Fundamentals of Compressible Flow
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Module – 03
Quasi-One Dimensional Isentropic Flow
Lecture - 08
Quasi-One Dimensional Isentropic Flow - II

Welcome again to this course Fundamentals of Compressible Flow. We are in the module 3 that is Quasi-One Dimensional Isentropic Flow.

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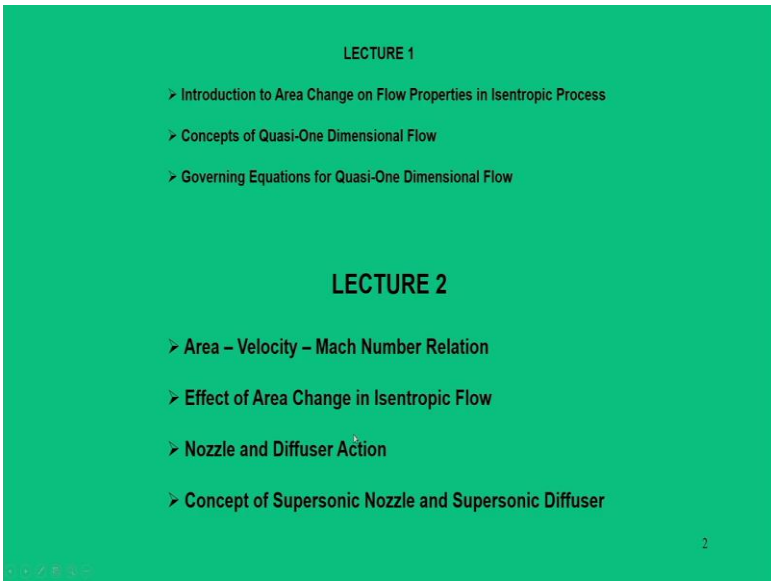

So, in the previous lecture, we introduced the topic of quasi-one dimensional flow and with respect to that quasi-one dimensional flow, we say that there is a area change and this area change leads to the property change in the flow properties. So, and in the entire flow process, we say that the thermodynamic properties associated with the change of state is said to be isentropic. Also, we discussed the fundamental equations for a quasi-one dimensional flow.

Now, in this lecture, we will also touch upon those topics, but in a most quantitative manner. So, the first fundamental equations that governs this quasi one-dimensional flow is the area, velocity and Mach number relations. Then with that relations, we are going to analyze what is the effect of area change by considering the entire process to be isentropic and in fact, this area change leads to two important consequences that is nozzle and diffuser actions.

In fact, nozzle and diffuser these are the two fundamental tools to expand the flow or increase the velocity or compress the flow to decrease the velocity. So, in this process, we are going to say that how a compressible flow can perform a nozzle or diffuser action by using this governing relations.

So, these concept also leads to the very fundamental topic of supersonic nozzle and supersonic diffuser. In fact, these are the typical devices, geometrical devices that are used in a wind tunnel setup either to increase the velocity or decrease the velocity. So, this is the very basic bottom line of this lecture. Now we will move to the topic.

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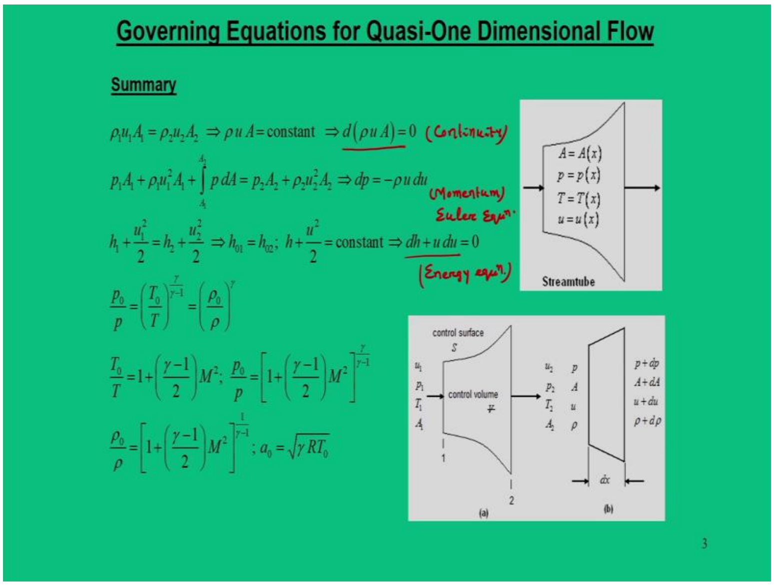

So, essentially just to go more into this topic, if you recall what we have summarized the governing equations for a quasi-one dimensional flow, what we say is that the flow is allowed to pass through a duct which is a hypothetical stream tube and that is flexible in nature. That means, area can change with respect to only one direction that is A is a function of x.

Now, while doing so, we expand this area in such a way or increase the area in such a way that the property change is associated with a very marginal increment like pressure changes by a small magnitude dp in a small distance dx. So, likewise, having making this assumption in this change, we can say that the entire process can be considered to be isentropic.

As so, with this logic, we use this continuity equations in a differential form, then we use this momentum equation and typically it is the Euler equation that is also in differential form and in fact, we have this energy equation which is essentially same what we have in one-dimensional flow.

Apart from this, we also try to correlate the property informations through stagnation as well as static properties such as stagnation to static pressure ratio, stagnation temperature to static temperature ratio, stagnation density to static density ratio. And in fact, since it is a moving flow so, we say all of them are functions of Mach number. So, this was the basics of quasi-one-dimensional flow equations.

