our G 2 is A inverse ok. So, we are going to use this property. So, what we do? We take the directional derivative on both sides of this equation in the direction U. So, we have directional derivative of A inverse in the direction U is equal to directional derivative of I in the direction U ok.

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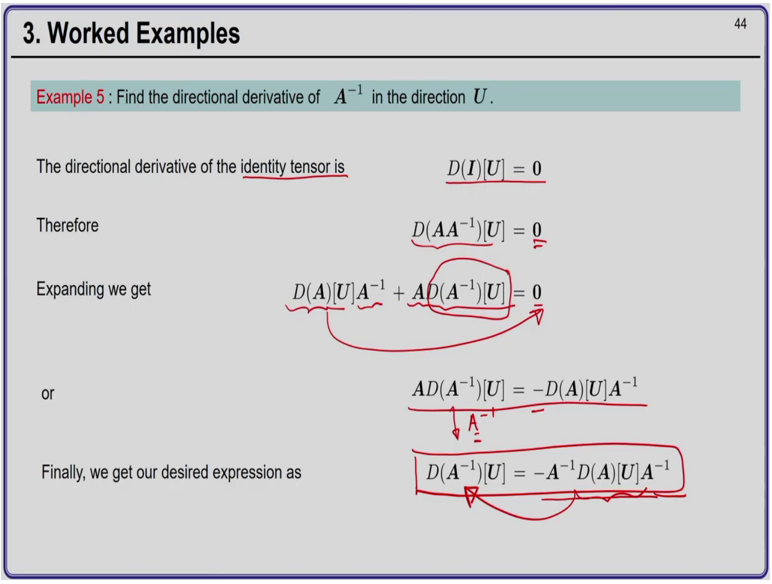

Now, the directional derivative of the identity tensor is 0 second order 0 tensor because I is a constant tensor. So, it will not change if you increment the value of A by I mean you increment the tensor A by U there is nothing going to happen for I ok. So, therefore, using this on the right hand side we get the directional derivative of AA inverse in the direction U will be equal to 0 ok.

Now, using our property that we discussed in the previous slide we will have the directional derivative of G 1 which is A here in the direction U times G 2 which is A inverse plus G 1 which is a times the directional derivative of G 2 in the direction U. So, G 2 is A inverse equal to 0 ok. So, that is our. So, if you look closely this is our directional derivative that we wanted to find out.

So, what we can do next is we can take this term on the right hand side ok. So, this is what you are going to get A directional derivative of A inverse in the direction U will be equal to minus of directional derivative of A in the evaluated at U times A inverse ok.

If you pre-multiply both sides by A inverse and using AA inverse equal to identity or A inverse A equal to identity then, so, you multiply A inverse on both the sides you will get our required expression for the directional derivative of inverse of A tensor in the direction U as nothing, but minus of A inverse directional derivative of A in the direction U multiplied by A inverse.

So, if A is a second order tensor and you increment A by U, then the change in the A inverse or the inverse of the tensor in the direction U will be nothing, but minus of A inverse. The directional derivative of tensor A in the direction U that is change of A in the direction U times A inverse ok.

So, if you could find out the directional derivative or the change in the value of A in the direction U you would be able to compute the change in the inverse of the second order tensor in the direction U.

So, that is your final expression. So, you could see here we did not follow the usual route that is d by d eta computing d by d eta and then substituting eta equal to 0, but we proceeded in a different way. So, it is not always that you proceed in that particular sense you have other means to derive the directional derivative ok. You have to be engineers some of the time.

