Computational Fluid Dynamics for Incompressible Flows
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Lecture 4
Types of Error and Accuracy of FD Solutions

Hello everyone, so today we will discuss Types of Error and Accuracy of Finite Difference solutions.

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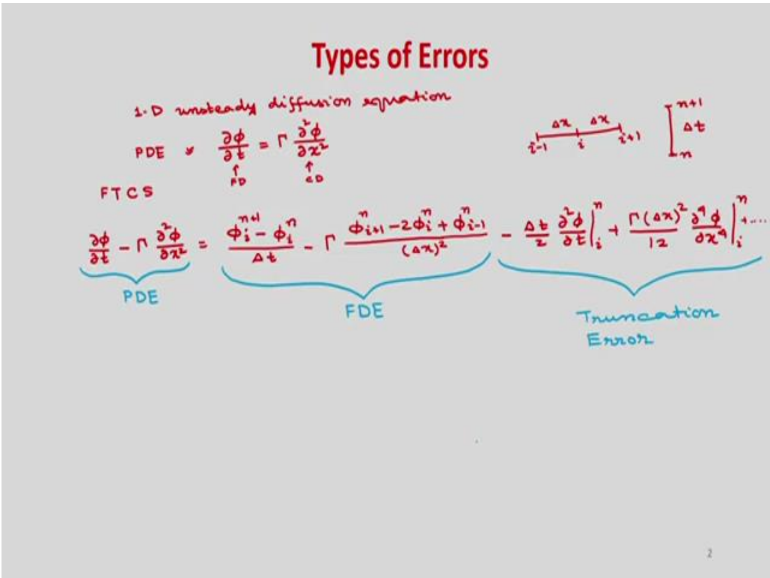

So, first let us considered 1 dimensional unsteady diffusion equation, 1 D unsteady diffusion equation. So, what is that equation? You know del phi by del t, we are writing for any general variable phi is equal to gamma, gamma is your diffusion coefficient and del 2 phi by del x square.

So, let us use some finite difference approximation to it, let us use forward time and central space. So, forward time and central space that means, the time derivative will use forward time and the spatial derivative will use central difference and this is your forward difference approximation will use, then obviously you can see that it is explicit method and order of accuracy is delta t delta x square.

So, if we use this finite difference approximation forward time and central space, then what we will get? So, you can write del phi by del t minus gamma del 2 phi by del x square and if we apply these finite difference approximation you are going to get, phi i minus by phi i, this is your n plus 1 and divided by del t.

So, this is time derivative minus gamma into the spatial derivative, we are using central difference, so it will be phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square and as it is explicit scheme, all these dependent variable will be at time level in. So, this is your i this your i plus 1, i minus 1 and this is your delta x, delta x and we are marching in time from n to n plus 1 with the time step size delta t. So, when we use this finite difference approximation will have other terms in the right hand side. So, what are the other terms which we neglected.

So, that if you write then you are going to get plus or you can write minus del t by 2 del 2 phi by del t square at point i, time level n, plus gamma del x square divided by 12, del phi by del x 4, at point i time level n, and other higher order terms. Because if you use Taylor series expansion and each derivative to represent with some approximation then you are going to get this equation.

So, if you see that whatever we have written here this is your partial differential equation, this is your PD. So, that you can represent so these as PD, partial differential equation. After discretization, whatever you have written here, so that is the finite difference approximation.

So, this is your finite difference equation, so this is your FD. So, whatever PD you have written, so use some finite difference approximation and after discretizing we got this finite difference equation and you are neglecting other higher order terms, these are you are neglecting the other higher order terms, and what is known as truncation error, already we have discussed, truncation error. So, you can see that this truncation error is the difference between the partial differential equation and finite difference equation.

