Fundamentals of Compressible Flow
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Module - 05
Lecture - 14
Expansion Waves and Oblique Shocks- I

I welcome you all for this course again. This course is on Fundamentals of Compressible Flow, in the last few lectures we covered up to module 4. Now, we are in the module 5, in which we will start a new topic that is Expansion Waves and Oblique Shocks.

In all previous modules if I just give you a glimpse of that, then what I can say is that we have understood about a compressible flow and how a subsonic flow is different from a supersonic flow, how the formation of sound waves and shock waves comes into existence. And whatever we covered so far it was based on the analysis with respect to one dimensional framework.

Now, we will move to tell about the dimensionality of this flow, we will bring another dimension into the system. So, what we are going to discuss on expansion waves and oblique shocks, those are in two dimensional in nature.

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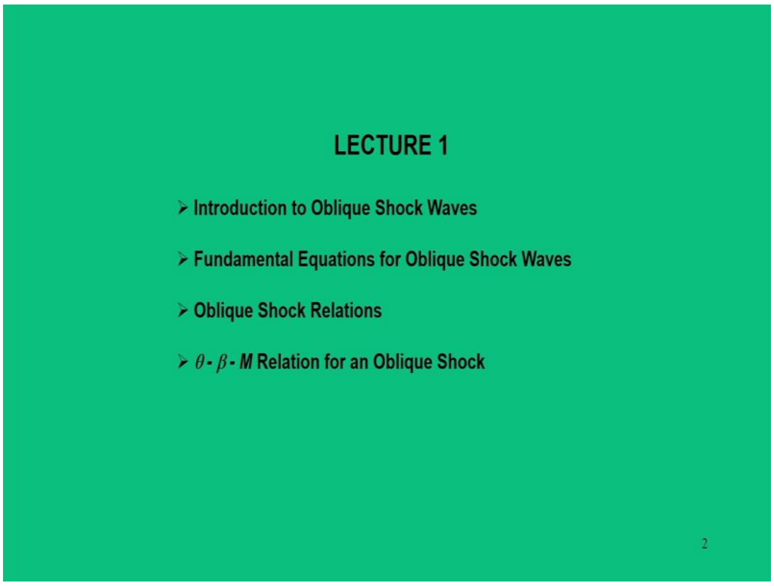

So, in this first lecture, I will introduce what is an oblique shock and how it is different from a normal shock or what is the resemblance between a normal shock and oblique shock. Then like in other sections, here also we have to derive the fundamental equations for oblique shocks. Using those equations, we are going to find out the oblique shock relations that essentially correlates between the flow properties across an oblique shock.

Now, since we have brought another dimension into the system like we are moving away from one dimensional system. So, in that aspects, since the oblique shock is analyzed in two dimensional systems; so we have to bring out another concept that is called θ-β-M relations. The terminology I will explain later, but in this terminology what I can say, θ is your flow deflection angle, β will be the shock wave angle and M is the Mach number of the flow. In fact, this is one of the vital relations that is mandatory requirement for oblique shock analysis.

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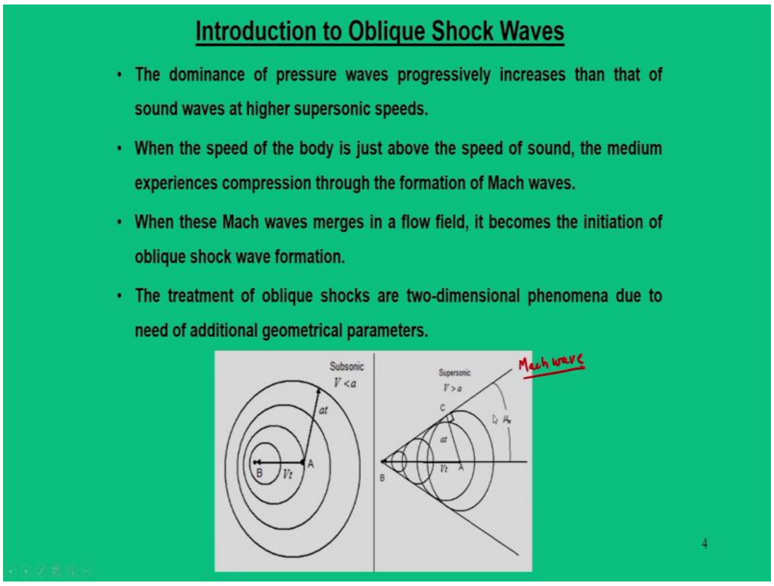

Now, let me start this topic. So, like in other situations, we have introduced the topic and something of that short if I just repeat those things what I can say; like in the beginning of our course discussion, we started the distinction between a subsonic flow and supersonic flow. And the subsonic flows are characterized when the velocity of the gas or body is less than the speed of the sound whereas the supersonic flow we say that velocity of the body is higher than the speed of sound.

Now, when I say speed of sound, this is the minimum or slowest possible pressure wave that is generated in a medium and it carries the entire information of the flow. So, based on these we say that, when we are in the subsonic regime, all the information about the medium is said everywhere. But such a privilege you do not have in a supersonic flow, because the body itself moves at very high speed and this location of the body is always away from the sonic circle.

So, the information is never propagated outside the sonic circle. So, based on that, we defined a cone which is called as Mach cone and it makes an angle Mach angle that is μ_m. In fact, this part we have discussed exhaustibly; but what the very basic point is or catchy point is that this is the Mach wave and that gives the initiation of an oblique shock formation.