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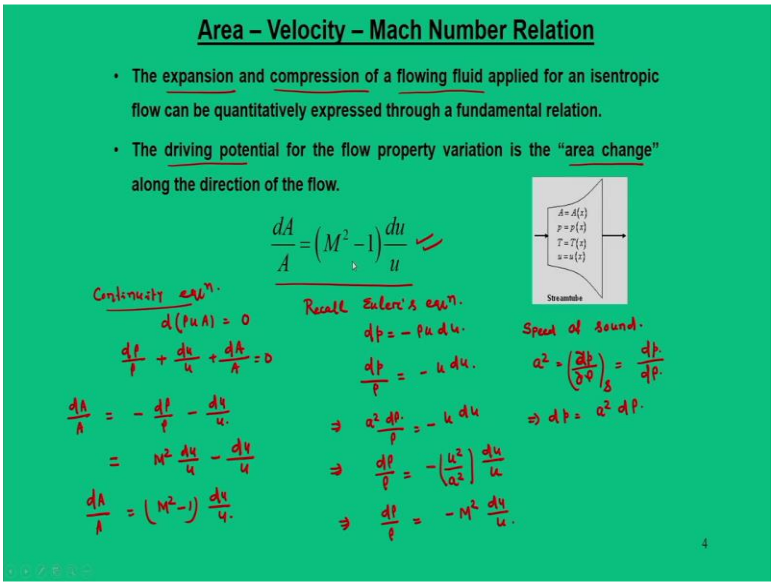

Now with these basics, we are now going to derive another relations which is called as area, velocity and Mach number relations. So, again we have this same nature of the flexible stream tube and in fact, this area, velocity Mach number relations is govern by this equations. So, let us see how these equations can be derived.

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{du}{u}$$

Now, another point needs to be emphasized here that this area, velocity Mach number relations is used whenever there is area change that is one can use this equations for an expansion process or for a compression process of a flowing fluid and we can represent them in a qualitative manner through this equations.

Now, since the there is no other source of property change except the increase or decrease in the area and the flow is isentropic so, this is the driving potential that is area change is the driving potential for this change to occur. So, to start this equations let me derive this particular area velocity Mach number relations. So, what you have to do you have to start with continuity equation.

So, in the continuity equations, we represent this continuity equation that is in differential form $d(\rho uA)=0$. So, this is one of the relations we used earlier. So, if you differentiate individually, we can write $\frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A}=0$. So, let us make a pause of it.

So, now, you say that you recall this momentum equation or I will say this Euler equation. So, that Euler equation you write $dp=-\rho udu$. Now from this equation, we have to write what is this dp. So, we say $\frac{dp}{\rho}=-udu$. Now this $\frac{dp}{\rho}=\frac{dp}{d\rho}\times\frac{d\rho}{\rho}$.

Now, for the time being now you move to another relation that is speed of sound. So, you can write $a^2=\left(\frac{\partial p}{\partial \rho}\right)_s$. So, since we are already saying, we can write this also as in ordinary differential form $\frac{dp}{d\rho}$. So, you can write this implies $dp=a^2 d\rho$.

So, now, putting this so, we can write $a^2\frac{d\rho}{\rho}=-udu$. So, we can find $\frac{d\rho}{\rho}=-\left(\frac{u^2}{a^2}\right)\frac{du}{u}$.

So, this turns out to be $\frac{d\rho}{\rho}=-M^2\frac{du}{u}$.

Now using this equation again in the continuity equations so, you can write $\frac{dA}{A}=-\frac{d\rho}{\rho}-\frac{du}{u}$. So, these $\frac{dA}{A}=M^2\frac{du}{u}-\frac{du}{u}$ or in other words, we can say $\frac{dA}{A}=\left(M^2-1\right)\frac{du}{u}$.

So, this is how the fundamental equations we derived $\frac{dA}{A}$ which signifies how area changes and $\frac{du}{u}$ say that how velocity changes and it is again to Mach number of the flow.

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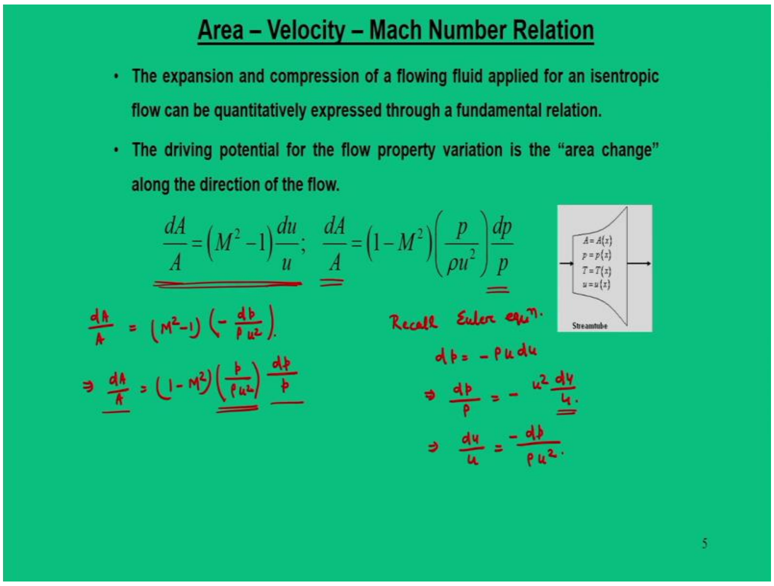

Now, when I say area change with velocity, there is another significant relations that this area change makes the change in the velocity, then what it does in terms of pressure. So, how that area change is related to pressure change. So, this is again a simple expression. So, we can also derive this from this area Mach number relation.