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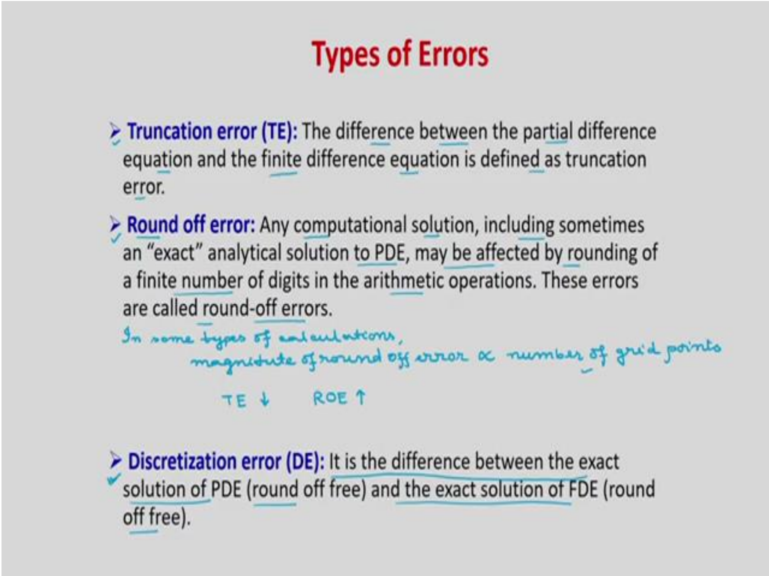

So, here we will discuss these errors on its truncation error, then round off error, then discretization error. So, truncation error already we have discussed and shown that that difference between the partial differential equation and the finite difference equation is defined as truncation error, and this you learnt earlier also.

What is round off error? Any computational solution including sometimes an exact analytical solution to partial differential equation may be affected by rounding of a finite number of digits in the arithmetic operation. These errors are called round off errors.

So, round off error, is the error caused by the approximate representation of numbered. So, when you are doing repetitive computation, arithmetic computation in the computer, so, it is rounding of the numerical numbers due to its precision and accuracy. So, due to that you are going to get some errors and that error is known as round off error. In some type of solutions, these round off error, magnitude of the round off error is proportional to the number of grid points.

So, some in some type of solutions, in some types of solutions, some types of calculations, magnitude of round off error is proportional to number of grid points. As number of grid points will increase your, step size delta x also will decrease and there will be increase in the round off error.

So, as number of grid points increases your round off error will increase, but your truncation error decreases, because truncation error obviously if you refine the mesh, then truncation error will tend to 0 that means it will decrease. However, your round off error will increase.

So truncation error decreases, but your round off error, round off error will increase in some types of calculations.

And what is discretization error? So it is the difference between the exact solution of PDE round off free and the exact solution of FDE round off free. So, you have the partial differential equation, if you can get the solution of exact solution of this partial differential equation round off free and you have the finite difference approximation, finite difference equation and if you have the solution of it round off free, the difference between these 2 is known as discretization error.

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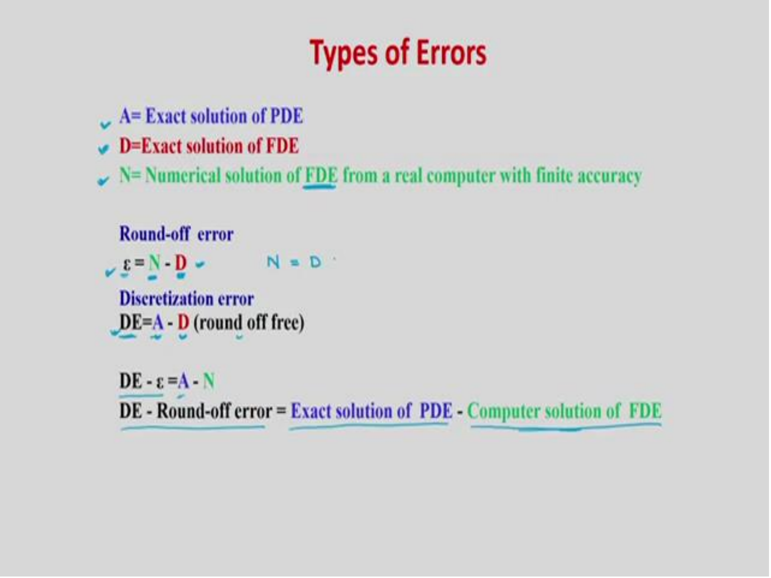

So, here if we see that, if A is the exact solution of PDE and if D is the exact solution of FDE. Whatever after discretizing the PDE whatever finite difference equation you got that exact solution if you represent as D and n is the numerical solution of finite difference equation from a real computer with finite accuracy, So, obviously, here round off error is included in this solution, so it is the numerical solution of the FDE, from a computer.