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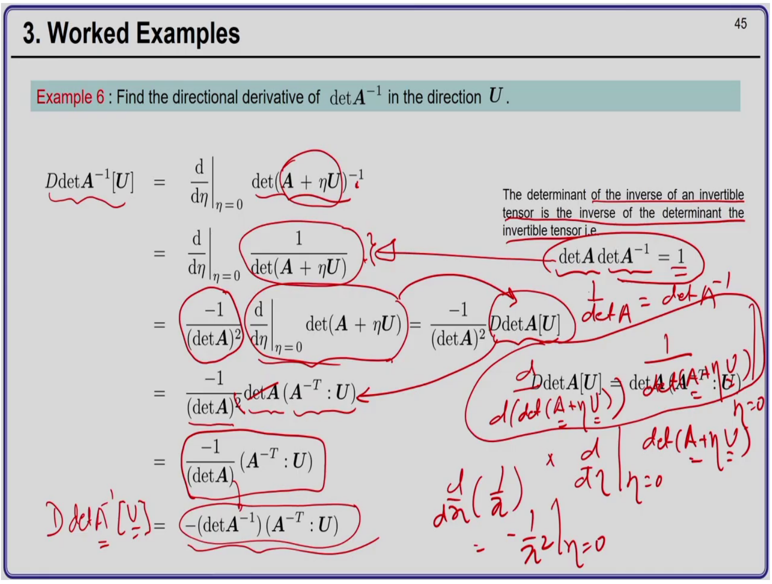

Next we come to finding the directional derivative of the inverse of a determinant of inverse of a tensor in the direction U ok. Suppose, you change A to A plus U; so, we wish to determine what will be the change in the determinant of inverse of the tensor in the direction U ok. So, as you change A its inverse also changes and therefore, it is determinant also will change. So, we wish to determine how much the inverse of the determinant of inverse of the tensor changes ok.

So, we can write the directional derivative that is we want to find out the directional derivative of determinant of A inverse in the direction U is nothing, but d by d eta evaluated at eta equal to 0 of determinant of A plus eta U the whole inverse ok.

Now, we know that the determinant of the inverse of an invertible tensor is the inverse of the determinant of the invertible tensor that is determinant of A into determinant of A inverse is 1 ok. So, you also know from the matrix theory that determinant of A matrix time determinant of the inverse of that matrix is equal to 1.

So, if we use this here so, determinant of this tensor A plus eta U inverse will be nothing, but 1 by determinant of A plus eta U we have used this relation here ok. Now, we wish to compute ok. So, now, what we do for this you can write d by d determinant of A plus eta U ok. I am into 1 by determinant of A plus eta U evaluated at eta equal to 0 times d by d eta evaluated at eta equal to 0 determinant of A plus eta U ok.

So, this term if you evaluate this term at eta equal to 0 you are going to get minus 1 determinant of A the whole square ok. So, you can just substitute determinant of A plus eta U as say x. So, this is like d by dx of 1 by x which is nothing, but minus 1 by x square x here is determinant of A plus eta U and then you have to evaluate that at eta equal to 0, when you substitute eta equal to 0 you are left with determinant of A.

So, the first term is minus 1 upon determinant a the whole square d by d eta of eta equal to 0 determinant of A plus eta U. So, determinant of A plus eta U is nothing, but directional derivative of A determinant of A in the direction U, this we have already computed ok.

So, from our previous slides we can see that this term over here is nothing, but the directional derivative of determinant of A in the direction U and since we have already computed this which was nothing, but determinant of A into A inverse transpose double contracted with U ok.

So, that is what we substitute here. So, substituting this what you get minus 1 upon determinant of A the whole square determinant of A into A inverse transpose double contracted with U ok. So, one determinant A cancels out and then you have minus 1 by determinant of A A inverse transpose double contracted with U ok. So, finally, you can and now determinant of A is nothing, but 1 by determinant of A inverse ok.

So, 1 by determinant A so, from here 1 by determinant of A will be equal to determinant of A inverse ok. So, that is what we substitute here and then we get our required expression which is the directional derivative of determinant of A inverse in the direction U is nothing, but minus of determinant of A inverse into A inverse transpose double contracted with U ok. So, this is the change in the value of determinant of inverse of the tensor when the tensor itself gets incremented by U ok.

So, when A gets incremented by U that will becomes A plus U then determinant of A inverse the change in determinant of A inverse will be given by this quantity over here.