So, what you try to do? Again, we recall this Euler equation which is $dp=-\rho u du$. So, So, $\frac{dp}{\rho}=-u^2\frac{du}{u}$. Now from this equation, we can replace this $\frac{du}{u}$ in this Mach number velocity relations. So, how do you do? So, this will talk about $\frac{du}{u}=-\frac{dp}{\rho u^2}$.

So, now, I can write from this equation $\frac{dA}{A}=\left(M^2-1\right)\left(-\frac{dp}{\rho u^2}\right)$ or we can write $\frac{dA}{A}=\left(1-M^2\right)\left(\frac{p}{\rho u^2}\right)\frac{dp}{p}$.

So, this gives the area change with respect to pressure, but here there is another term that drops in apart from Mach number that is $\frac{p}{\rho u^2}$ this is nothing which includes pressure as well as velocity.

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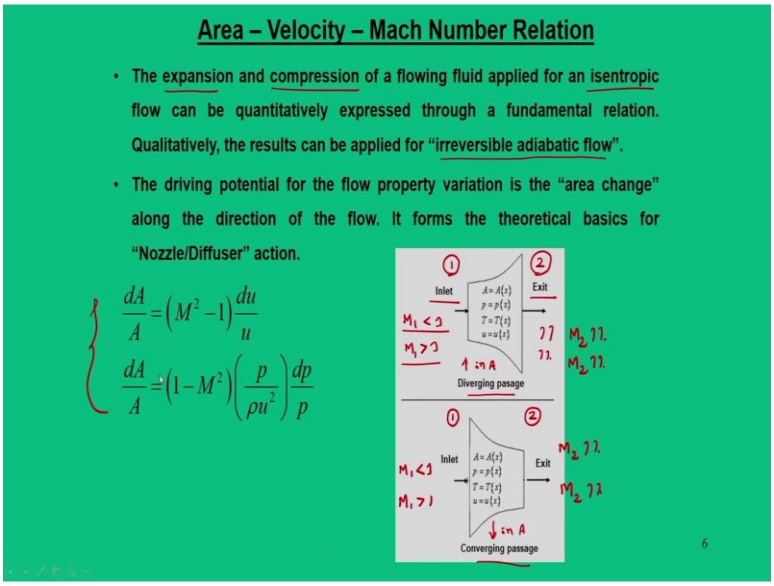

So, these are the two basic equations that governs the flow that enters in a passage. So, essentially one can analyze this expansion and compression of the flowing fluid in these isentropic process, but same results one can use it for a adiabatic flow as well. Although because in many situations, flow is not treated to be isentropic rather it can be considered to a major extent, the flow to be adiabatic like we explained in the last class about its property change in a Mollier diagram.

Now, we are going to analyze these two basic equations for a situation that in a diverging passage and in a converging passage. When I say diverging passage, I say increase in area. When I say converging passage, I say decrease in area. So, let us talk one by one. So, what I am saying only say that we have a diverging passage, the flow is entering and leaving. So, we assign this as inlet and exit. So, we assign the conditions as 1 and 2.

Now, when I say this inlet and exit; so, my inflow conditions that is M_1 it can be less than 1 that is flow is subsonic, it can be also it can be greater than 1 or it that is flow can be supersonic. So, we can have a situation that the either a subsonic flow can enter a diverging passage, or a supersonic flow can enter a diverging passage.

When it enters, what happens in the exit? So, for both the situation, what happens in the exit? Now taking this as a reverse situations, when I say this is 1 and 2 that means, flow is entering at the higher area and leaving in the smaller area. So, at the inlet also we have similar situation M_1 can be also subsonic less than 1 we can have M_1 also can have can be greater than 1. So, in both the situations, we will find out what is the condition of M_2.

So, essentially there are four conditions and two passages. So, all the four conditions we are going to analyze through this fundamental relations.

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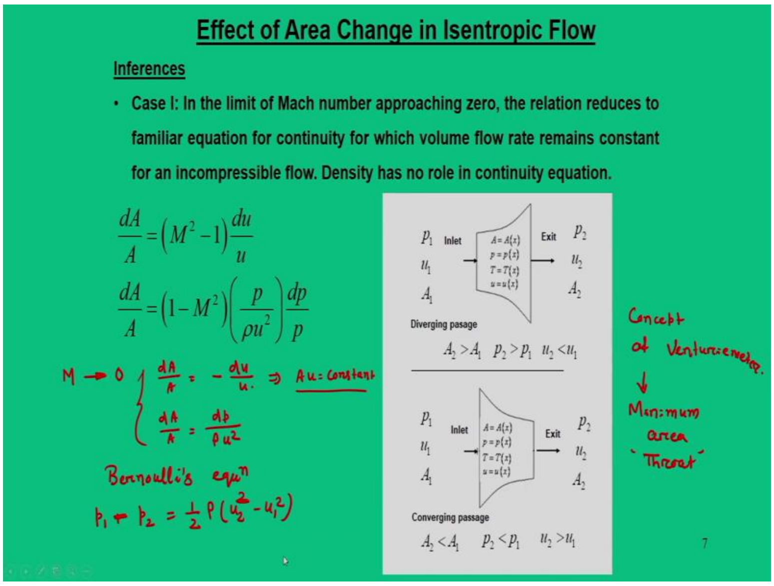

So, for that let us see the first situation, what I will say inference 1 that is case 1: in the limit of Mach number approaching to zero. So, when I put M tends to 0. So, what happens here? We say $\frac{dA}{A} = -\frac{du}{u}$ and in the second equation, we say $\frac{dA}{A} = \frac{dp}{\rho u^2}$ and this turns out to be Au is equal to constant.