So, how you will define the round off error? So, round off error obviously the difference between the numerical solution of FDE from a real computer minus the exact solution of the FDE. So, that is your round off error. So, you can represent epsilon which is round off is N minus D, where N is the numerical solution of the FDE from a real computer with finite accuracy minus exact solution up FDE.

And discretization error now we can define as the exact solution of PDE, round off free minus the exact solution of FDE round off free, this is the difference between the exact solution of partial differential equation and the exact solution of finite difference equation round off free, so that is your discretization error.

So, now, this DE minus epsilon if you write, then DE minus epsilon is nothing but A minus N, because D D will cancel. So, A minus N that means discretization error minus round off error you can write as exact solution of PDE, which is A minus the computer solution of the FDE which is your numerical of FDE from a real computer. So, here you can see this is epsilon is equal to N minus D or numerical solution N, you can write as DE which is your exact solution of FDE plus epsilon round off error.

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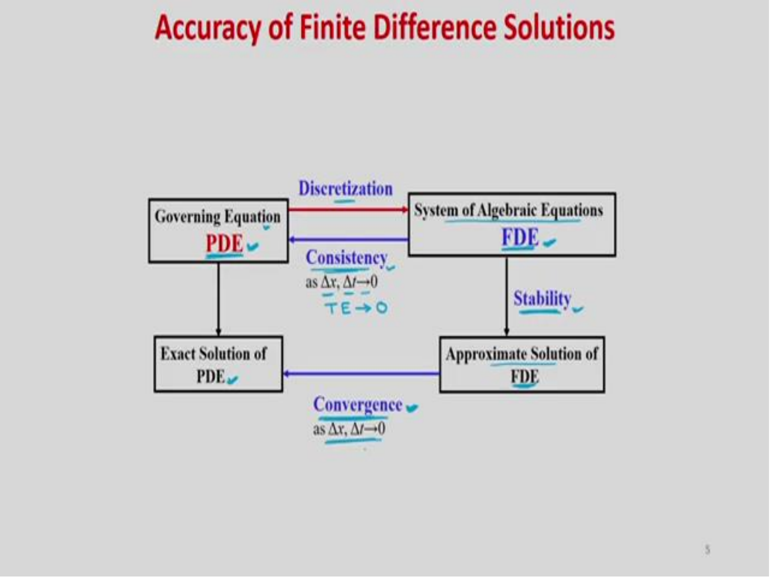

Now, we will discuss about the accuracy of finite difference solutions. So, our goal is the accuracy of solutions of partial differential equation. To achieve this goal, this finite difference equations should satisfied, three criteria, one is consistency, then stability and convergence.

So that will discuss now. So, you have governing differential equation, which is your partial differential equation, and what we do after that? We use some numerical scheme to discretize this partial differential equation. So, when you discretize using some method or scheme, then you will get system of algebraic equations which is known as finite difference equation. So, you had PDE, you use some discretization method, and you wrote the FDE, finite difference equation.

So, the scheme will be said as consistent if this finite difference equation approaches to the partial differential equation as grid is refined. That means, delta x and delta t tends to 0, then the scheme is known as consistent. So, if this finite difference equation approaches to the partial differential equation as your truncation error tends to 0. So, finite difference equation approaches to partial differential allocation as grid is refined and times step tends to 0.

So, what does it mean if your grid is refined and delta t tends to 0? That means your truncation error will tends to 0 that means the difference between the partial differential equation and the finite differential equation which is known as truncation error will tends to 0 that means truncation error will tend to 0. So, if it is a consistent scheme, the new truncation error will tend to 0.

Now after using discretization method, you have finite difference equation. So, if you solve it. So, although this what we are now describing stability that is mostly applicable for time marching problem. So, now when you have the finite difference equation, when you are solving it, you will get the approximate solution of FDE.

So, during the solution of this finite difference equation if the error remains bounded then the scheme is known as stable scheme that means with your time marching process the error will not grow. So, that is that then the scheme numerical scheme will be known as stable scheme.