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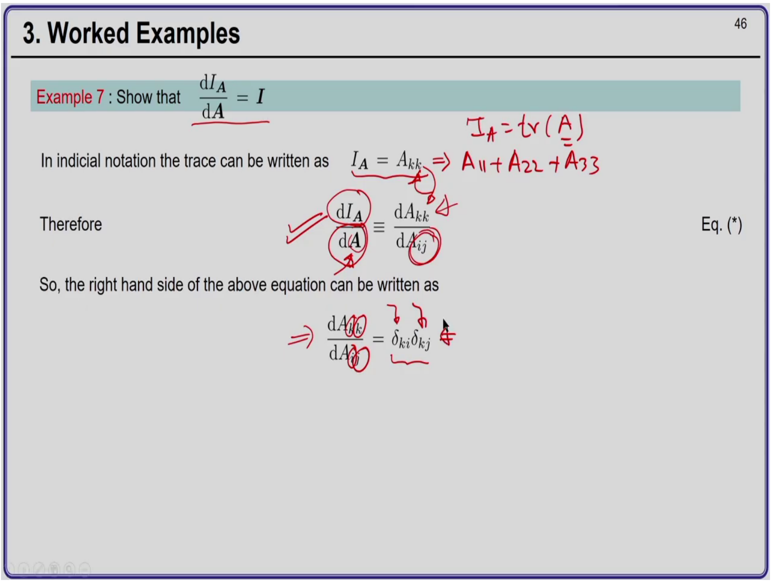

So, moving on to our next example, ok. So, the next three examples that will do will come handy when we are discussing the constitutive relation for compressible hyper elastic material in hyper elasticity it will be very handy if you remember this proofs ok. So, there instead of driving these proofs we will directly use the results for what will drive in the next three examples.

So, the first example asks us to show that the derivative of the first invariant of a second order tensor with respect to the second order tensor is a second order identity tensor ok. So, you are ask to show that d I upon d A will be equal to I which is second order identity term.

So, to proceed what we have to do is we will derive our required result using indicial notation ok. So, remember, the trace of a second order tensor the first invariant is nothing, but the trace of a second order tensor.

So, I A is nothing, but trace of A and then in indicial notation you can write that as A A kk. So, this is nothing, but A 11 plus A 22 plus A 33 ok. So, because if you remember the rule for indicial notation whenever an index is repeated twice, then a summation is implied unless until you have specifically said that the know sum rule is valid because we have not written know sum rule which means that a summation over k is already implied ok.

So, trace of a can be written as A subscript kk that is the first invariant of a second order tensor. Once we have this we can write our given expression on the left hand side the derivative of the first invariant of the second order tensor with respect to the tensor itself in equivalent indicial notation ok.

So, we can write so, I A is A kk. So, we have dA kk divided by d A ij ok. So, why ij because if you look on the left hand side the left hand side happens to be a second order tensor you have a denominator in denominator you have a second order sensor. So, it is a second order tensor. So, there will be two free indices ok. So, therefore, we will have two free indices in our term and then because they are free indices they will occur only once.

So, therefore, we should for the other indices we should have some other symbol. So, we have chosen k so, we have dA kk by d A ij ok. Now, if you remember the expression for derivative of a tensor with respect to a vector or another second order tensor you will notice that dA kk by d A ij will be nothing, but delta ki ok.

So, first index over here and first index over here gives you the first Kronecker delta delta ki and then the second index over here and the second index over here gives you the second Kronecker delta kj ok.

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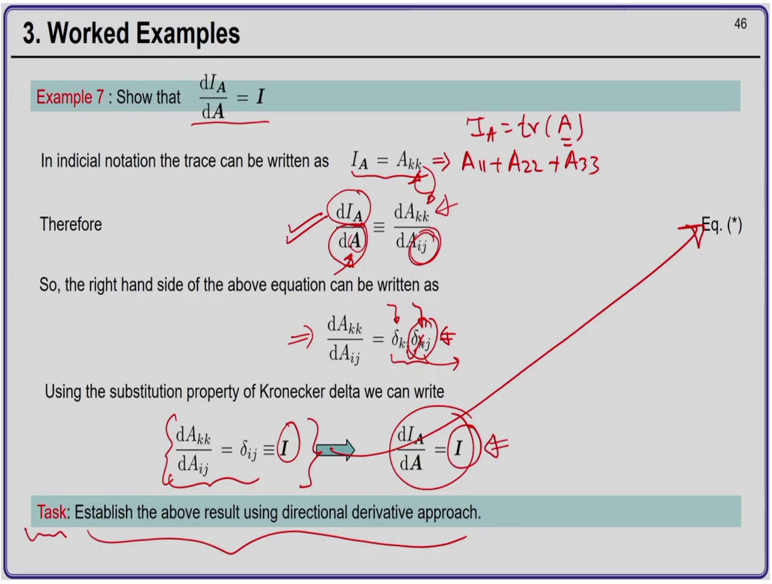