So, what is this physical significance of this? So, when M tends to 0; M tends to 0 that means, the area, velocity and Mach number relation turns out to be Au is equal to constant. There is no parameter density here. So, Au typically represent the volume flow rate and this is essentially what happens in an incompressible flow.

So, that is what it says that in the limit of Mach number approaching the 0, the relation reduces to familiar equation for continuity for which volume flow rate remains constant for an incompressible flow. In fact, density has no role in the continuity equations.

Now, when such a situation happens, we can say that for a diverging passage when incompressible flow enters, then with increase in the area, the pressure must increase so, it increases. So, velocity must decrease so, it decreases. So, this is how what happens in a diverging passage.

Now in the reverse situation that is in converging passage, when a incompressible fluid enters, the area decreases; when this area decreases, pressure also decreases by satisfying this equation, but velocity increases. So, this is how we simply govern by the Bernoulli's equation for an incompressible flow between two states.

That is, we can write $p_1 - p_2 = \frac{1}{2}\rho\left(u_2^2 - u_1^2\right)$. Of course, we do not consider alleviation into account here. So, this is how the equation is going to satisfy to make this pressure and velocity changes and based on this, we design the concept of venturimeter for which there is a minimum area known as throat.

So, this is how what we did in the incompressible flow. But this is not at the part of our interest, main part of interest is towards the compressible flow, but what it turns out to be the case that our main relations when it turns to incompressible limit, still the Bernoulli's equation and the concept of venturimeter still gets satisfied.

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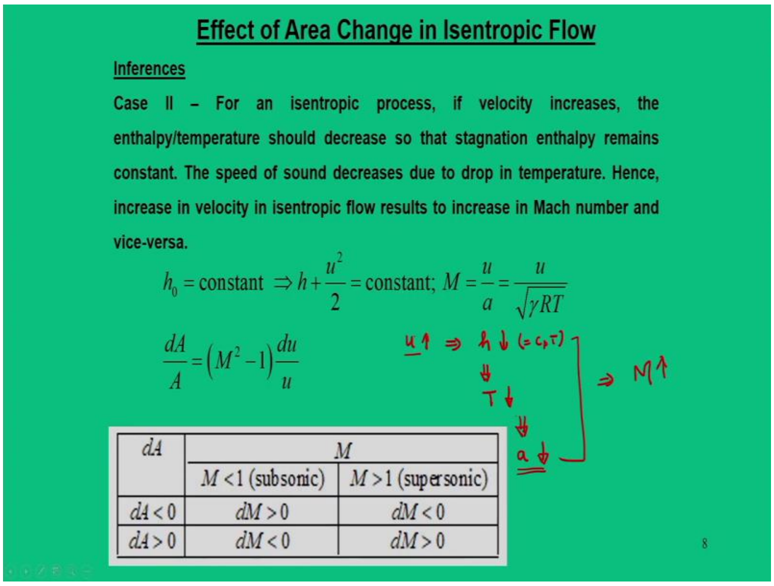

Now, let us move to the cases that when we progressibly increase the velocity or Mach number. So, instead of Mach number going to the 0 limit, we use to keep increasing towards the compressible flow limit. So, we will try to move the Mach number from subsonic to supersonic side.

So, before you do that, we will say that what happens if I increase the velocity from inlet to exit what happens to other parameters? So, when velocity increases so, if you see this Mach number it is a relative function of the speed of your gas velocity and the speed of sound.

So, Mach number can be increased by increasing u or decreasing the speed of sound. Decreasing speed of sound means we must decrease the temperatures. So, this is how as a consequence of increase in u or decrease in T, the Mach number will always increase.

So, let us see that what happens if I increase u. So, if I am increasing u which implies I must decrease the h, h means static enthalpy. Why? Because along a streamline, we can say that the total enthalpy remains constants which constitutes the static part and the dynamic part. Static part is h and dynamic part is u.

So, and I am increasing u, h must decrease and since $h = C_p T$ for a calorically perfect gas. So, this will implies temperature also decrease. So, decrease in the enthalpy will also decrease the temperatures.

Now, when temperature decrease, this will imply speed of sound also will decrease. So, increase in u will lead to drop in speed of sound and in both increase in u and decrease in a, this will imply the Mach number will always increase.

So, what it says is that for an isentropic process, if the velocity increases, the enthalpy or temperature should decrease so that, stagnation enthalpy remains constant. The speed of sound decreases due to drop in temperature. Hence, the increase in the velocity in the isentropic flow results in increase in the Mach number and vice versa.

So, why I am saying this consequence because in the subsequence slides, we will say that while increasing the velocity, the Mach number also increases; because here we are in this equation talks about both Mach number as well as the velocity. So, based on these, we can prepare a matrix that talks about what happens for different possible conditions.