Now, you have the approximate solution of the FDE. If you solve this PDE, and if you get the exact solution of PDE, then if your approximate solution of FDE approaches to the exact solution of PDE as grid is refined and delta t tends to 0 then your numerical scheme will be said as convergent.

So, that means, if that difference between the exact solution of PDE and the approximate solution of FDE tends to 0 as grid is refined then your numerical scheme will be said as convergent. So, consistency and stability are the prerequisite of convergence. The scheme should be consistent then it should be stable then only you will get the convergence.

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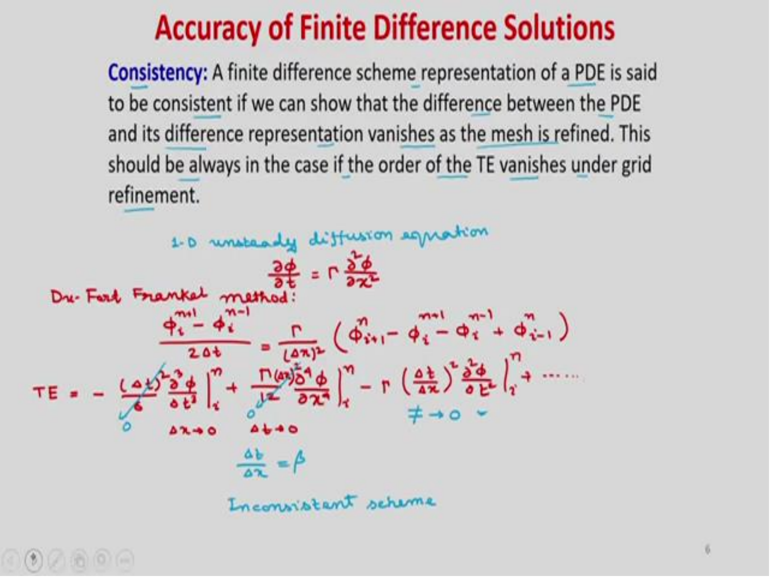

So in detail now let us discuss. So, consistency, a finite difference scheme representation of partial differential equation is said to be consistent, if we can show that the difference between the PDE and its difference representation vanishes as the mesh refined. This should be always in the case or case if the order of the truncation error vanishes under grid refinement. So, all the numerical schemes are not consistent, but add grid is refined and time step tends to 0 your truncation error should tends to 0, otherwise the numerical scheme will be said as inconsistent.

So one inconsistent scheme we will just discuss now. So we will consider the same equation 1 D unsteady diffusion equation. So, what is your governing equation? del phi by del t is equal to gamma del 2 phi by del x square. So, if you use Du Fort Frankel scheme which we will discuss later then it will be phi i n plus 1 minus phi i n minus 1 divided by 2 delta t 2 delta t is equal to gamma by delta x square phi i plus 1 n minus phi i n plus 1 minus phi i n minus 1 plus phi i minus 1 n. So, this scheme is known as Du Fort Frankel, this is your Du Fort Frankel scheme.

For this scheme, if you find the truncation error, then it will be truncation error, first few terms will be minus delta t square by 6 del cube phi by del t cube i n plus gamma by 12 del 4 phi by del x 4 i n delta x square and y will have minus gamma del t by del x square del 2 phi by del t square i n and other higher order terms.

So, now you can see here, that if your delta x tends to 0 and delta t tends to 0, then your this term will tend to 0 and this term will tend to 0, but your this terms will not tend to 0, why?

Because it is del t by del x, if this decreases in the same proportionate then del t by del x will be some constant beta, then this term will not go to 0.

So, you can see that Du Fort Frankel scheme when you apply to a 1 dimensional unsteady diffusion equation, it is a inconsistent scheme. So, it is a consistent scheme, because truncation error is not approaching to 0 as grid is refined and the time step tends to 0. So, the schemes would be consistent. As grid is refined and delta t tends to 0 your truncation error will tends to 0.

