Fundamentals of Compressible Flow
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Module – 03
Quasi-One Dimensional Isentropic Flow
Lecture - 09
Quasi-One Dimensional Isentropic Flow - III

We are back again in the another module of this lecture that is in the same module 3, we will be discussing about Quasi-One Dimensional Isentropic Flow.

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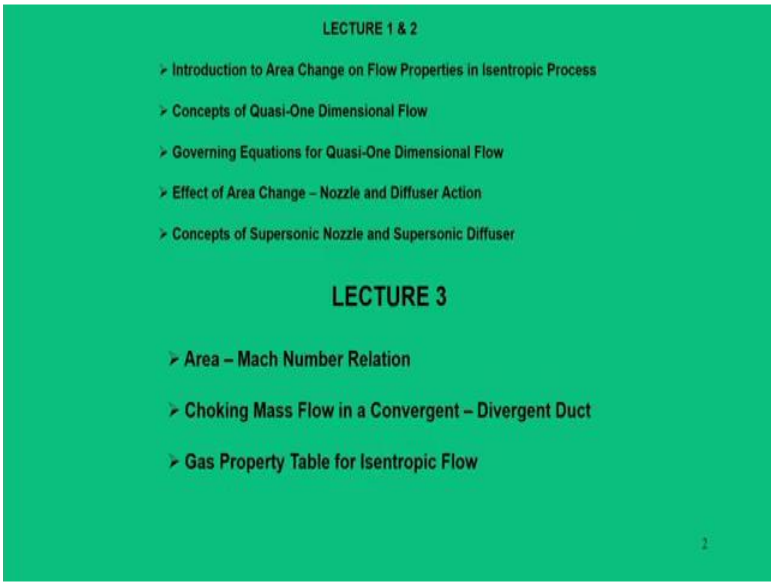

And if we recall the previous lectures; so we have discussed on this topic about the effect of area change on the flow properties. Then we introduced the concept of quasi one dimensional flow and for this quasi one dimensional flow, we arrived at the various governing equations that is mass, momentum and energy. Then we introduced the concept of area change and that essentially talks about the nozzle and diffuser action.

Subsequently, the concept of supersonic nozzle and supersonic diffuser also introduced. Now, in this module will now talk about the another lecture that is 3rd part of this lecture and here, we shall discuss about important topics such as area Mach number relations.

Another important topic is choking conditions, so we will estimate the choking mass flow rate for a convergent divergent duct. And apart from this which is the most important segment of this isentropic flow, here we will discuss about the gas property table. So, this gas property table will essentially tell us that how to refer gas tables for an isentropic flow. We will talk about these things in the subsequent slides.

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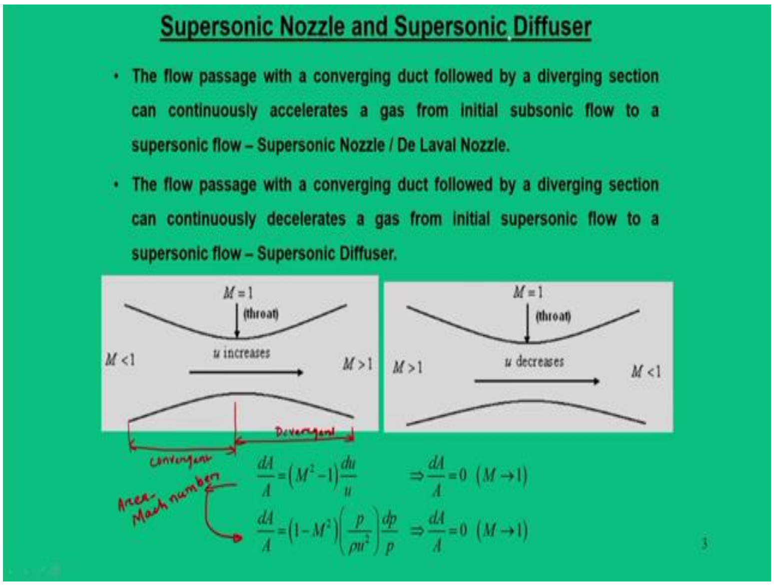

Now, if I just recall what has been discussed in the last module. So, we came with a conclusion in the last module that a supersonic nozzle and diffuser is possible by using two segment of a passage; that is the convergent passage and then we have a divergent passage. And theoretically, we prove that it is possible to accelerate a subsonic flow entering in a convergent passage and finally, getting a supersonic flow at the end of the divergent section which is again attached to the convergent duct. And essentially, at a minimum area that is the location of throat, we are going to get a sonic flow. And similarly, in a same convergent divergent passage if our inflow is supersonic, then we can decelerate the flow so that at the exit, we will be able to get a subsonic flow.

So, all this concepts were governed by this relations which is the Area Mach number relation. Now, this area Mach number relations also is connected with respect to pressure change that will talk about whether it is a nozzle action or it is a diffuser action.