So, what does this means in this matrix, how do you look at? In the one axis, I can put dA either it can be less than 0 or greater than 0. What means in the direction of the flow; that means flow can enter in a converging passage for dA less than 0. So, flow can enter in a diverging passage for which dA greater than 0. So, in fact, the first row is the converging passage analysis and the second row is the diverging passage analysis.

Now, when I am doing this dA less than 0 and if M is subsonic that is M is less than 1, then your du must increase so, dM must increase. So, likewise using this correlations which I derived here, this matrix can be formed. In fact, this matrix is the benchmark for our all subsequent analysis. So, this is how once I frame this matrix, I can tell that what is going to happen to my flow.

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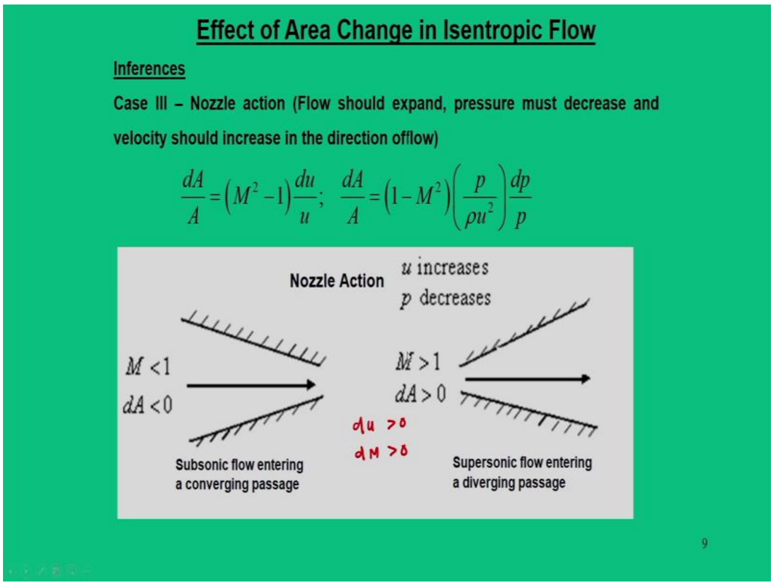

So, this is my next section that is case 3 which is a nozzle actions. So, by definition the nozzle action means the flow should expand, pressure must decrease and velocity should increase in the direction of the flow. So, under what circumstances in that matrix I can have. So, if I have to perform this nozzle action, then I must have the Mach number should be greater than 0. So, dM should be greater than 0 that is the diagonal consequences can be written.

So, in this case, you say that your du must increase and or du and dM they are greater than 0. So, this is how the nozzle action talks about. So, what does this mean? I can perform a nozzle action that means, I can increase the velocity of the flow by allowing a subsonic flow in a converging passage or I can allow a supersonic flow in a diverging passage.

So, this is a very contradictory statement that we had in the incompressible flow. So, in incompressible flow, we are mostly concentrated on a converging passage. To increase the velocity we must talk about the converging passage. But in a other case that if we have a supersonic flow, then if you want to increase this velocity, then you must use a diverging passage. So, this is one of the important consequence or significance for a nozzle action in a supersonic flow.

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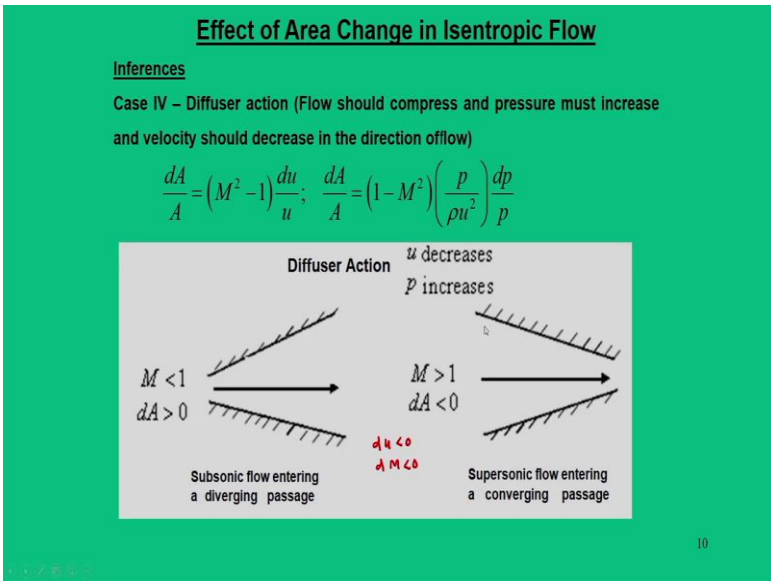

The exactly opposite to that situation like I will tell du less than 0 and dM less than 0 that is a diffuser action. So, velocity must decrease, pressure must increase that is in the direction of the flow. To have this, if we have a incoming flow is subsonic and then, we want to perform a diffuser action, this means we have to use a diverging passage.

In other scenario, for a supersonic flow, if you want to perform a diffuser action, then we must use a converging passage. So, this is how the we say the nozzle action and diffuser action with respect to area change.