So, for example, if your speed is such that the pressure disturbances create waves which are stronger than Mach waves, then it creates a oblique shock. And here now our analysis or treatment of oblique shock will be a two dimensional phenomena, because there is a need of additional geometric parameters. So, in this case we will define another angle which is called as shock angle, that is the requirement here and this shock angle is always higher than this Mach angle μ_m. So, we will discuss those things later.

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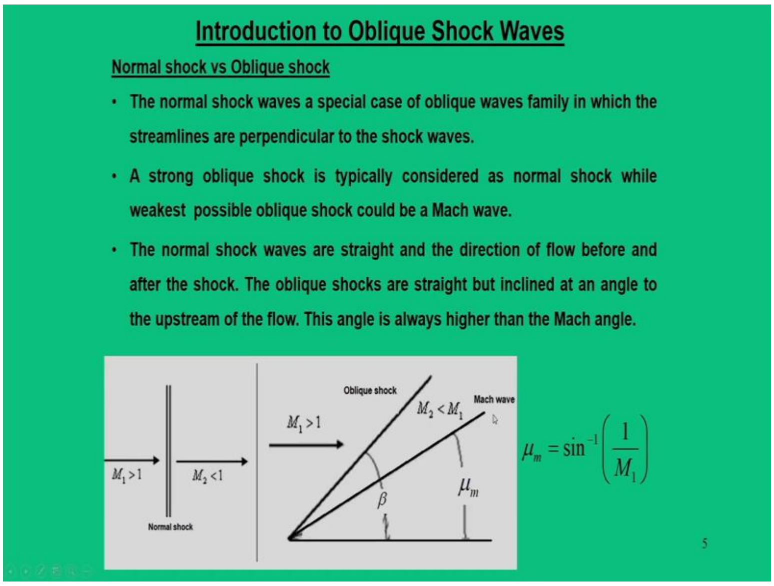

Now, just to give a brief a distinction between a normal shock and oblique shocks; what I can say is that, normal shock waves are special case of oblique shocks family in which stream lines are perpendicular to the shock waves. In general, the shock wave initiation start with a Mach wave which is a two dimensional phenomena. And one particular situation, the shock wave is normal to the flow; that means the direction of the stream line and the shock waves, they are perpendicular to each other.

So, it is one particular instant we say that the analysis we did through a normal shock analysis, and in fact these are straight. So, normal shock is a special case for oblique shock. So, in natural, actually oblique shocks are formed and in certain situations the flow conditions are such that, it is as if the shock wave is normal to the flow. So, that particular treatment we call this as a normal shock, but all other situations you treat it as a oblique shock.

A strong oblique shock is typically considered as a normal shock and the weakest possible oblique shock could be a Mach wave. So, it means that when the Mach wave becomes strong, it starts forming oblique shocks and if strength of the oblique shock increases, then it becomes a normal shock.

So, the normal shock waves are straight; the direction of the flow is straight before and after the shocks. But the oblique shocks are straight, but inclined to an angle to the upstream of the flow. That means, if you say this is the shock waves and this is the flow directions upstream; the shock wave is sitting at an angle β, so you call this as a shock wave angle. And had there been a Mach wave, there would have been a Mach angle defined by mu M that is nothing, but μ_m is equal to $\sin^{-1}\left(\frac{1}{M}\right)$.

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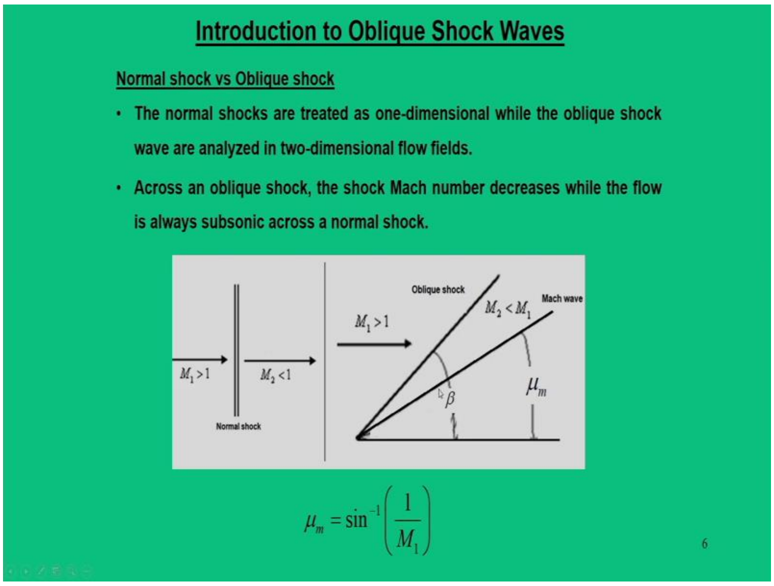

So, I also mentioned that the normal shocks are treated as one dimensional, whereas oblique shock analyzed in two dimensional flow fields. So, across an oblique shock, the shock Mach number decreases. So, what you see in a oblique shock; so, if you have an oblique shock, the Mach number after the shock wave decreases; but whereas in this case, the Mach number always remains subsonic. So, not necessarily in an oblique shock, Mach number will be subsonic; but this Mach number value will be less than the upstream Mach number.