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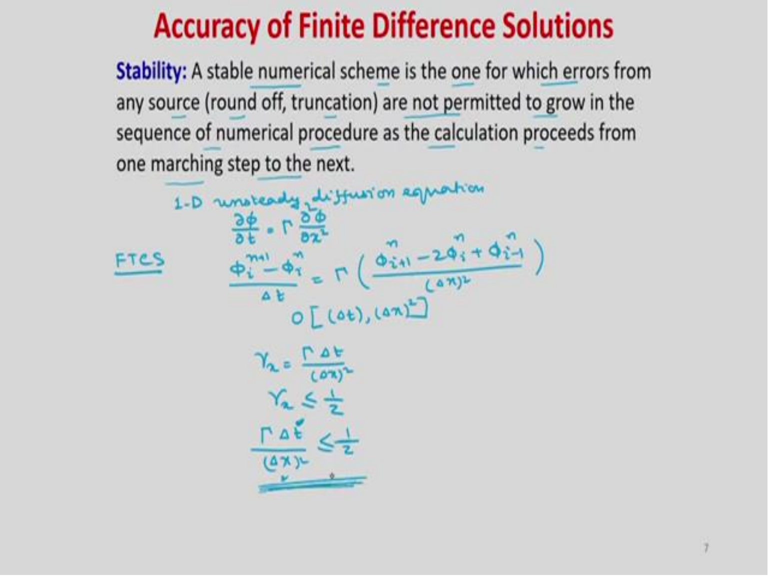

Stability. So, a stable numerical scheme is the one for which the errors from any source round off or truncation are not permitted to grow in the sequence of numerical procedure as the calculation proceeds from one marching step to the next. That means the error remains bounded. It should not grow as time progresses, then the numerical scheme will be said as stable and we will discuss later that for different schemes there are some stability criteria to choose the time step.

So, as you are considering 1 D unsteady diffusion equation and if we use forward time and central space method. So, if you have del phi by del t is equal to gamma del 2 phi by del x square. So, this is your 1 D unsteady diffusion equation. You if you use a FTCS method, then it will be phi i n plus 1 minus phi i n divided by del t is equal to gamma phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square at time level n, so order of approximation is your order of delta t and delta x square, so this scheme is stable.

If you do the Von Neumann stability analysis later will show that the scheme is stable if your gamma x which is gamma delta t by delta x square is less than equal to half. That means gamma delta t by delta x square should be less than equal to half then your error remains bounded. So, that means he your grid is fixed that means delta x is fixed, so you have to choose the delta t such way that these criteria will be satisfied, then the numerical scheme will be stable.

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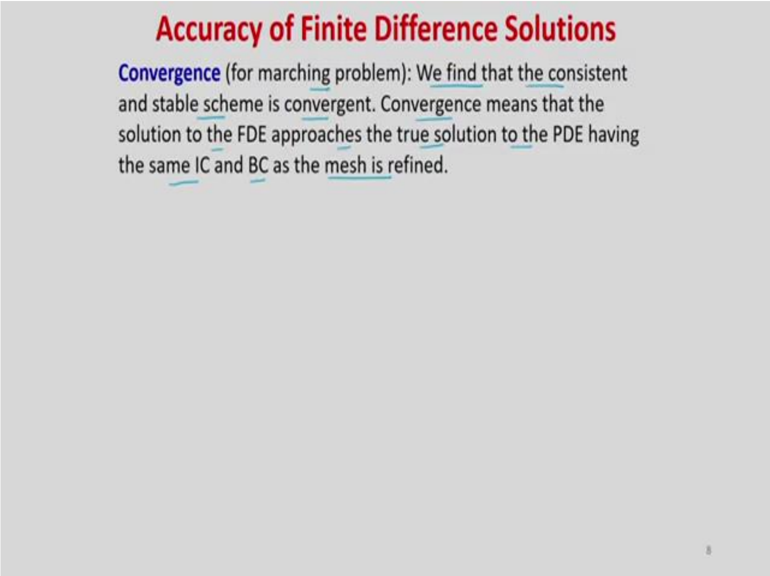

So, convergence, so it is for marching problem as we discussed. We find that the consistent and stable scheme is convergent. Convergence means that the solution of the FDE approach the true solution to the PDE having the same initial condition and boundary conditions as the mesh is refined. So, as mesh is refined and delta t tends to 0, the approximate solution of FDE should approach to exact solution of the PDE, then the numerical scheme will be said as convergent and we have already discussed that consistency and stability are prerequisite to convergence.