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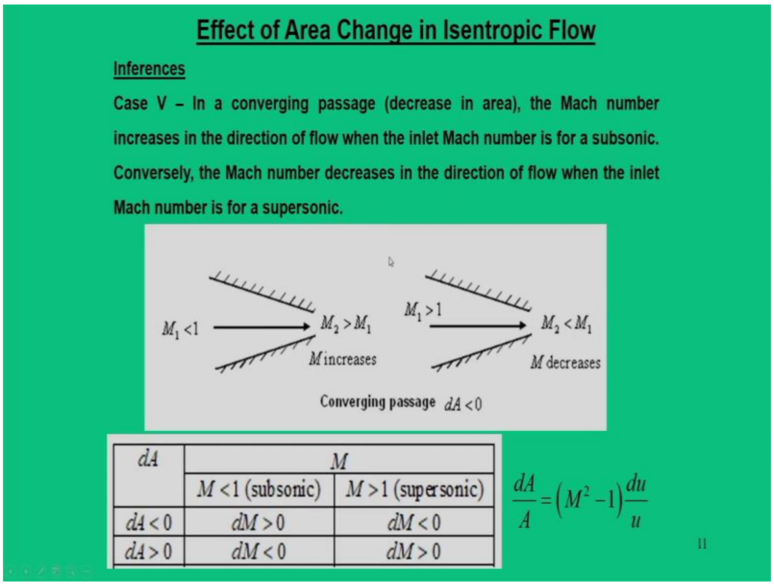

The next important segment that means, same interpretation we are analyzing in a different aspects. Let us use always a converging passage that means, when it is only a converging passage dA less than 0, then from this we can use the first row where dA less than 0. So, when dA less than 0, now if I allow a subsonic flow when it enters a converging passage; so, in the downstream the Mach number will increase.

But in other case, when a supersonic flow enters a converging passage, the Mach number decreases. So, what it says in a converging passage which is means decrease in the area, the Mach number increases in the direction of the flow that is when the inlet Mach number is subsonic.

But in other scenario, the Mach number decreases in the direction of the flow, when the inlet Mach number is supersonic. So, this is how the significance of a converging passage.

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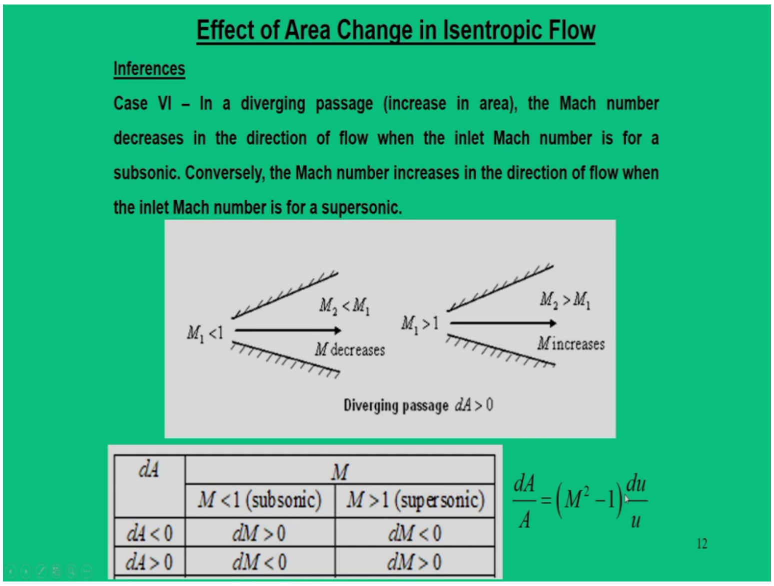

Then, moving further in a reverse situation we will say what happens in a diverging passage. So, as I say if it is a diverging passage, your dA must be greater than 0. So, I should refer this second row. So, in second row in one case I get dM less than 0 and other case, I get dM greater than 0. So, subsonic case, we say dM less than 0, supersonic case we say dM greater than 0.

So, what it means to us that if a subsonic flow enters a diverging passage, the Mach number decreases in the direction of the flow. This is similarly the consequence we get in a incompressible flow situations. But in a reverse situation, when we have a supersonic flow, when it enters a diverging passage the Mach number increases. So, the Mach number increases. So, this is in contrast with the incompressible flow analysis, but it is true with respect to area, Mach number and velocity relations.

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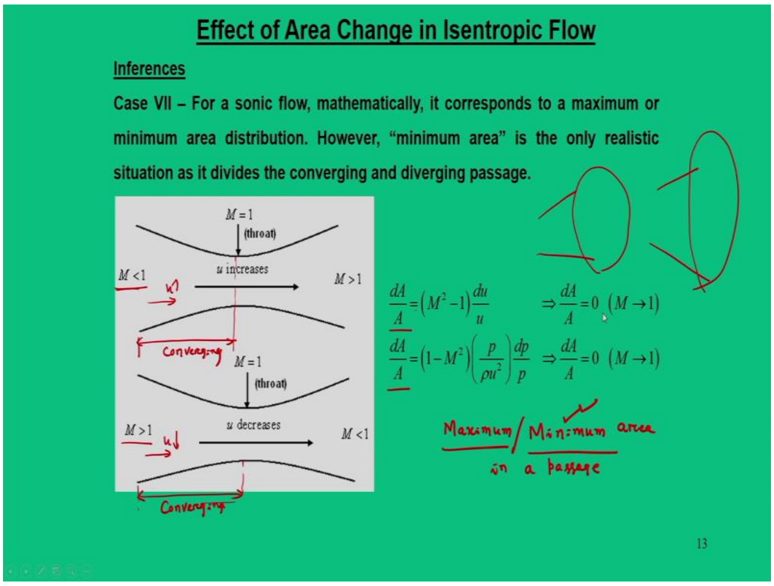

Now, we will talk in another important inference that is in a situation when the flow is sonic. So, when I say flow is sonic so, if I try to use this equations $\frac{dA}{A}=\left(M^2-1\right)\frac{du}{u}$, $\frac{dA}{A}=\left(1-M^2\right)\left(\frac{p}{\rho u^2}\right)\frac{dp}{p}$. So, if I put M is equal to 1 in this equations, I will get $\frac{dA}{A}=0$.