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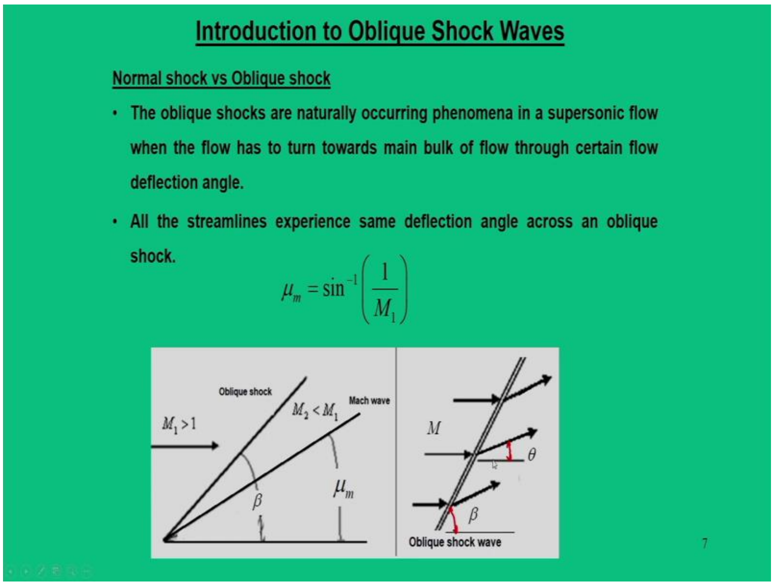

The most important phenomena of an oblique shock is that, this is a mechanism across which the flow can turn. So, what happens in a normal shock, flow does not turn; because this streamlines before the shock and after the shock they are perpendicular to each other, so they do not turn.

But in a oblique shock; that means, if you want to turn the flow in certain conditions and we have to think about an oblique shock that can turn to a particular angle to the flow. So, this is an another significant features of an oblique shocks, these are naturally occurring phenomena in a supersonic flow when the flow has to turn towards the main bulk of the flow. So, means, if we refer to this figure, there is an oblique shock sitting on to this; the flow Mach number is M that is upstream of the flow and at the upstream the flow is almost parallel to the horizontal.

But what happens across this shock wave, the flow gets turn. Why it is turn? Because such is the correlation between Mach number, β and θ is such that the flow has to turn to an angle. So, that is known as flow deflection angle θ. And this shock wave is making an angle β with horizontal.

So, β is your shock wave angle, M is your upstream Mach number and θ is the flow deflection angle. So, now next point is that it turns such that the flow turns towards itself; which means that initially the flow direction was horizontal and again the flow gets deflected after the shock in such a way that its main bulk or main part of the flow turns towards the main flow path or what you say towards itself.

And in fact when you analyze this oblique shocks, the flow deflection is not only limited to one particular stream line. In fact, it also applies for all this stream lines across the oblique shocks that means the entire slug of mass gets turned towards itself.

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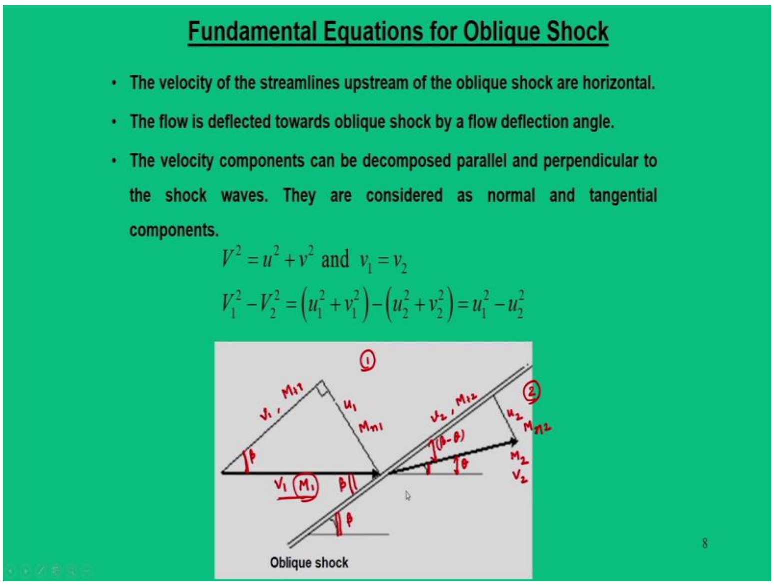

Now, we are going to analyze the fundamental equations for an oblique shock. So, to study this fundamental equations of oblique shock what we can say, we have to consider an oblique shock; this oblique shock is sitting at an angle β with respect to horizontal. And this is one particular streamlines which has a Mach number M_1, velocity V_1 and it after it crosses the oblique shock, the flow turns; that means stream line turns to an angle θ.

So, when it turns the final Mach number, we can write as M_2 and the flow velocity becomes V_2. So, what I can say is that we want to analyze this particular information. Since we bring this concept of θ here and this stream lines are not in same directions as that of normal shock. So, what you have to analyze; the streamline profiles with respect to their component analysis.

So, what I can say is that this velocity component which is the absolute velocity has two parts; one is perpendicular to this flow what I can say it is u_1 and other part is parallel to this shock wave. One is the perpendicular to the shock wave; other is the parallel to the shock wave that is v_1. So, this I assign as region 1, this I assign as region 2. So, similarly for this case also, we can say one has to be perpendicular that is u_2; other would be parallel that is v_2.

So, when I say this. So, I can frame this equations like $V^2 = u^2 + v^2$. So, these are the components that are perpendicular to the shock wave and parallel to the shock wave. So, one can write down this equation for region 1 and region 2, so that we can find out $V_1^2 - V_2^2 = \left(u_1^2 + v_1^2\right) - \left(u_2^2 + v_2^2\right)$. So, later on we will show that the components v_1 and v_2 will be equal for this oblique shock situation.