So, del A kk d A kk by d A ij is nothing, but delta k i delta k j ok. We can use the substitution property and using this substitution property delta ki delta kj. So, k is common. So, one of the Kronecker delta can be replaced say we choose the second Kronecker delta to stay and then the k in the second Kronecker delta will get replaced by this I ok. So, eventually del d A kk by d A ij is nothing, but delta ij which in direct notation is nothing, but the second order identity tensor ok.

So, if we substitute this relation in this equation equations start over here ok. We get the derivative of the in first invariant of a second order tensor with respect to the tensor itself as the second order identity tensor ok. So, that is a very useful relation which will be used later when we are discussing the hyper elasticity ok.

So, now, you can establish the above result using directional derivative approach also. So, you can start with the our usual definition of directional derivative and you can proceed ok.

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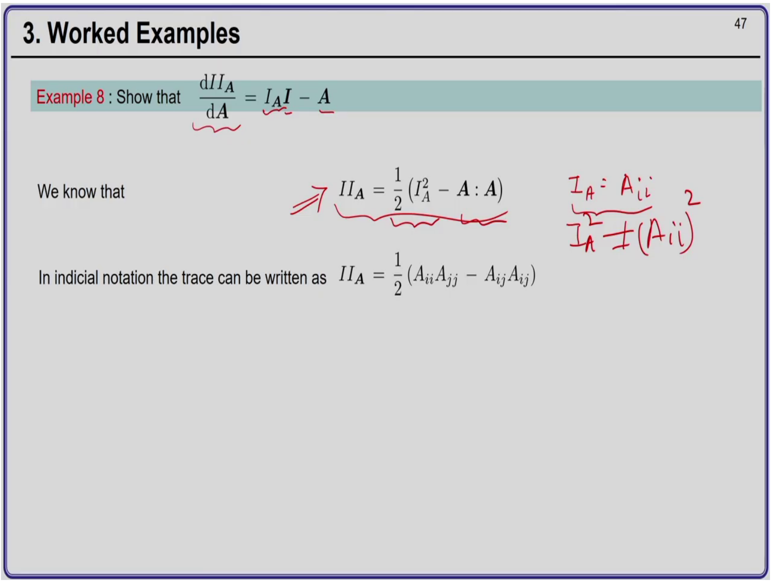

So, this is left as a task for you ok. The next example that we are going to consider we want to show that the derivative of the second invariant of a second order tensor with respect to the tensor itself is nothing, but the first invariant of the tensor multiplied by the second order identity tensor minus the tensor itself, where the second invariant of the tensor is given by following expression 1 by 2. The first invariant square minus A double contracted with A itself ok. So, that is a.

So, again we start with indicial notation ok. So, when we start with indicial notation. So, remember it is I square ok. So, I square I A was A ii ok. See on the left hand side, you do not have any index it is a scalar quantity ok. So, each of this product term or the symbol group; so, we have two symbol groups on the right hand side each of these symbol group will only have the mean indices or repeated index there will be no free index ok.

So, from our previous example we know that the first invariant of a second order tensor can be written as A ii or A kk whatever index you choose I choose specifically A ii ok, then I A square will not be A ii square ok. So, that will be wrong to write if you write A ii square ok. So, I told you that you should not write square of a tensor in indicial notation ok. So, this will not be true ok.