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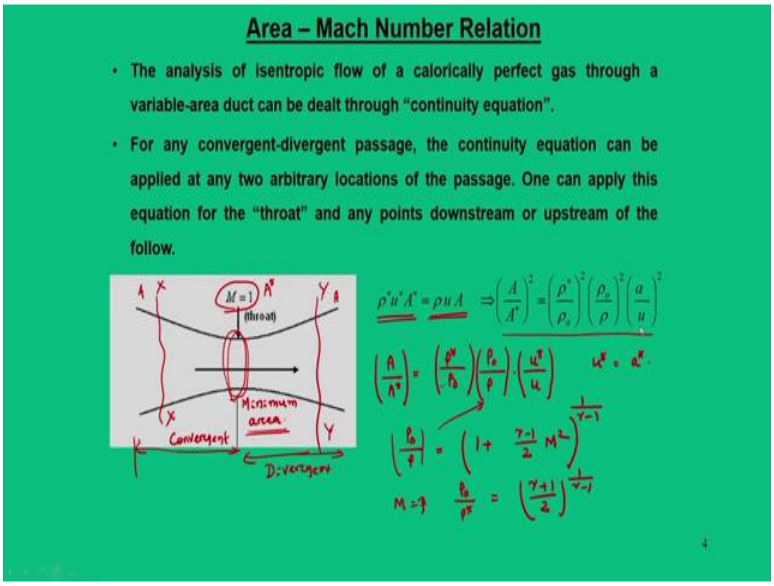

Now, moving further we are now going to analyze the; the same continuity equations, but with a different context. Now, what is that context is that basically if you look at this convergent divergent passage; throat is the minimum area section.

So, when we really want to design a nozzle or diffuser; then we must take into account this throat area or this minimum area. And that has to be taken has a reference conditions because whether you go for a supersonic nozzle or diffuser; we have to come across a minimum area.

So, with respect to this minimum area other locations like we can have a sections which is at convergent part. So, basically we can have a cross section which is at a convergent part and we also can have a cross sections which is at divergent part. So, what we are trying to see is that; how to correlate any section X-X or Y-Y with respect to the minimum area section. And this minimum area sections we represent as A^* and any other arbitrary sections; we represent as A.

So, if you look at the relation between cross sections between X-X and throat; then we can say it is between A and A^*. Similarly, if you look at the cross sections Y-Y and the throat, then we also this part also we can put it as A.

So, area at that location is arbitrary; so correspondingly we get the mass flow rate $\rho u A$ at that section. And with respect to throat, the mass flow rate is $\rho^* u^* A^*$ because at this condition, you have sonic flow. So, by continuity equations; we can relate this two expressions and further, we can rewrite this expressions as in this manner like

$$\rho u A = \rho^* u^* A^*$$

$$\left(\frac{A}{A^*}\right) = \left(\frac{\rho^*}{\rho_0}\right)\left(\frac{\rho_0}{\rho}\right)\left(\frac{u^*}{u}\right)$$

Now, making it square we arrive at this particular expression, but here one thing to be noted here, we have put u^* as a^*. So, we can replace this u^* as a^*; at that point your u^* is happens to be sonic velocity.

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_0}\right)^2\left(\frac{\rho_0}{\rho}\right)^2\left(\frac{u^*}{u}\right)^2$$

Now, we can recall our isentropic relations $\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}$. So, $\frac{\rho^*}{\rho}$ can be put here and when you put M is equal to 1, this entire expression will be $\frac{\rho_0}{\rho^*}$. So, this turns out to be $\frac{\rho_0}{\rho^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}}$; this we can get when we put M is equal to 1; so this is one expression.

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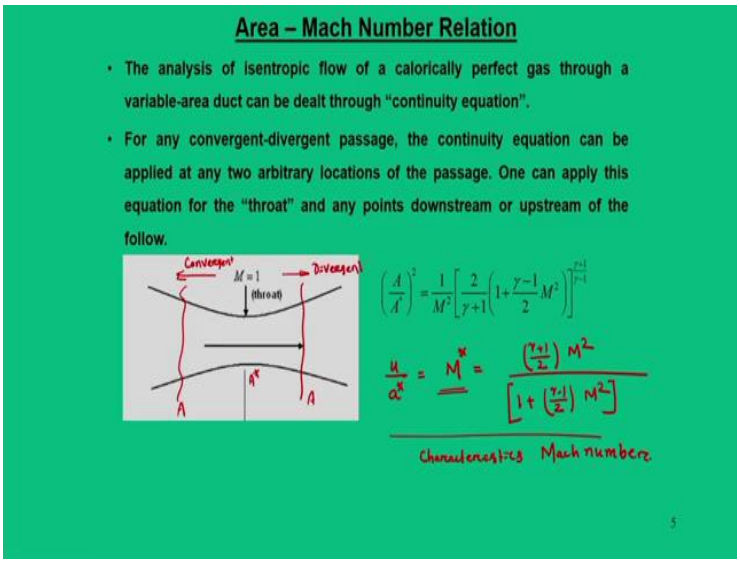

Another expression, we can also get between $\frac{u}{a^*} = M^* = \frac{\left(\frac{\gamma+1}{2}\right)M^2}{\left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]}$

So, this expression we get it as the formula which is known as characteristics Mach number.

So, how do you get it? Because this concept tells that at any location when the flow is reached to sonic velocity, then we call this as characteristics Mach number. Now, by putting that expressions; we will arrive at the relation between $\frac{A}{A^*}$; that is A is any location and A^* is the throat location.

So, that $\frac{A}{A^*}$ after simplifications, it turns out to be this particular expressions.