In fact, that means in other words the tangential components do not contribute anything for the analysis point of view. Once after doing this; similarly corresponding to the velocity if you have Mach number M_1, we can also write is normal Mach number M_{n1} and we can also write tangential Mach number M_{t1}. Similarly for Mach number M_2, we can write normal Mach number component M_{n2}, tangential Mach number component M_{t2}

So, these terminology you were going to use exhaustibly for this analysis. And in fact, when you say this is θ and this particular angle would be β; so what I can say is the difference between this particular angle will be β-θ. So, these are the certain geometrical information. If this angle is β, I can also say it is also this particular angle will be β and this particular angle will be also β. All these information are simply the geometrical representation or geometrical correlations based on which one can frame.

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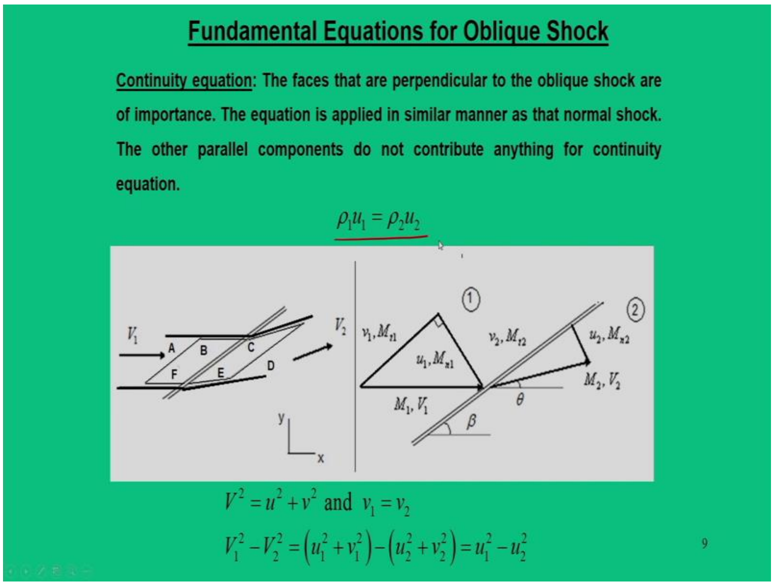

Then we will start the equation analysis one by one. So, as we do it for all other situations, the first equation that is the continuity analysis. So, when I say continuity analysis, it talks about the total mass flow rate remains constant. So, here we can say that the entire shock wave we can imagine to be an imaginary adiabatic duct containing the streamlines, and both inflow and outflow have equal area that can be analyzed in this framework as shown in this figure.

So, when we see this one particular view; if you say that this is the passage of the flow or upon particular plane and in this particular plane, the flow comes and turns across an oblique shock. Now, I can say that looking at this figure we can see the flow encounters almost six faces A, B, C, D, E, F. And with respect to flow direction, the face A and face D are perpendicular and with respect to flow direction the face B, C, E and F are parallel

So, when I say the parallel components with respect to velocity vector, their dot product will be zero. So, in other words what I can say, the component of velocity that are parallel to this face has no role; because they do not contribute anything to the continuity equations, but those components that are perpendicular to this face, they only contribute in the continuity equations.

So, the continuity equation is written only for one normal component like $\rho_1 u_1$, and since area remains same in both region 1 and region 2. So, this gets canceled. So, the continuity equation that remains is $\rho_1 u_1 = \rho_2 u_2$. And with same nomenclature, I have I have written all those terminology in this fashion.

So, this is the first fundamental equations that we get from the continuity equations. In fact, this particular equation we shall be using it in for all subsequent situations.

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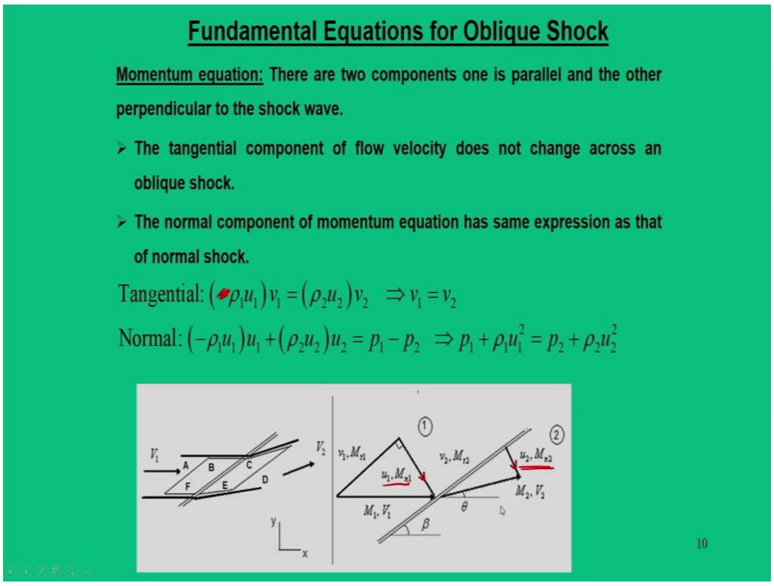

Now, moving to these momentum equations; since it is also a vector, so we also can say, there will be two components; one is tangential components, other will be normal components.

And from this analysis what I can say that, looking at the tangential component like v_1 and M_{t1}; that is both for region 1 and region 2, they are equal and opposite. Because, why equal? Because already you know that the mass flow $\rho_1 u_1$ and $\rho_2 u_2$ they are same; so that means $-v_1 = v_2$,

So, v_1 and v_2 they are equal. So, $(\rho_1 u_1)v_1 = (\rho_2 u_2)v_2$. Since $\rho_1 u_1$ and $\rho_2 u_2$ are same from the continuity equations; so you can say $v_1 = v_2$. So, in other words we can say tangential component of flow velocity does not change across this oblique shock; but what changes is the normal component. So, when the normal component has same expression as that of normal shock. So, what I can say, if I assume this to be a normal shock; then correspondingly the flow should see is the M_{n1} and u_1.