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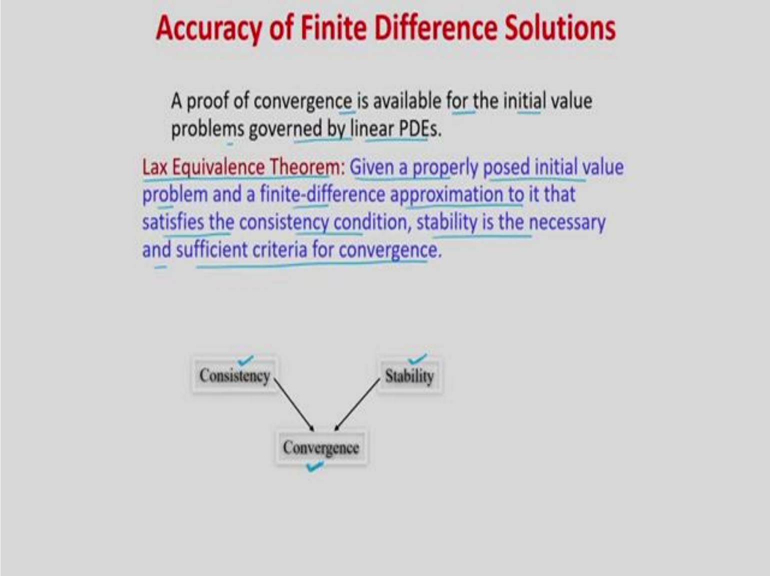

So, a proof of convergence is available for the initial value problems governed by linear PDEs. So, these theorem due to lax is stated here without proof. So, this is your known as this is known as lax equivalence theorem, which states that given a properly posed initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient criteria for convergence. That if you have a consistent scheme, if you have a stable scheme then anyway it will give convergence. So, this is known as lax equivalence theorem.

So, now we will write some difference operator just to represent the finite difference equations in a compact form. So, these are generally used in some books. So, that just will write down few difference operators and while discretizing the equation we will use some of the operators.

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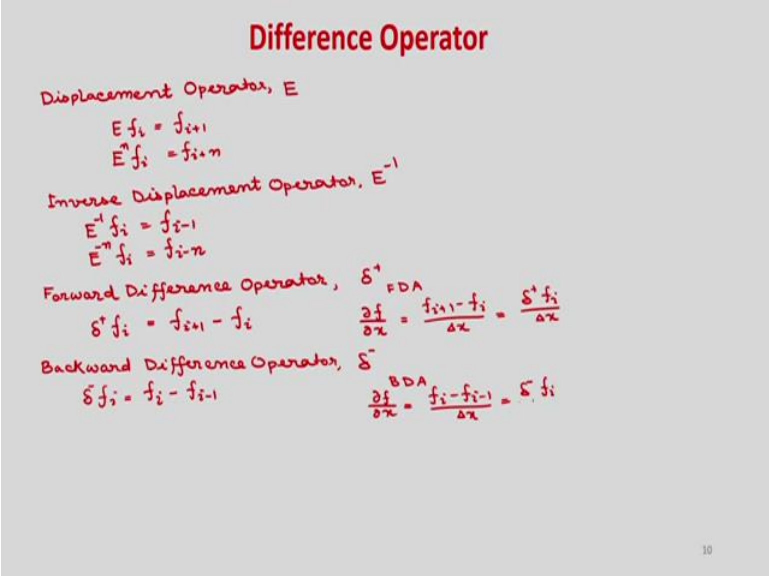

So, first we will use displacement operator, displacement operator. So, this is denoted by E, so if you apply to E to f i then you will get f i plus 1. Similarly, if you write E n f i, then you will get f i plus n. Similarly inverse displacement operator, inverse displacement operator. So, it is E inverse, so if you apply to f i then you will get f i minus 1. Similarly if you write E minus n f i then you will get f i minus n, so this is known as inverse operator.