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

So, what we can tell is that for any convergent divergent passage; the continuity equation can be applied at any two arbitrary locations of the passage and one can apply this equation for the throat and any point downstream or upstream of the flow.

So, I mean downstream means if you look at this side; this will be the divergent part and this part will be convergent part and this A can refer to any two locations of this passage.

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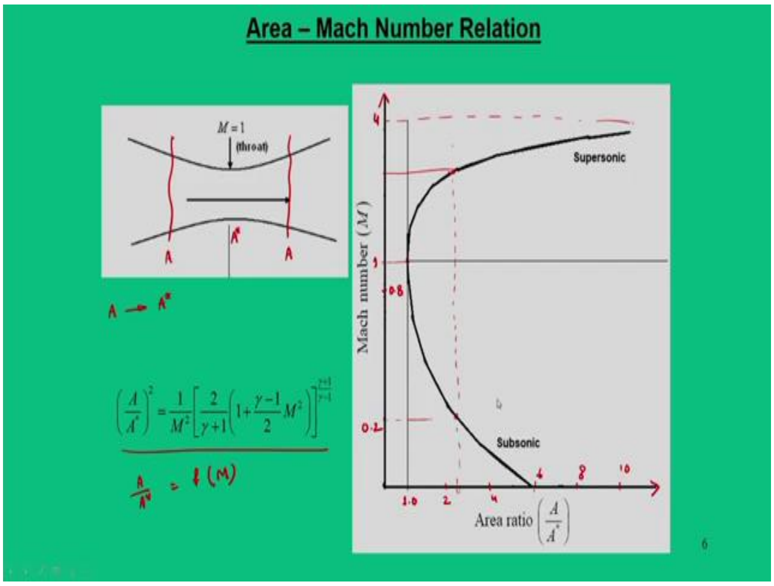

Now, one of the important segment from this expression is that here we say $\frac{A}{A^*}$; we can write as a function of Mach number. So, in one way of calculating this area ratio; that means, with respect to this throat and with respect to any arbitrary area A, if you know the Mach number; then it is easy to calculate $\frac{A}{A^*}$ by using the expressions. Because by simplification by just putting the Mach number and the value of gamma for the gas, then we can find out this area ratio. But the relation between the Mach number and area ratio is implicit in nature means that if you know area ratio; then it is almost difficult task to calculate Mach number. Because, the nature is not simple you have to use the some trial and error concept to solve that equations.

Now, this is one part; second part of this expression is that one way of arriving at this difficulty in calculating Mach number from area ratio; a plot is given or a graph can be generated which is between this Mach number and area ratio which is shown in here. So, what we can see here that; if I just put some standard number, what is the standard number? We can have minimum area when A is equal to A^*.

So, A goes to A^*, that is the minimum conditions and that area ratio happens to be 1.0. And when this is 1; obviously, your Mach number will be fixed at one and this particular point on the curve; we can have, this is the Mach number. Now, then what we do? We increase the area, try to solve these equations. Now, when I increase this area; the curve goes in this manner which is a kind of a elliptical in nature.

Now, when you say elliptical in nature and if I just draw a apex point; that is at this point that is the apex point of this ellipse. Then, I can say there are two parts the first part is the subsonic part and second part is the supersonic part.

So, what I can say is that; if you look at this plot, at any area ratio; if you put any fixed area ratio and draw a vertical line; this line will intersect, this curve at two points that is one in the subsonic domain, other would be in the supersonic domain.

So, what I can say? For a given area ratio $\frac{A}{A^*}$, there are two possible solutions which one is in the subsonic domain, other is in the supersonic domain. So, just putting this number; if you put some realistic number, what I can say; this area ratio, we can say up to this point may be 6, this point may be 8, this point may be 10 and this particular point may be 2, may be 4.

Here also Mach number, we can have around 4 and this Mach number may be 0.2, may be 0.8. So, this will give a rough indication that how area ratio and Mach number is related and using this curve, one can find out what is the requirement.

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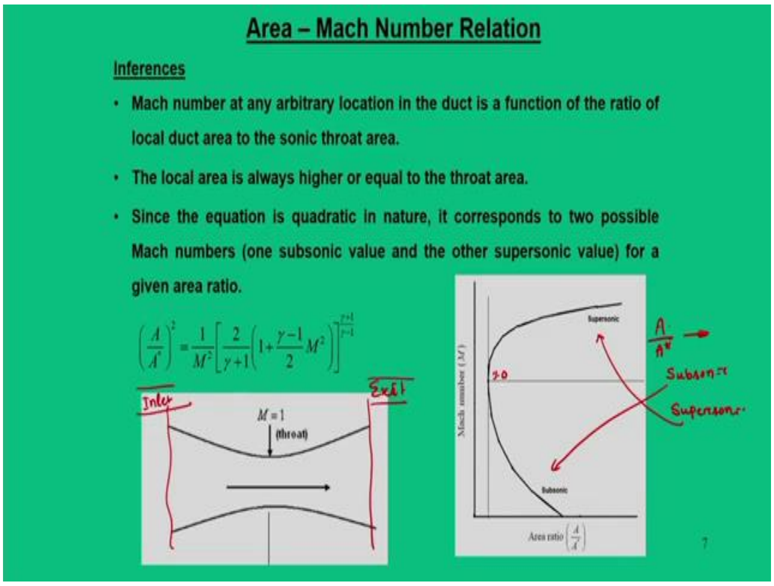

So, here what we see that for a given area ratio, there are two possibilities; one is in the subsonic, other is in supersonic. So, the subsonic solution goes to the bottom part of the curve, supersonic solution goes for the top part of the curve.