So, these are the two normal components that the flow should see. And similarly that is before the shock, and for the after the shock, the flow that will see as normal shock is u_2 and M_{n2}. So, as if we can say this is an oblique shock; but it is seeing a component of the main flow which is perpendicular to it.

So, in other words what we can analyze that as if this oblique shock can be treated to be normal shock with respect to a effective Mach number M_{n1} and upstream and effective Mach number M_{n2} downstream. So, this analysis is to be used as if it is a normal shock situations.

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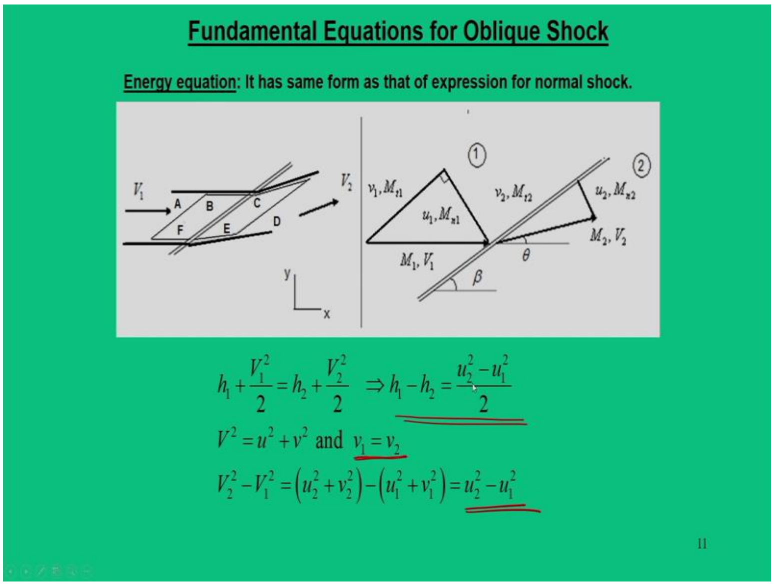

Again moving further, we can go to the energy equations that is for the completeness. So, and it is the same form that the form as that of expression for the normal shock; why? Because if you recall our energy equation that is $h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$. So, you can find out h_1 and h_2; but $h_1 - h_2 = \frac{u_2^2 - u_1^2}{2}$.

So, why you say this? Because the we all already proved that, the $V^2 = u^2 + v^2$ and we know that the tangential component do not change, that is $v_1 = v_2$. So, $V_2^2 - V_1^2 = u_2^2 - u_1^2$.

So, if you look at this particular equation, this is nothing, but the same equation as you did it for energy equation for normal shock. So, only thing that we have to take the component of flow velocity.

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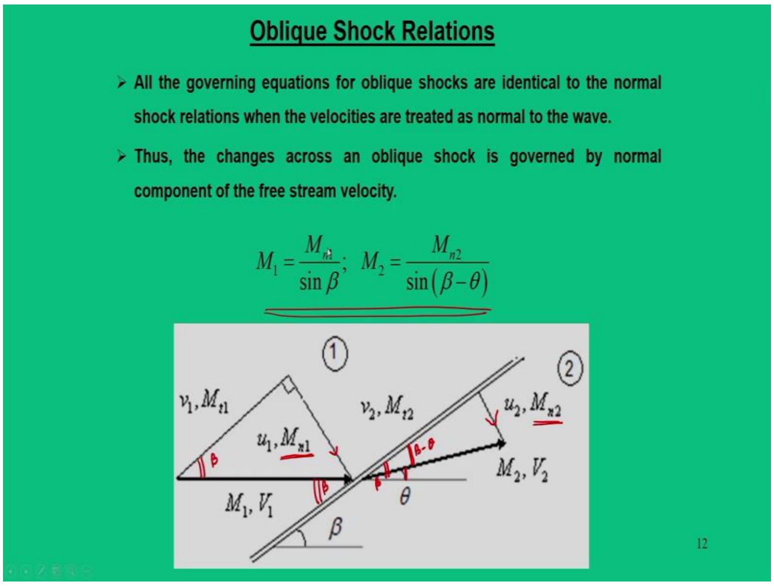

Now after saying this, so we have analyzed continuity equation, momentum equation and energy equations. So, as I mentioned that we are only looking at effective Mach number M_{n1} and M_{n2} now; treat this particular oblique shock as a normal shock, in which it sees an effective Mach number M_{n1} upstream and it sees effective Mach number M_{n2} downstream.

But then what is that M_{n1}? So, M_{n1} and M_1 can be related through a shock angle β. So, from this geometry of the figure what I can say; if I say this angle is β, I can say this is also β and from the geometrical analogy we can say, this angle will be also β.

So, what we can say? $\sin\beta = \frac{M_{n1}}{M_1}$. Similarly this angle would be β that is shock wave angle, the flow deflection angle is θ; but difference in these two angle will be β-θ. So, $\sin(\beta-\theta) = \frac{M_{n2}}{M_2}$.

So, this particular lesson is very vital in this oblique shock analysis. Why? Because in a given problem, you will be given a obsolete Mach number M_1; but you also need to know β and θ, so that we can get the normal components M_{n1} and M_{n2}. Once we know this normal component M_{n1} and M_{n2}; so this oblique shock is treated as if it behaves a normal shock. And entire analysis were subjected to the fact that it sees there is an effective Mach number M_n, M_n which is normal to the shock wave.