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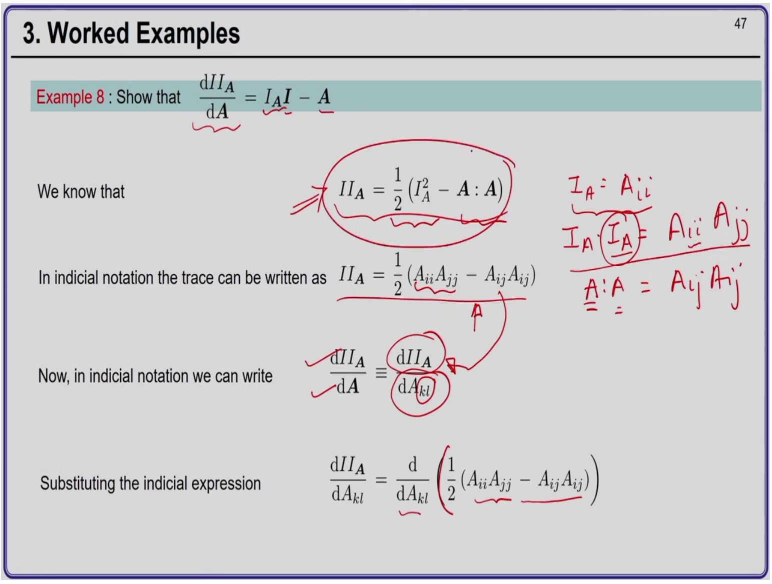

So, if you have to write I A square what you have to do is you have to write I A into I A and then you have A ii. Now, for the second I what do you do see there will be no free index and there will be only repeated index. So, the second I ok, over here we have to choose some other index, because one repeated index also cannot occur more than twice. So, we already have i twice so, i cannot occur again.

So, we take instead of i we take j ok. So, I is I square A will be nothing, but A ii A jj. So, that is your first expression in the right hand side in indicial notation. Now, we want to write a double contracted with A in indicial notation and if you remember this was nothing, but A ij A ij ok.

So, once we have our second invariant in terms of the tensor in indicial notation this expression we have, then in indicial notation we can write the derivative of the second invariant of the tensor with respect to tensor as dII A two A divided by dA kl.

Why kl? Because our expression for the second invariant already has two i's and two j's ok. So, now, in any given expression we cannot have more than two repeated index. So, then we have to choose some other symbol for subscript and we have chosen k and l ok. So, you have d II A by dA kl.

So, next we can substitute the expression for the second invariant of a second order tensor in the numerator ok. So, that is what we have done we can write d by dA kl of 1 by 2 A ii A jj minus A ij A ij ok.

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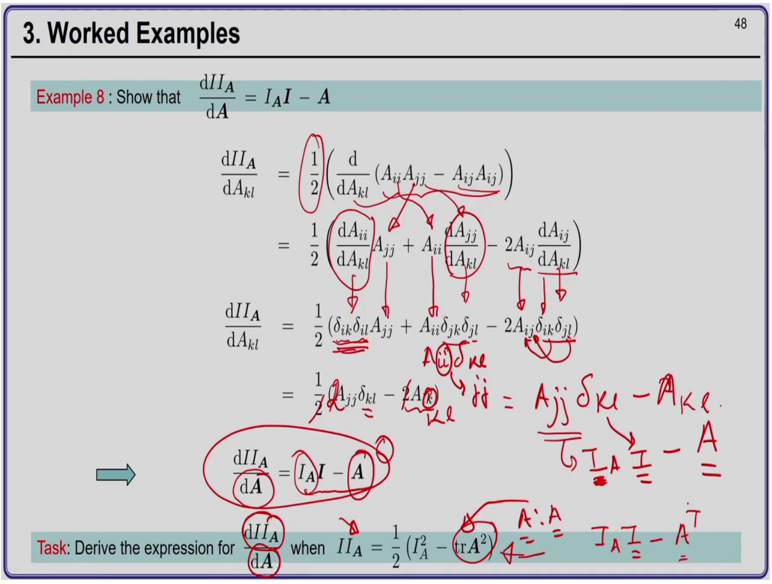

So, now, you can take 1 by 2 outside the bracket it is a constant term and then you just have to take the derivative of this term inside the bracket with respect to kl ok. So, you can take the derivative inside and you can use the chain rule ok.