Now, whether the flow will be subsonic or supersonic; that again depends on the nature of pressure ratio that is maintained between the inlet and exit. So, that part, we will talk separately when we bring the concept of pressure ratio into this analysis. So, at this stage the very basic bottom line is fixed that the area ratio decides the Mach number.

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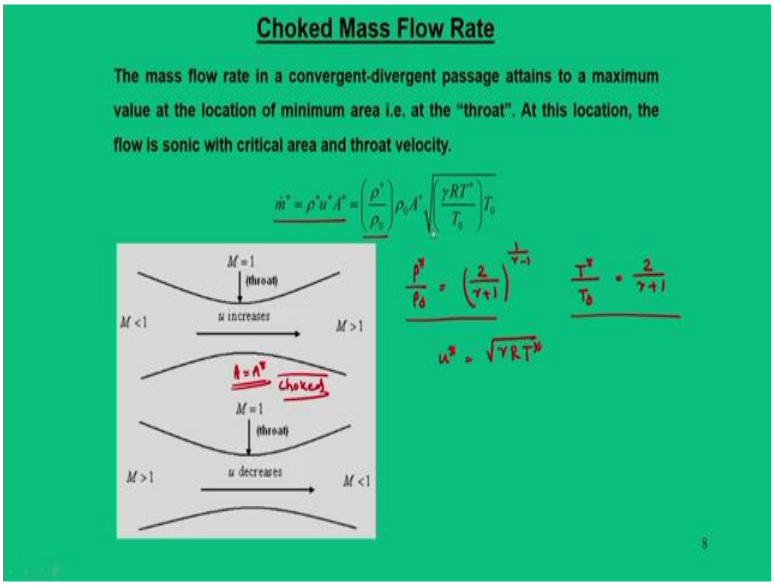

Now, the another important segment of this convergent divergent passage is the choked conditions. The choked condition is essentially decided by the fact that the flow is sonic at the minimum area. So, when the flow is sonic at this minimum area, that is $A = A^*$; then we say it is a choked condition. So, when I say choked condition, this is possible for whether your inflow is subsonic or whether the inflow is supersonic.

So, in the last two analysis, we say that when a subsonic flow enters in a convergent divergent passage. At the beginning of the passage, its velocity increases because that is a convergent duct, but one point it becomes maximum when it is maximum; that means, it is at the sonic flow; that is velocity is maximum at this throat; the velocity has sonic at the throat and that point the flow is choked.

Now, if you see other situation also, when the Mach number is supersonic, the flow decelerates in the same convergent passage and it again reaches to sonic. So, irrespective of what flow condition it is entering; always a sonic flow is ensured at the throat.

So, while discussing about a converging diverging passage; or we can say convergent divergent nozzle or convergent divergent diffuser, we have to ensure that we are allowing some certain minimum mass flow rate that has to be entered into the passage. To have this minimum mass flow rate, we must have a fixed area and that fixed area should be logical in nature so that a slog of mass at least can enter into this passage. So, this is one of the important design aspect where one has to calculate the choking mass flow rate. When I say choking mass flow rate, quantitatively it is expressed $\dot{m}^* = \rho^* u^* A^*$.

Now, this particular expression can be written in this another fashion where what I did is, instead of ρ^* we put $\frac{\rho^*}{\rho_0}\rho_0$, A^* remains as it is, u* I have put as $\sqrt{\gamma R T^*}$. Now, in that square root term; we have introduced a temperature T_0 in the numerator as well as denominator, so that these two terms gets canceled. And ultimately, if you simplify this; we get, will get $\rho^* u^* A^*$.

Now, why I am doing this? Then that is because we want to find out what is this ratio because this ratios are known; what are this ratio? One ratio is $\frac{\rho^*}{\rho_0}$ which is known; that is in our earlier expression, we can say this is $\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$. In the other ratio, that is $\frac{T^*}{T_o} = \frac{2}{\gamma+1}$.

Now, when we put this two expressions there and here also as I mentioned; we replaced this $u^* = \sqrt{\gamma R T^*}$.

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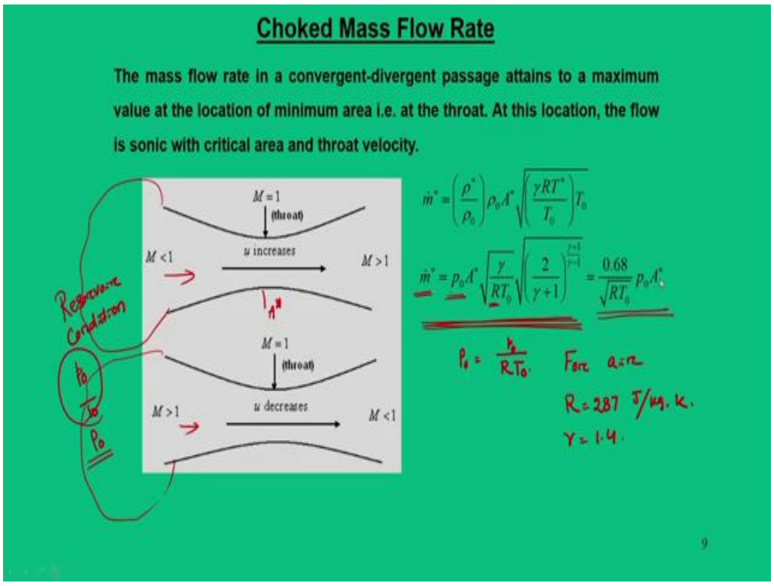