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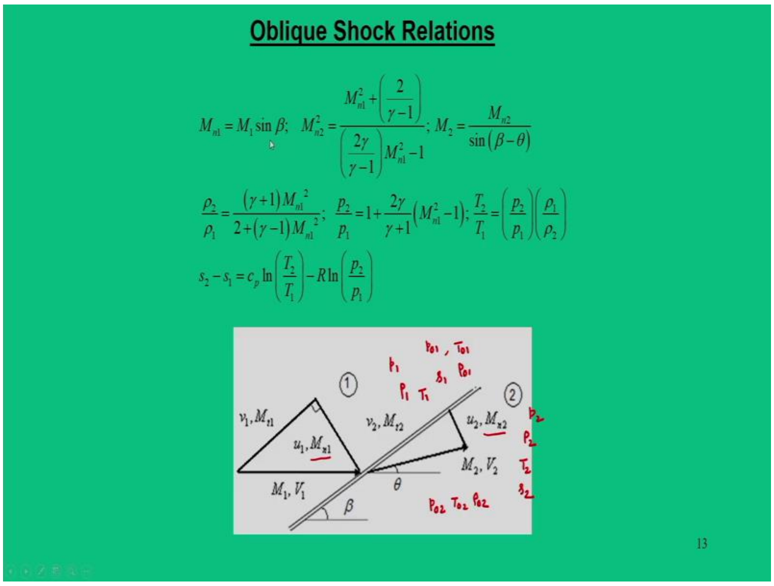

So, when we do this, then we can rewrite all the property relations; property relation I can say, pressure p_1, pressure p_2, density ρ_1, density ρ_2, temperature T_1, temperature T_2, entropy s_1, entropy s_2. In fact, I also can say p_{01}, T_{01}, ρ_{01}; also you can have p_{02}, T_{02} and ρ_{02}. So, all these properties can be entirely calculated based on the fact that as if this is treated as a normal shock with shock Mach number M_{n1} before the shock and M_{n2} after the shock.

So, if you look at these relations $\dfrac{\rho_2}{\rho_1}$, $\dfrac{p_2}{p_1}$; what it carries is the M_{n1}, instead of M_1. So, in earlier normal shock relations, these were replaced with M_1; but here it is replaced with M_{n1}, and this M_{n1} is we can find out from M_1 using the shock angle.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{n1}^{\;2}}{2+(\gamma-1)M_{n1}^{\;2}}; \quad \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n1}^2 - 1\right)$$

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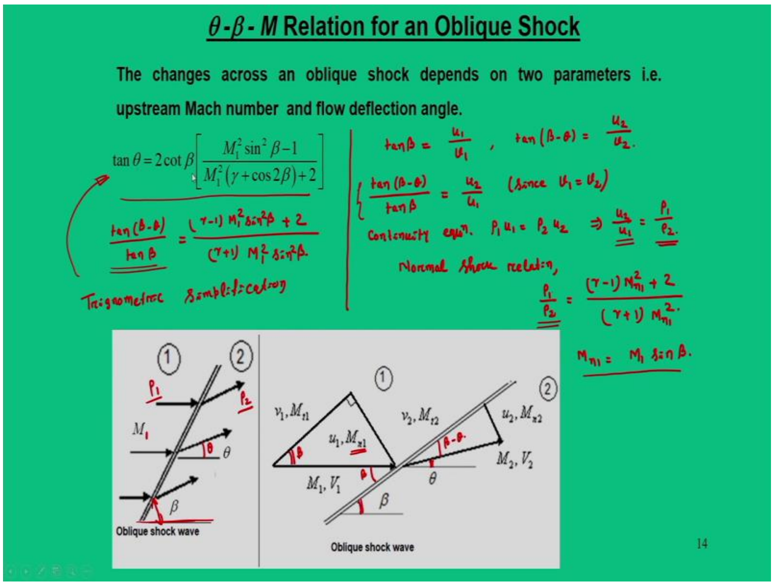

Now, the question that remains again that, we require M_{n1}; but to calculate M_{n1}, we need to have the Mach number M_1, β that is the requirement at the region 1, upstream of the flow and to get one of the information; to get M_2, we also require θ. So, without the knowledge of this θ, β and M_1; we cannot find out all this properties. So, our main philosophy that we have to find out M_{n1}, then only we can compute all the property ratio.

To do that, one important relation that was normally derives; we call this as θ-β-M relation. To analyze this θ-β-M relation, what I can say the terminologies like we have to say that oblique shock wave angle is beta, flow deflection angle is theta and upstream region your Mach number is M_1. So, with this terminology we have to find out. So, the changes of an oblique shock depends on two parameters, that is upstream Mach number and flow deflection angle.

So, we need these two information. So, this particular relation that is $\tan\theta = 2\cot\beta\left[\frac{M_1^2\sin^2\beta - 1}{M_1^2\left(\gamma+\cos 2\beta\right)+2}\right]$. So, this is this relation is known as θ-β-M relations. Let us see how we can derive this relation. So, to derive this relations, we have to again recall our analysis, geometric analysis; if this angle is β and this angle is θ, then we can say this angle will be β-θ. Now again if this angle is β, I can say this angle will be β and this angle will be β.

So, what I can write that, $\tan\beta = \frac{u_1}{v_1}$. And $\tan(\beta-\theta) = \frac{u_2}{v_2}$. Then we can find out, what is the $\frac{\tan(\beta-\theta)}{\tan\beta}$? So, $\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1}$, since we have proved that v_1 is equal to v_2.

Now, when I say continuity equation, I can write $\rho_1 u_1 = \rho_2 u_2$. Now, we require $\frac{u_2}{u_1}$, so I can write $\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}$. Now, this $\frac{\rho_1}{\rho_2}$ is nothing but densities across the shock wave 1 and 2, and if you treat this as a normal shock, so as if the component of M_{n1} comes into picture.