So, you have d by d A ii dA kl into A jj into plus A ii into d A j A jj by d A kl that is the first term when it is taken derivative with respect to A kl and then the second term you have 2A ij ok. There are 2A ij. So, it will be twice of A ij dA ij by dA kl ok.

Now, from our previous example you can substitute the derivative of the terms in terms of Kronecker delta all these terms can be written in terms of Kronecker delta. So, the first term here is delta ik delta il A jj A i A ii delta jk delta jl minus 2 ij delta ik delta jl, and next what we can do is use the substitution property of Kronecker delta.

So, delta ik delta il i is if you see here delta ik delta il will be nothing, but delta kl and then you have ok. So, you will have A ii A delta jk delta jl which will be nothing, but A ii delta kl and because i is a dummy index can be replaced by j ok.

So, this becomes twice of A jj delta kl minus delta 2 A ij delta ik delta jl they do not have any index which is common, but they will replace i here and j here. So, the first Kronecker delta will replace i with k and the second Kronecker delta will replace j with l. So, that is what you will have II A lk ok.

So, next this two gets cancelled out and you are left with. So, you will be left with A jj delta kl minus A lk sorry this will be kl ok. So, it will be A kl. So, there is a typo here it will be A kl ok. So, A jj is nothing, but the first invariant which is I delta kl is nothing, but second order identity tensor and A kl is nothing, but A ok.

So, writing in direct notation you can write the derivative of the second invariant of A tensor with respect to the tensor as I A into second order identity tensor I minus A ok.

So, next you can derive what will be the derivative of second order second invariant of the tensor with respect to the tensor when you use following definition of the second invariant. So, remember the last term over here instead of having A contracted with A we have trace of A square ok. So, you will notice that instead of A here you should get A transpose, but that is left as a task for you.

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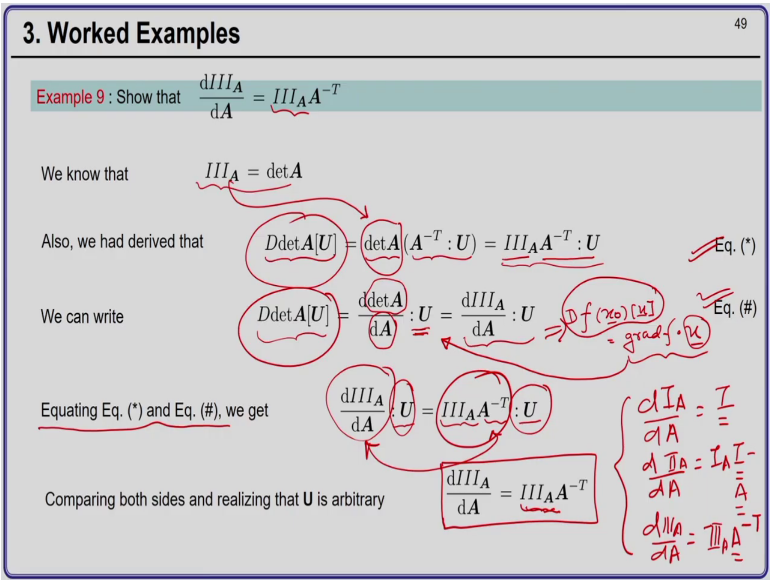

So, next we move to our next example ok. So, this is going to be our final example. So, we want to evaluate the derivative of the third invariant of a second order tensor with respect to the second order tensor ok. We want to show that it is the third invariant of a second order tensor multiplied by A inverse transpose ok. So, we know that the third invariant of a second order tensor is given by determinant of A ok.

Now, if you remember we already have derived that the directional derivative of determinant of A in the direction U is nothing, but determinant of A into A inverse transpose double contracted with U ok. So, I can write determinant of A as III A ok. So, III A into A inverse transpose double contacted with U ok. So, let us say this is equation star ok.

So, next we can write and this is a important concept that you should very clearly understand that the directional derivative of determinant of A in the direction U is nothing, but the derivative of determinant of A with respect to the tensor A in the direction of U ok; in the direction of U means double contacted with U ok. So, how do we write this? So, this has parallels with how we compute the gradient ok.