So, by putting this expressions; we can arrive at a working formula which is known as, mass flow rate in the convergent divergent passage and that attains to a maximum value at the minimum area location; that is throat. So, when I put this expression and finally, after simplification we arrived at this particular relations.

$$\dot{m}^* = p_0 A^* \sqrt{\frac{\gamma}{RT_0}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

So, if you look back this equations; we started with this particular expression and we now arrived at this expression. Here, we have brought a term p_0 and this p_0 is nothing, but $\rho_0 RT_0$. I will write in a other way like $\rho_0 = \frac{p_0}{RT_0}$; that is when I put this expression here and simplify this, we get a working relation of choked mass flow rate.

So, what is this physical significance of this expression here? That, normally when a flow enters in a convergent divergent passage; so, it has to be correlated with respect to some reservoir condition. And this reservoir conditions is nothing but your total pressure or stagnation pressure, total temperature; that is stagnation temperature and total density stagnation density.

Roughly, we do not calculate density; we specify in terms of pressure and temperature. So, that is the relation the choke mass flow conditions is expressed in a working relation form that involves p_0 and T_0. And rest of the part are known that is R is characteristic gas constant and gamma is the specific heat ratio for the gas that is entering in the passage.

Now, we will move to the minimum area part; for minimum area part A* is known; so which is minimum area location is known. Now, once I know this; then for air, if you put R is equal to 287 Joule per kg Kelvin and gamma is equal to 1.4. We arrived at the choking mass flow rate in a very simplified form that is $\dot{m}^* = \frac{0.68}{\sqrt{RT_0}} p_0 A^*$

So, here what he says is that for a given reservoir conditions and if you know the minimum area that flow has to pass, then we can find that what is the maximum flow rate we can have. So, this is a very important working formula for this analysis.

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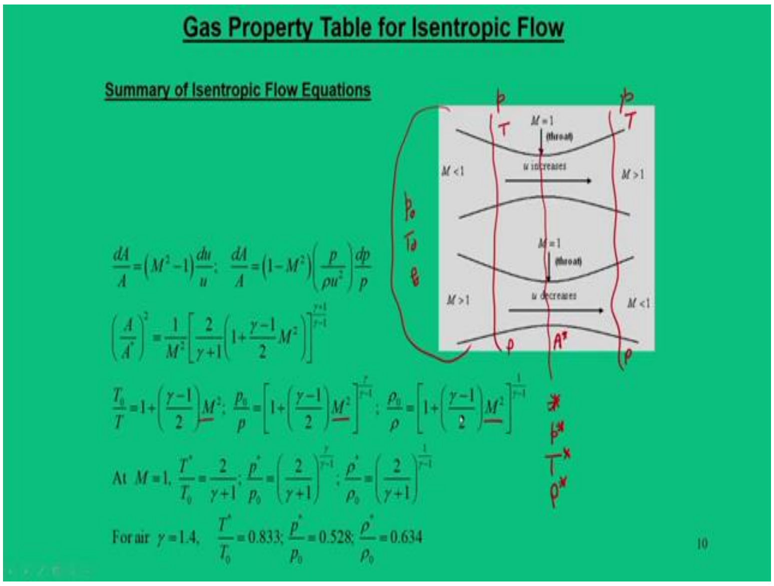

So, now with this; we will now summarize this isentropic flow equations what we have known so far. So, the first thing is that; this isentropic flow conditions what we discussed here and with respect to quasi one dimensional analysis, we introduced a supersonic nozzle and supersonic diffuser. Then for that equation, they are governed by this area Mach number relations and area pressure relation and then it seems the entire flow passage is analyzed in isentropic manner, these are the isentropic relations.

So, what it means that p_0, T_0 and ρ_0; these are the reservoir conditions from which the flow is initiated. And at any arbitrary locations; that is pressures are p, T and ρ; this at any arbitrary locations for which the conditions are p, T and ρ. And these are the essentially the reservoir condition p_0, T_0 and ρ_0; this are this location we say p, T and ρ.

So, when I say and this particular condition which is minimum area; that is star, that is p*, T^*, ρ*and this is at the area A*, the conditions are defined by these expressions; And finally, when putting gamma to be 1.4; we get a fixed relations.

So, this is how the isentropic flow relations are calculated, but what we are going to introduce here? A gas property table. So, gas property tables; if you look at this equation, all this expressions are function of Mach number. So, if you see $\frac{T_0}{T}$ or $\frac{p_0}{p}$ or $\frac{\rho_0}{\rho}$ is also a function of Mach number. So, if you know Mach number then we can generate a lot of data just by changing this Mach number and keeping gamma to be 1.4. So, if you keep doing it we can generate a gas table.