Then we will discuss about forward difference operator, forward difference operator. So, this is denoted by delta plus. So, if you apply delta plus to f i, then you will get f i plus 1 minus a f i. So, this is a forward difference approximation or representation, so f i plus 1 minus f i. So, if you write say del f by del x, del f by del R1, if you use the forward difference approximation, forward difference approximation if you use, what you will write? f i plus 1 minus f i divided by delta x. So, that in compact form you can write as delta plus f i divided by delta x.

Similarly, backward difference operator, backward difference operator. This is denoted by delta minus and if you operate to f i. So, you will get backward difference that means f i f i minus 1.

Similarly, if you use the backward difference approximation of the first derivative del f by del x, then you can write as f i minus f i minus 1 divided by delta x you can represent it as delta minus a f i by delta x.

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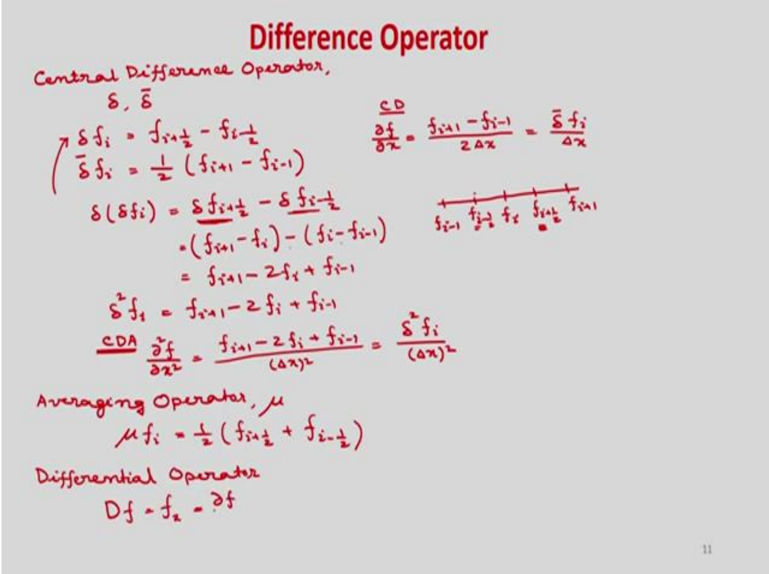

Then we will discuss about central difference operator, central difference operator. So, it can be denoted as delta and delta bar, where delta if you operate on f i then you will get f i plus half minus f i minus half, so this is a central difference and similarly, delta bar f i, if you write then you will get half of f i plus 1 minus f i minus 1.

And now if you write say first derivative, central difference if you write, central difference of first derivative del f by del x, what you write? f i plus 1 minus f i minus 1 divided by 2 delta x. So, it will be you can represent as delta bar f i by delta x and if you apply delta on this first relation then you will get delta on delta f i, if you apply then you will get delta f i plus half minus delta f i minus half.

So, if you represent you see, so this is your f i, this your f i plus half, this your f i plus 1, this your f i minus half and this your f i minus 1, if you apply delta on f i plus half. So, this is your f i plus half. So, in the f i plus half if operate this delta then you will get f i plus 1 minus f i, so this is your delta f i plus half. And delta f i minus half here.

So, if you operate it, central difference operator, then you will get minus f i minus f i minus 1. Then you can write f i plus 1 minus this f i and this f i, 2 f i, and this minus minus plus, so plus f i minus 1. So, that means you can write delta square f i is equal to f i plus 1 minus 2 f i plus f i minus 1. So, this is your central difference operator.

Now, if you use the central difference approximation for the second derivative, then how will I write? So, if you write delta 2 f by delta x square, then we represent the central difference approximation on this second derivative, then you will get f i plus 1 minus 2 f i plus f i minus

1 divided by delta x square, so that you can present at delta square f i divided by delta x square.

Now, averaging operator, averaging operator. So, it is represented as mu, so if you operate on f i then you will get half into f i plus half plus f i minus half and differential operator you already know, differential operator. So, this is your D on f if you write then you can write it as f x which is nothing but del f by del x. So, whatever differential operator we had we have discussed here, so there are some relations, so that let us write that you can derive and write few relations we will show it.