So, what does this mean when I say $\frac{dA}{A}=0$? It means is a situation of maximum or minimum area in a passage. So, let us analyze this situation that when a subsonic flow entering in a converging passage.

So, we are analyzing that in one situation, the subsonic flow is entering a converging passage. In another situation, the supersonic flow is entering in a converging passage. So, continuously our result show that in the for subsonic case, the u increases and the supersonic flow u decreases. So, this must increase or decrease.

Now, this flow cannot increase subsequently by keeping on increasing this area. So, in our analysis, one can say this area has to decrease. So, this means this is a minimum area situations. Now, if I cannot have a situations where the area will have maximum for a sonic flow to occur. So, what does this mean?

That means, as long as the flow becomes sonic so, it only is possible for a minimum area situations because we cannot have in a diverging passage, we cannot have continuously increase in the velocity. The there is no limit of maximum area.

So, the practical application point of view, the minimum area is the most realistic situation because it divides a converging and diverging passage. But if it is a only diverging passage, then there is no limit of this area which will lead to a situation $\frac{dA}{A}=0$. So, this essentially very important consequence that leads to a concept of minimum area and this minimum area is a situation that divides the converging and diverging passage.

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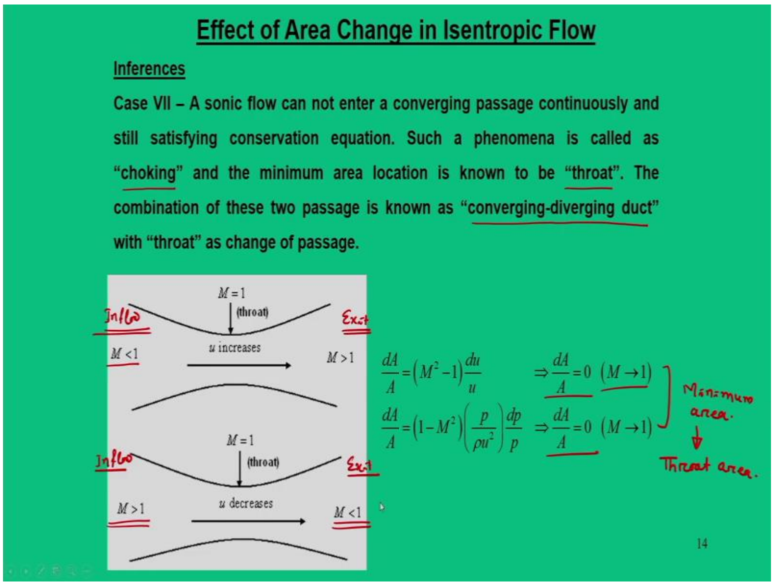

Now this particular concept gives another important phenomena which is called as choking. So, when you say in the last slide that when M goes to 1, dA goes to 0 and this is a situation for minimum area and this minimum area is known as throat area.

So, in a passage, the throat is the minimum area locations and when the flow reaches to this throat, then it is said to be choked because the flow has attained a sonic velocity at this point and no further area change is possible. And, when I say when there is no further area change possible; that means, a sonic flow cannot enter a converging passage continuously and still satisfying the conservation equations. So, by satisfying the area velocity, Mach number relations, the sonic flow cannot enter a converging passage.

So, such a phenomena is known as choking and the minimum location at which it happens is known as throat. So, to satisfy these things, what important consequence we have? We can have a converging-diverging duct. Although, the word converging duct or diverging duct they mean two different entities for different flows, but here by using this concept, we can bring both the duct together so that, we can have a minimum area in combining both of them and finally, we can treat either a subsonic flow or a supersonic flow.

Now, when I do so, then all this equations: area, velocity, Mach number and area Mach number and pressure relations gets satisfied to achieve the concept of nozzle or diffuser. Now, let us see how it happens. So, now, in this case what I have merged here a converging passage and a diverging passage, they are merged together. Here also converging passage and diverging passage are merged together.

Now, let us analyze two basic distinct situation where one case inlet condition is subsonic, other case the inlet condition is supersonic. So, let us say when a subsonic flow enters a converging passage; so, obviously when it enters a converging passage, the velocity increases. How long the velocity can increase? The velocity can increase up to the minimum area that is throat locator. So, when it increases to this throat means it has reached a sonic point. So, that is the sonic Mach number is reached at the throat.

Now a sonic flow, as I said that sonic flow cannot enter in a converging passage because the flow is said to be choked. So, we have to attach a diverging section with same throat on both sides. So, the diverging passage will have minimum area that is throat and the when the sonic flows sees a diverging passage it further increases. So, at the end, at the exit, we get the Mach number greater than one.

So, what does this mean that we started with inflow was subsonic and we get exit flow as supersonic. So, this is a very important tool that say that a subsonic flow can be accelerated by using a converging diverging duct and so when I say, then it can be accelerated to a supersonic flow.