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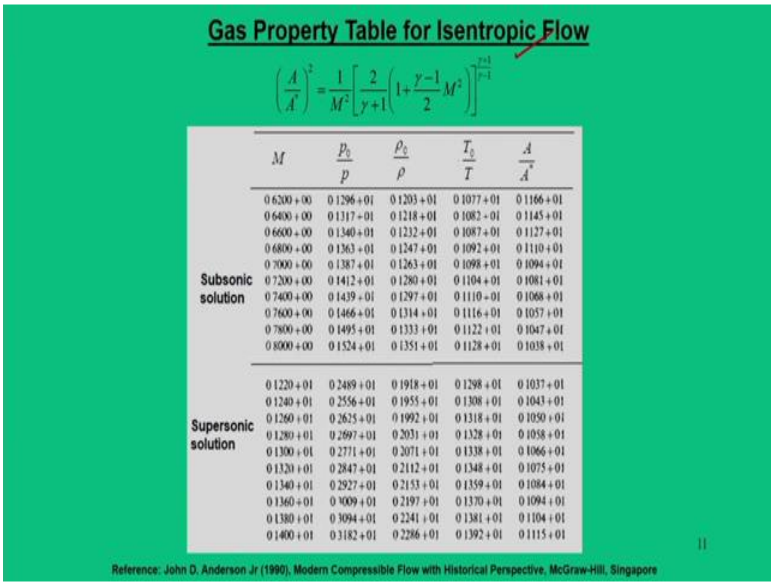
Gas Property Table for Isentropic Flow

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

	M	$\frac{p_0}{p}$	$\frac{\rho_0}{\rho}$	$\frac{T_0}{T}$	$\frac{A}{A^*}$
	0.6200+00	0.1296+01	0.1203+01	0.1077+01	0.1166+01
	0.6400+00	0.1317+01	0.1218+01	0.1082+01	0.1145+01
	0.6600+00	0.1340+01	0.1232+01	0.1087+01	0.1127+01
	0.6800+00	0.1363+01	0.1247+01	0.1092+01	0.1110+01
	0.7000+00	0.1387+01	0.1263+01	0.1098+01	0.1094+01
Subsonic solution	0.7200+00	0.1412+01	0.1280+01	0.1104+01	0.1081+01
	0.7400+00	0.1439+01	0.1297+01	0.1110+01	0.1068+01
	0.7600+00	0.1466+01	0.1314+01	0.1116+01	0.1057+01
	0.7800+00	0.1495+01	0.1333+01	0.1122+01	0.1047+01
	0.8000+00	0.1524+01	0.1351+01	0.1128+01	0.1038+01
	0.1220+01	0.2489+01	0.1918+01	0.1298+01	0.1037+01
	0.1240+01	0.2556+01	0.1955+01	0.1308+01	0.1043+01
	0.1260+01	0.2625+01	0.1992+01	0.1318+01	0.1050+01
Supersonic solution	0.1280+01	0.2697+01	0.2031+01	0.1328+01	0.1058+01
	0.1300+01	0.2771+01	0.2071+01	0.1338+01	0.1066+01
	0.1320+01	0.2847+01	0.2112+01	0.1348+01	0.1075+01
	0.1340+01	0.2927+01	0.2153+01	0.1359+01	0.1084+01
	0.1360+01	0.3009+01	0.2197+01	0.1370+01	0.1094+01
	0.1380+01	0.3094+01	0.2241+01	0.1381+01	0.1104+01
	0.1400+01	0.3182+01	0.2286+01	0.1392+01	0.1115+01

Reference: John D. Anderson Jr (1990), Modern Compressible Flow with Historical Perspective, McGraw-Hill, Singapore

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So, what it sees is that what I have introduced a concept which is called a gas table and this data was taken from this book John Anderson, that is Modern Compressible Flow and this is just an extract from this book; just to explain that how this gas tables are referred.

So, what it says is that the parameters which is a Mach number, which is known to us and if you know the Mach number, we can calculate the stagnation pressure to static pressure ratio; stagnation density to static density ratio and stagnation temperature to static temperature ratio. So, instead of calculating it; if you do not want to calculate using this expression; one can directly refer to the table and take the data from the table directly from itself.

So, this is what we say the gas table for isentropic flow and in this gas table for isentropic flow, there is another parameter $\frac{A}{A^*}$. So, as I mentioned; this $\frac{A}{A^*}$ is a function of Mach number and as I can tell you that from this datasheet also, one can say that for a given $\frac{A}{A^*}$; we can find two different solutions, one is in this subsonic region, other in the supersonic region.

For instance, if you say as area ratio as may be 1.11; so for this area ratio, the Mach number is in the subsonic range will be about 0.68. In fact, for the similar area ratio of 1.115, the Mach number turns out to be 1.4; which is in the supersonic region. In fact, from the datasheet also; one can directly refer, instead of going for the implicit way of calculating the Mach number from the area ratio. So, this is how the simplicity of gas table that has been generated for isentropic flow.