So, if you remember the directional derivative of a scalar function at x 0 in the direction U was nothing, but the gradient of the function times U dotted with U ok. So, you can write directional derivative of a scalar function field evaluated at x 0 in the direction U as gradient of f evaluated at x 0 dotted with U ok.

So, when we extend this concept to second order tensor ok. So, we can write so, if you see this term over here is same as this quantity ok. So, therefore, we will have to have the gradient of something similar to gradient of f which is nothing, but and our f is nothing, but determinant of A ok.

So, we will have the determinant of A the derivative of determinant of A with respect to A and instead of dot product because these were vector. So, we had a dot product, but because now we have a second order tensor we have to take double contraction ok. So, it is a double contraction in the direction U similar to this here ok.

Now, once we have this let us say this is equation prime ok. Now, we can compare equation star and equation prime ok. So, on the left hand side you see we have directional derivative of determinant of A in the direction U from equation star this is nothing, but the determinant of A into A inverse transpose double contracted with U and from equation prime we have this as the derivative of the determinant of A with respect to the tensor A double contacted with U and remember U here is a arbitrary vector.

So, if we compare our equations star and equation prime so, you will get the derivative of the third invariant of a second order tensor with respect to the tensor double contracted with U is equal to the third invariant of the second order tensor A inverse transpose double contracted with U ok. So, after this double contraction we have the same quantity on both the sides.

Therefore, the quantity with which U is being contracted with should also be same ok. So, this quantity over here should be equal to this quantity over here ok.

So, if we compare both the sides we can write the derivative of the third invariant of a second order tensor with respect to the tensor itself is nothing, but the third invariant of the tensor multiplied by A inverse transpose ok. So, this relation again is a very important relation which will be used later.

So, I will again go back to the last three examples and we will see what results we got ok. So, let me move back to. So, a tensor has three invariants I A, II A and III A 1 2 3 ok. So, the first derivative of the first invariant of the tensor with respect to the tensor itself is nothing, but identity second order identity tensor ok. So, d I by d A is nothing, but I the second invariant of the tensor II a.

So, the derivative of second invariant of the second order tensor with respect to the tensor itself is nothing but the first invariant of the tensor multiplied by the second order identity tensor minus the tensor itself.

Remember, for this case your second order I mean your second invariant has to be defined in this way ok. So, again instead of that expression for second invariant if it is defined in this particular manner ok, then the final relation will be different. You will notice that you will get I A I minus A transpose ok.

So, it did not have a transpose here, but if you use this expression for yours second invariant of the tensor you will get transpose ok. And, the derivative of the determinant of the tensor which is the third invariant of the tensor with respect to the tensor itself is nothing, but the third invariant of the tensor multiplied by A inverse transpose.

So, dI A by d A is I d II A by d A is I A I minus A and d III A by d A is III A A inverse transpose. So, these three expressions should be very clear to you because we are going to use these expressions very often in our later discussions when we are developing the constitutive relations for hyper elastic material which means relation between stress and strain ok.

So, with this example we have come to the end of this part of discussion on tensors ok. So, for last six lectures we have been discussing about various aspects of tensor we discussed how simple tensors operations can be written in indicial notation, direct notation, we discussed about the eigenvalues, the invariants of the tensors then finally, we discussed what is meant by directional derivative and linear and directional derivative of the second order tensor.

And, then we looked into how to compute the gradient divergence of second order tensor, first order tensors and scalars and finally, we look into looked into one important integral theorem for scalars, vectors and second order tensors and specifically we saw the important theorem of Gauss divergence theorem. So, this theorem Gauss divergence will be again used later when we are deriving the we are when we look into the weak form ok.

So, with this we have come to close for this part of the course. And, next we will move to the next part of the course which is on kinematics that is where we consider deformation without actually considering the cause for that deformation which means we will not take force into account and we will look into how the deformation can be characterized ok.

So, we will close today and we will see you in the next module of this course which will be on kinematic.

Thank you.