And in reverse situation, when your inflow is supersonic and exit we can have a sub sonic Mach number. So, what does this mean? When a supersonic flow enters a converging passage; so, when it enters, the rule says that velocity must decrease. How long it will decrease? It will decrease to a sonic point where M is equal to 1. Then after that what will happen? It sees a diverging passage. So, the velocity continuously decreases to desired subsonic Mach number.

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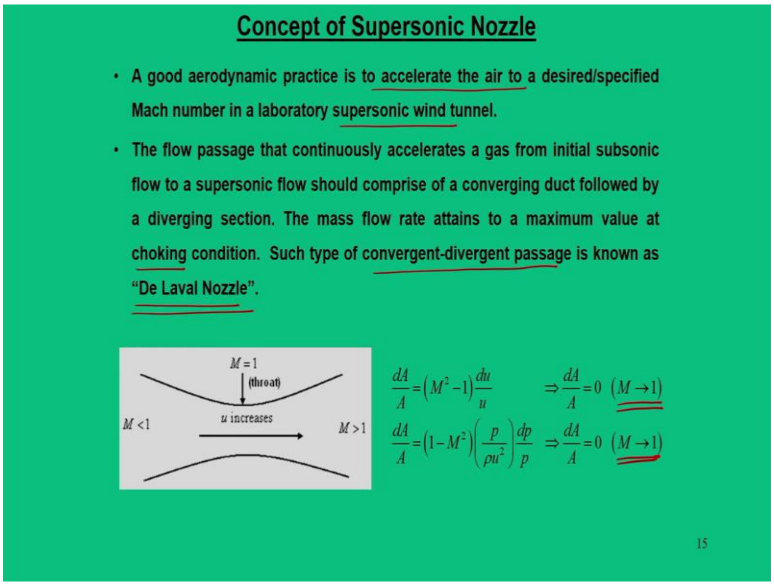

So, this leads to this particular concept of supersonic nozzle. What it says? This supersonic nozzle has a very good laboratory tool in subsonic wind tunnel. What for it is used? It is used to accelerate the air and the how does it does?

That the flow passage that continuously accelerates gas from a subsonic to supersonic value should comprise of a converging duct followed by a diverging sections and the mass flow rate attains to a maximum value which is known as choking conditions. And at this choking condition, the Mach number attains to sonic by satisfying this equations. And such a converging diverging passage will say is a De Laval Nozzle and in fact, this De Laval Nozzle we typically call it has a supersonic nozzle.

So, the supersonic nozzle flow means that a subsonic flow can be accelerated to a supersonic value.

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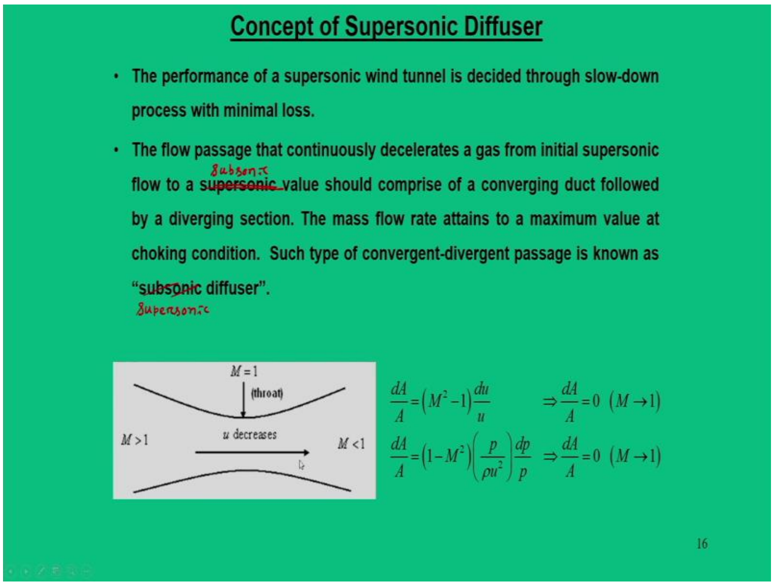

The reverse situation that occurs that means, in most of the wind tunnel applications, when you first we increase the flow to a supersonic value, but we must slow down the process with minimal loss; that means, when we release that gas to the exit atmosphere, it should be such that it should encounter minimum loss. So, to have this minimum loss, we cannot dump that high speed air just like into the atmosphere.

So, we have to decrease the velocity that is where the concept of diffuser comes in. So, we call this has a concept of supersonic diffuser. So, what it does is? It just does the exactly the opposite function of nozzle; that means, here the flow passage, the gas continuously decelerates from its initial supersonic value to a subsonic value. And, it comprises of converging duct followed by diverging sections.

So, I will read out a most efficiently in the way that the flow passage that continuously decelerates the gas from initial supersonic flow to a subsonic value should comprise of the converging duct followed by diverging sections.

So, in this process, the mass flow rate attains to a maximum value that is choking condition at the throat and such a type of convergent divergent passage is known as supersonic diffuser.

So, when I use a word supersonic diffuser, it means that my inflow condition is always supersonic. When I say supersonic diffuser, your inflow condition is supersonic. So, along the direction, the velocity must be subsonic.

So, with this I will conclude this talk for the today. So, in the subsequent analysis, we will talk about the more details of the choking conditions which happens exactly at the throat.

So, thank you.