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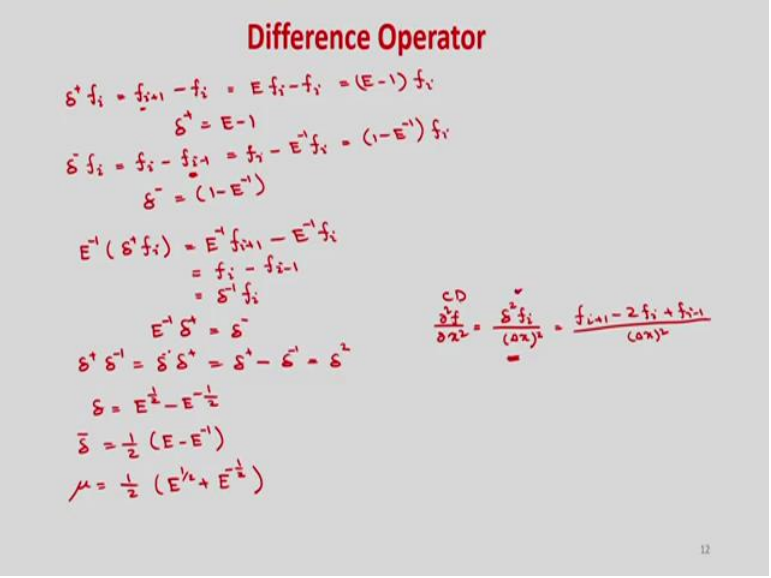

So if you operate the forward difference operator delta plus on f i then you will get f i plus 1 minus f I, and if you use the displacement operator, then what you can write? You can write as E f i minus f i that means you can write E minus 1 f i, so that means you can write delta plus as E minus 1.

Similarly, if you operate the inverse displacement operator on f i, then you will write f i minus f i minus 1. So, you can write f i and this you can use the inverse displacement operator then you can write E inverse f i. So, you can write 1 minus E inverse f i, so delta minus you can write 1 minus E inverse.

Now, let us apply the inverse displacement operator inverse on delta plus f i. So, we have delta plus f i, so there you are applying. So, you can write E inverse f i plus 1 minus E inverse f i, you can write it. So then you can write. So, E inverse f i plus 1, so what is that? , E inverse f i plus 1, so you will get f i, right, and E inverse f i you can will get f i minus 1.

So, that means, you will get delta minus f i. So, you can write E inverse delta plus is equal to delta minus, and you can show it that delta plus and delta minus will be delta minus delta plus is equal to delta plus minus delta minus is equal to delta square.

Similarly, you can show delta is equal to E half minus E minus half, and also you can show that delta bar is equal to half into E minus E inverse, and mu you can show that it is half into E half plus E inverse. So, these are some relations you can show it for few relations we have derived and shown. When we will discretize the equations we will mostly use that central difference operator, delta square.

So, you can write the central difference, so that is your delta square f by delta x square. So that you can write as delta square f i divided by delta x square. So, you can see that you can write in compact form, because your original original representation is this one, del x square, but in compact form you can write delta square f i by delta x square, so that in compact form if you write then it will be easy to write the finite difference equation, so for that is in we have just discussed this difference operator.

So, what we have done today? First we have discussed three different types of error, one is truncation error, which you learned earlier and it is the difference between the partial difference equation and the finite difference equation. Then we discuss about the round off error. So, it is the limitation of the computer accuracy, so, you get some round off error. Then we discuss about the discretization error, the discretization error is the difference between the exact solution of the PDE and the approximate solution of the FDE, so that is known as discuss discretization error.

Then we discussed about the accuracy of the finite difference solutions. So, there we discussed about the consistency, stability and the convergence. So, as your grid is refined and the time step tend to 0, if your finite difference approximation approaches to the partial differential equation, then the numerical scheme is known as consistent, when you are marching in time.

So, if the error remains bounded if it does not grow with time, then the numerical scheme is known as stable and if your numerical scheme is consistent and stable, then you will get the convergence. So, the solution of the approximate solution of the finite difference equation will approach to the exact solution of the PDE. So, and we discuss about the lax equivalence theorem. Thank you.