So, by doing normal shock analysis; so, what I can write this $\frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_{n1}{}^2}{(\gamma+1)M_{n1}{}^2}$. Now we know what is M_1, $M_{n1} = M_1 \sin\beta$.

So, we know all these equations, we know $\frac{u_2}{u_1} = \frac{\rho_1}{\rho_2}$. So, once we put this, this particular equation in this main expression; then what I can write $\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{2+(\gamma-1)M_1{}^2\sin^2\beta}{(\gamma+1)M_1{}^2\sin^2\beta}$.

So, then we have to expand this, do trigonometric simplification; of this equation we will take you to this particular expression which is nothing but the θ-β-M relations. So, what this relation says that, the tanθ which is the flow deflection and β is your shock wave angle and M_1 is the upstream.

So, one of the parameter is required. So, any two of the parameter will tell you what is the shock wave angle. In general in generic flow field, normally your M_1 is given, that is inflows Mach number is given and we say that flow deflection is angle is given; because why we say that normally a oblique shock is generated by putting a surface such that it the flow gets deflected through an angle. So, in a given situation, normally M_1 and θ are given. So, using this expression, you can calculate what is β.

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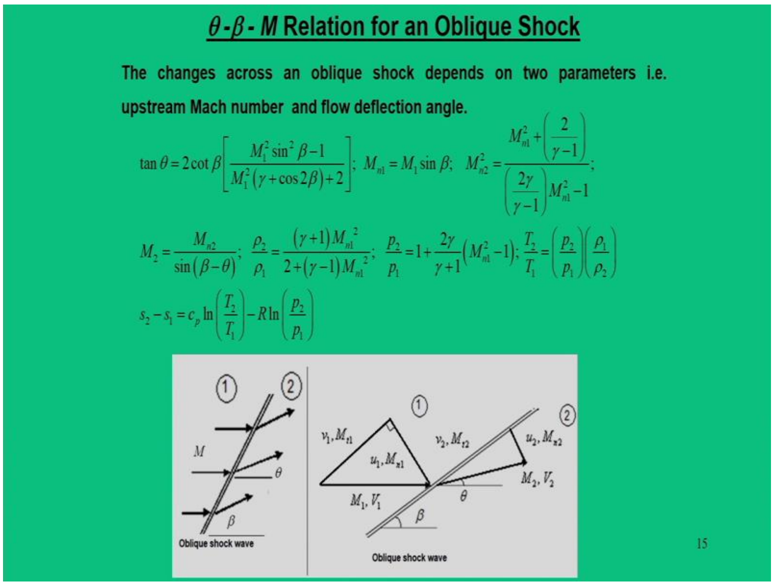

Now, this is how I can summarize that, how do you treat this oblique shock analysis. So, first thing is that, θ-β-M relation in addition to this normal component calculation will take you to all the flow parameter estimation across a oblique shock.

So, the standard procedure is like this; in a normal situation you will be giving M_1 and we will be given the flow deflection angle. So, from M_1 and flow deflection angle, one can calculate this shock wave angle β. Once the shock wave angle is known the normal component of the flow Mach numbers are known M_{n1}.

So, when M_{n1} is known. So, correspondingly prandtl relations we can say, because this is normal correspondingly component M_{n2} is also known. So, we know M_{n1} and M_{n2}, then we can find out M_2, we can also find out M_2 from M_{n2}.

So, flow Mach number at the downstream is known. All other property relations $\frac{\rho_2}{\rho_1}$, $\frac{p_2}{p_1}$ and $\frac{T_2}{T_1}$, they are now governed with effective normal component of the flow Mach number that is with M_{n1}. So, this is how all the flow properties are controlled across an oblique shock.

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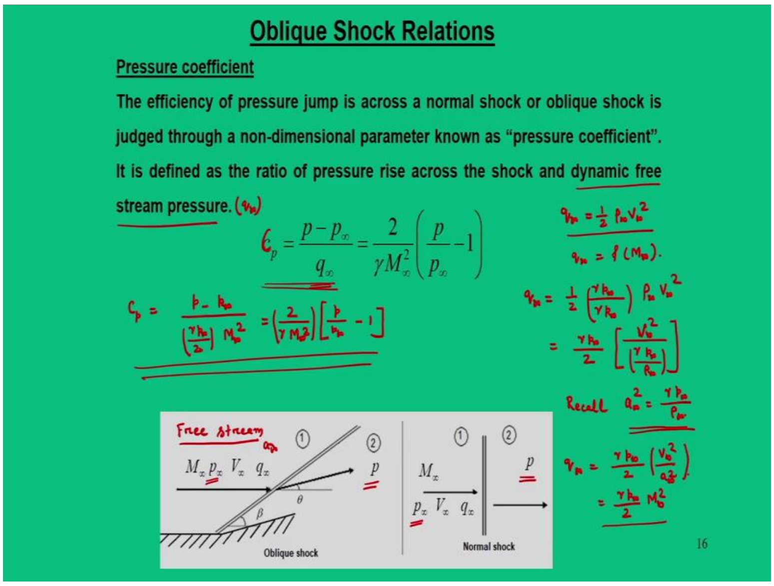

The next important topic is a term which we call as a pressure coefficients. So, as I mentioned in my previous lecture that a normal shock is treated as a compression device; because the static pressure always rises. And in fact across an oblique shock, the static pressure also rises.

So, if you look at a situation where we use a word which is called as free stream. Free stream means, a stream of flow which is moving at certain velocity and those are denoted by a putting a subscript as infinity. So, if I say free stream Mach number; so I can say M_∞. So, free stream means, if a body is travelling at certain altitude of pressure temperature; this is the corresponding altitude condition that will decide what pressure and what temperature it is going to encounter.