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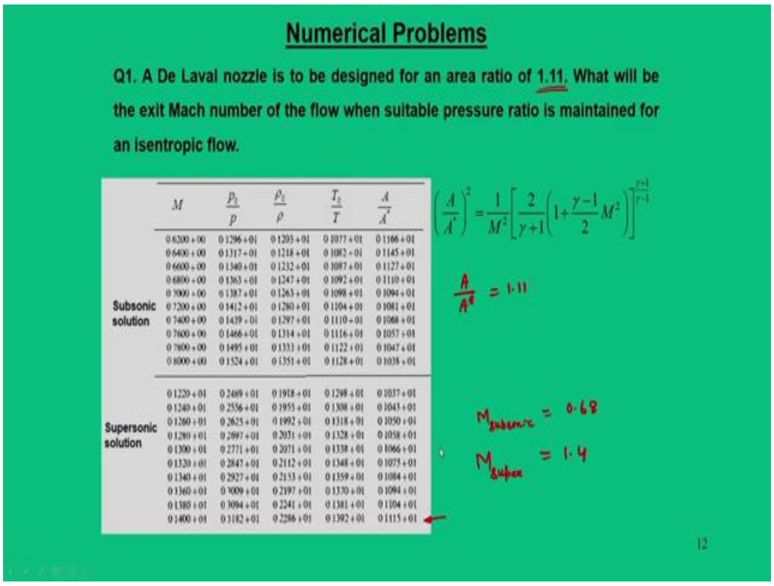

So, with this I will just try to solve some sample problems in this case; like for example, the first problem which is just now I explained that how Mach number has to be calculated from the area ratio.

Now, for an area ratio of $\frac{A}{A^*}$ as 1.11; as one is subsonic, that is 0.68, M is supersonic that is 1.4. So, this is how the table has to be referred for these things and in fact, this table can be used for all possible area ratio that comes under subsonic or supersonic domain.

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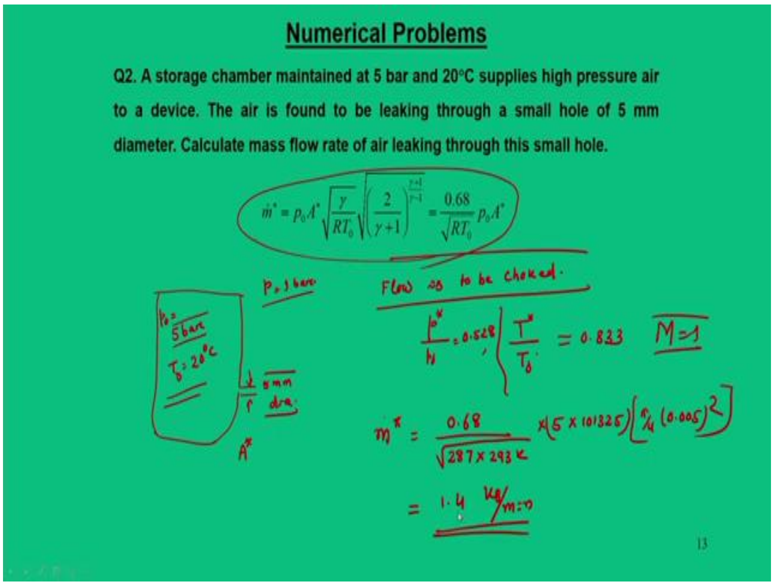

The another simple problem that talks about a storage chamber is to be maintained at 5 bar and 20°C; that supplies high pressure air to a device. The air is found to be leaking through a hole of 5 mm diameter. Calculate the mass flow rate of air leaking through the small hole.

So, this particular problem refers to the fact that we have a storage tank that has 5 bar pressure and temperature is 20°C and this temperature is 20°C; we can say it is a stagnation. In fact, since it is a storage device; we can say stagnation pressure has to be this. So, what it says is that; there happens to be a small hole somewhere in here; this hole is happens to be 5 mm dia.

So, if you look at this system and assuming this atmosphere is to be 1 bar. So, taking this small hole, we can say the flow is going to be choked. Why it is to be choked? Because you have to calculate what is the value of $\frac{p_0}{p^*}$ and what will be value of $\frac{T_0}{T^*}$.

So, putting these as atmospheric pressure; one can check that $\frac{T^*}{T_0}$ be 0.833 and $\frac{p^*}{p_0}$ 0.528. So, this is the conditions for M is equal to 1 and when this ratio is maintained, the flow is choked; when the flow is choked, we can directly use this particular expression to calculate mass flow rate.

So, you can calculate $\dot{m}^* = \frac{0.68}{\sqrt{287 \times 298}} \times (5 \times 101325)\left[\frac{\pi}{4}(0.005)^2\right]$

So, if you put this turns out to be about 1.4 kg/min. So, this says that how a leak in a storage vessel can generate; that means, due to very large pressure difference; it can generate a leak of 1.4 kg/min.

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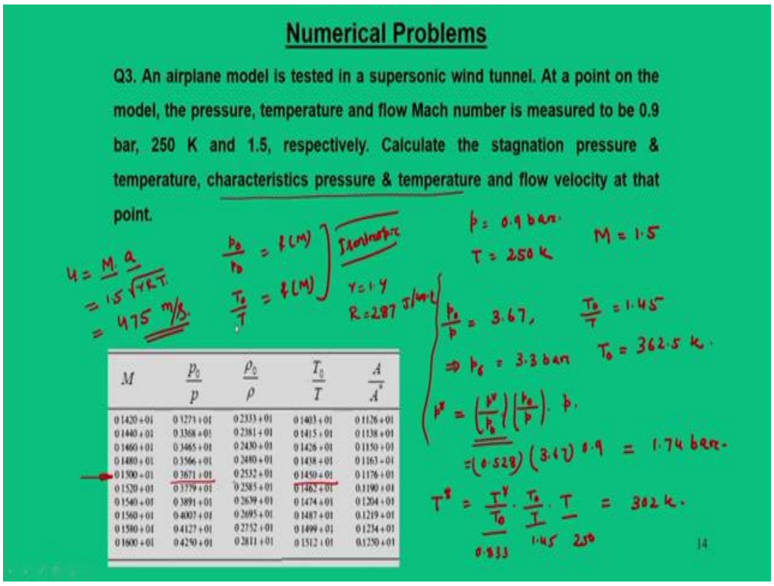

The third problem which I am talking about is that; how to refer a gas table? So, what has been given that an airplane model is to be tested in a supersonic wind tunnel where at any point in the model, the pressure is given as 0.9 bar, temperature is given as 250 Kelvin, Mach number is given as 1.5.

So, one way to calculate this problem, that we can directly get $\frac{p_0}{p}$, $\frac{T_0}{T}$ as a function of Mach number; that is from isentropic relation that we have talked elaborately previously.

Now, what I am trying to say that same thing we going to do it using gas table. This is one of the extract for the gas table where we say that if you fix this Mach number as 1.5, we can directly get this $\frac{p_0}{p}$ has 3.67; $\frac{T_0}{T}$ 1.45. So, this two numbers can be directly taken from this gas table.

Now, once I say this; then this will give you as what will be p_0 and we know p 0.9, so we can get p_0 to be 3.3 bar. T_0 would be; this is 1.45 multiplied by 250; so this is about to 362.5 Kelvin and then we also wanted to have characteristics pressure and temperatures.

So, we also require p* ; we can write $p^* = \left(\frac{p^*}{p_0}\right)\frac{p_0}{p}p$; so $\frac{p^*}{p_0}$ is 0.528, $\frac{p_0}{p}$ is 3.67 and p is about to 0.9 bar. So, if you multiply; we can get 1.74 bar.

Similarly, $T^* = \left(\frac{T^*}{T_0}\right)\frac{T_0}{T}T$; so when I put this number, this is 0.833, this is $\frac{T_0}{T}$ 1.45 and temperature is about 250; when I put this number, we get 302K. Apart from this, we also require the flow velocity; so the flow velocity u; we can write M times a; that is Mach number times a. So, Mach number we know 1.5; a is $\sqrt{\gamma RT}$; so we know γ 1.4, R 287 J/kg-K.

So, when you put this expression here; we can get u to be about 475 m/s. So, this is how from a given data; how we can calculate the characteristics condition and stagnation conditions.

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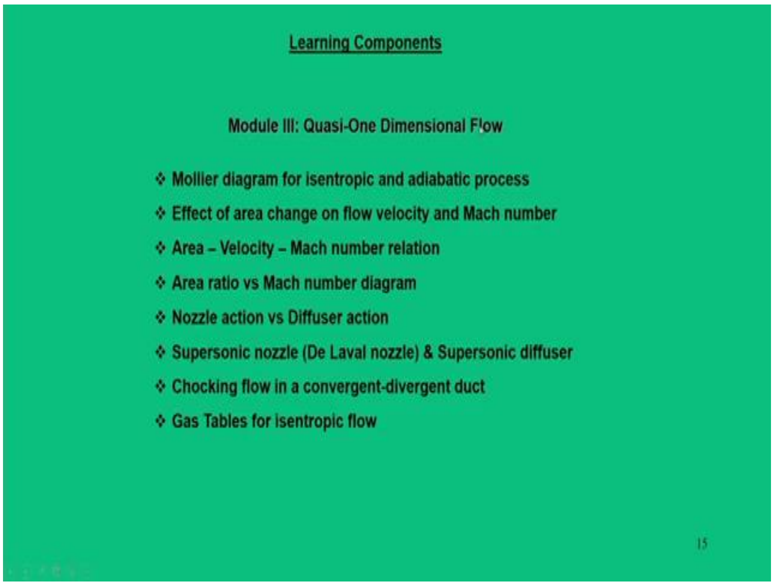

So, this is all about the entire learning module for the quasi one dimensional flow. Now, if I just want to recall that at the end of this model; what you should remember? That we should be knowing, the Mollier diagram which is to be used for isentropic and adiabatic process. A non isentropic process is normally referred as adiabatic process; here then we have to know about what is the effect of area change on the flow velocity and a Mach number, when there is a flow in a convergent divergent passage.

So, we also have a very important relation; we call as area, velocity, Mach number relations or subsequently we have also have area ratio versus Mach number diagram; that is from this diagram itself, one can find out implicitly Mach number from the area ratio. And looking at this area ratio, we also elaborately analyzed; how a nozzle action and diffusion action can be performed in a convergent divergent passage.

Then, we have the concept of supersonic nozzle and then we have a choking flow conditions in a convergent divergent duct. And finally, I introduce the topic of gas tables which; in which one can directly find the necessary values from the given data. So, with this I come to the end of module 3, I hope; I have clarified most of your doubts.

Thank you.