So, correspondingly we can say that if your body is moving at that certain speed, we can say its free Mach number is M_∞. And we will try to see that for a given free stream conditions; if a flow has to undertake an oblique shock or across a normal shock, how much pressure is going to change?

So, the efficiency of pressure jump across a normal shock or oblique shock is judged through a non dimensional parameter known as pressure coefficient. So, it is defined as the ratio of pressure rise across the shock wave to the dynamic free stream pressure. So, here I define this term as C_p. So, this is not specific heat, C_p that is pressure coefficient $p - p_\infty$.

So, p is your pressure after the shock, whether it is a oblique shock or it is a normal shock; p_∞ is the pressure corresponding to the free stream conditions, whether it is a oblique shock or normal shock, q_∞ is known as this dynamic free stream pressure. And this dynamic free stream pressure we normally write as, $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$ that is due to the velocity.

Now, this is how the pressure coefficient is defined. Now many situation what happens, all this parameters are not known; we are basically known with the flow free stream Mach number and pressure. So, you want to compute the pressure coefficient. So, we can write this q_∞ as this. We can now represent this q infinity as a function of M_∞.

How do you do this? So, what I can write is that, from this basic definition of q_∞, we can write $q_\infty = \frac{1}{2}\left(\frac{\gamma p_\infty}{\gamma p_\infty}\right)\rho_\infty V_\infty^2$. So, multiply on both side numerator and denominator as γp_∞; then what I can write $q_\infty = \frac{\gamma p_\infty}{2}\left[\frac{V_\infty^2}{\left(\frac{\gamma p_\infty}{\rho_\infty}\right)}\right]$.

Now, we can recall $a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}$, that is speed of sound at this location. So, when I put it. So, I can write $q_\infty = \frac{\gamma p_\infty}{2}\left[\frac{V_\infty^2}{a_\infty^2}\right]$, and that is nothing, but $q_\infty = \frac{\gamma p_\infty}{2} M_\infty^2$. Now, when I put this particular expression in from the main expression; so, I can rewrite $C_p = \frac{p - p_\infty}{\left(\frac{\gamma p_\infty}{2}\right) M_\infty^2}$.

So, if we can simplify this, I can $C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$. So, this is a expression we can now use for calculating pressure rise across a normal shock. In fact, this is an efficient term that are routinely used for normal shock as well as oblique shock analysis. And in fact, it talks about the effectiveness of pressure rise across a normal shock in some situations.

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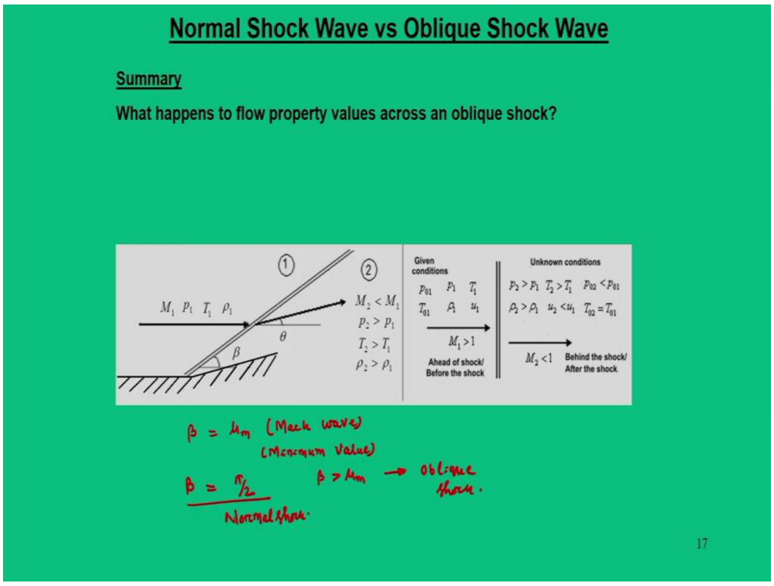

Now to some of what we have analyzed so far; I can say that the flow property across normal shock and oblique shock, to some extent they are same and in certain situation they are different.

So, what it means is that, when I say oblique shock; we require the additional information of β and θ because this flow field is a two dimension in nature. Now, when you say the strength of the oblique shock, it is mostly decided by the fact that what is the shock wave angle?

Now, one particular instant if I say β is zero, so there is no shock wave or the minimum value of β is a Mach wave. So, β starts with the Mach wave. So, what I can say, this may be the minimum value and it cannot be zero; but β can be as high as possible. So, it is π/2. So, when β greater than μ_m, we call this as an oblique shock and when beta is π/2, we call this as a normal shock.

Now, looking at the strength of the flow properties, we know that irrespective of the fact whether it is a oblique shock or normal shock, Mach number is always supersonic. But downstream of the flow in an oblique shock, Mach number is not necessarily to be subsonic; whereas in a normal shock, it is always subsonic. This is the first basic difference between these two.

And regarding property values like static pressure, static temperature, static density; both oblique shock and normal shock, they increase. But the rise or strength is higher for a normal shock; because oblique shock becomes stronger and stronger when β values goes to 90° or towards the normal shock. But, the strength of a normal shock is always higher than that of oblique shock; but the occurrence of a normal shock is very limited in a actual flow situations.

In most of the flow situation, it occurs with an oblique shock only. And in fact, in the actual flow field, we will find the normal shock is a very specific region where the flow field can be treated to be a normal shock. So, with this I will conclude for this lecture today.

Thank you for your attention.