Computational Fluid Dynamics for Incompressible Flows

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Lecture 3

Finite difference in non-uniform grid

Hello, everyone, so today we will study Finite Difference in Non-Uniform Grid. So in earlier lectures we used uniform grid, and we used finite difference approximation of derivatives. But today we will use the non-uniform grid and we will write the finite difference approximation of derivatives.

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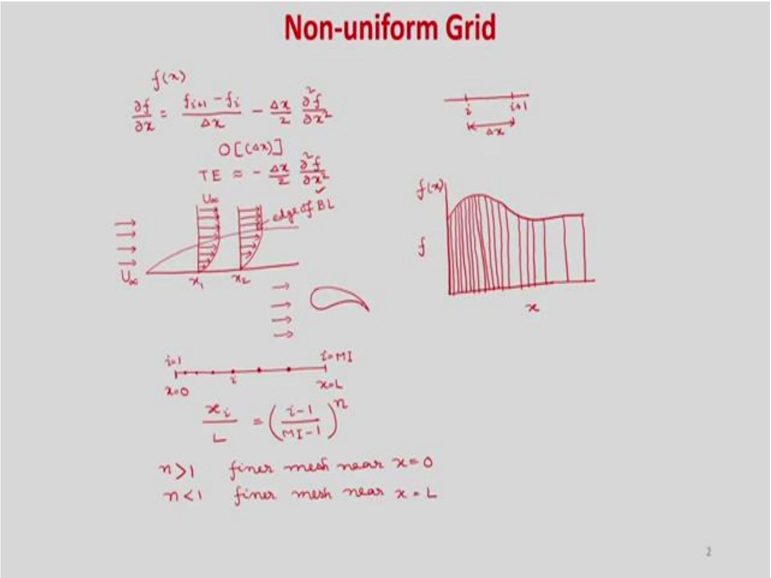

So, you can see, so first let us write the finite difference approximation of first derivative, in uniform rate. So, what is that? If you have a function f x 1, and you use del f by del x then if you have a grid this as i and this is your i plus 1 and this is your delta x, then what will be your finite difference approximation?

If you use forward difference then it will be f i plus 1 minus f i divided by delta x minus del x by 2, del 2 f by delta x square. So, what is the truncation error? And what is the order of accuracy? Obviously, order of accuracy is delta x here you can see, and the truncation error is this one but leading term is in truncation error leading term is minus delta x by 2, del 2 f by delta x square.

So, here you can see that this truncation error not only depends on the step size delta x, but also the derivative. So, first derivative of second derivative. So, obviously, you can see that if you have the flow variations in a domain, then the gradient will be different at different places, it is obviously, although if you use the uniform grid size or step size delta x but you will not get uniform error.

Hence, we need to use non-uniform grid, so we will use smaller delta X, where you have larger gradients in the domain and we will use higher delta x where you have smooth gradient in the region.

So, let us say you have a gradient, so let us say you have f x, which varies like this. So, this is your x and this is your f. So, you can see that in some region you have a higher gradient, this region you can see that you have a higher gradient.

But in this region you can see you have smooth gradient. So, to have the uniform error in that domain, you can use very small delta x where you have larger gradients that means here you have larger gradient, so you use very fine mesh where you have delta x is very small.

Now you can see in this region you have a smooth gradient. So, gradient will be smaller in these regions, so obviously you can use higher delta x, larger delta x you can use, so you can use, larger delta x. So, this is known as non-uniform grid, so you are using smaller delta x where you have larger gradients in the domain and you are using larger delta X, where you have smaller gradient in the domain. So, you will get a uniform error on the domain.

So, to use it now, you can see there are many examples in the flow physics, where you can see that near to the solid region obviously, we have higher flow gradients. Say velocity gradient will be mode. So, consider flow over flat plate, it is in undergraduate fluid mechanics course it is a you have learned this problem flow over flat plate and what is the boundary layer edge of the boundary layer will go like this.

So, you can see that what is the variation of the velocity? So, if you have a uniform velocity, free steam velocity. So, obviously outside this boundary layer age you will have the U infinity, so, there will be no variation in the velocity outside, but inside this boundary layer edge velocity is changing, so, you have mode gradients of velocity.

If you find that del U by del x it will be mode, in two different places you can see that inside the, into different x location, x 1 and x 2 inside the boundary layer you have mode gradient of velocity. So, obviously near to the solid region, you need to use mode finer mesh that means smaller delta x. So, that you can capture accurately this pro-gradients.

There are many other examples, so flow over a wing, flow over aerofoil, flow over aerofoil, you can see. So, obviously there will be boundary layer development on this surface and you can use very finer mesh near to the region, solid region.

Now, so, let us see how we can generate non-uniform grid first. So, you have this region, x equal to 0 and x equal to L, we want to generate finer mesh, let us say near to the x equal to 0 region and coarse mesh where delta x is larger near to the x equal to L region. So, if you find, so obviously there will be mode gradients and gradually it will vary.

So, you can see it gradually it is varying. So, these are the grid points. So, this is your i is equal to 1 and this is your I is equal to M I . In simple way you can find x i at any point i by L is equal to i minus 1 divided by M I minus 1 to the power n, you can use this relation to generate non-uniform grid. So, for n greater than 1, finer mesh you will get near x equal to 0, and n less than 1 you will get finer mesh near x equal to L, so this way you can generate.

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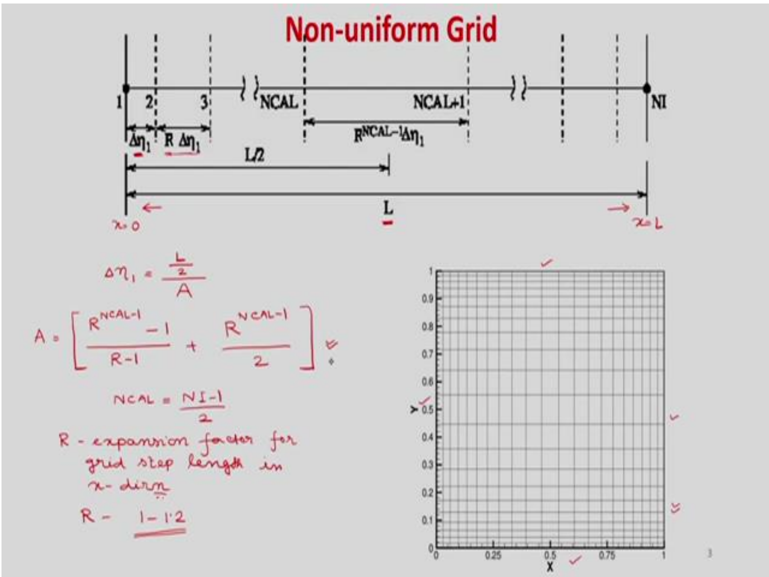

Here I have shown another methods. So, you have a x equal to, here x equal to 0 and this is equal to x equal to L, this is the domain and you want to generate non-uniform mesh, but you want to generate finer mesh near x equal to 0 and x equal to L. There are 2 solid boundaries at x equal to 0 and x equal to L.

So, you want finer mesh near to this solid region where x equal to 0 and x equal to L. So, you can use this expression where you can use delta eta 1, you can see delta eta 1 is the first grid point, this step size, then you use a factor R, constant factor R into delta eta 1 and the next is just R into delta eta 1.

So, that way if you write, then delta eta 1 you can write as L by 2, L is the total length of the edge deduction, so L by 2 is the half width divided by A and A you can calculate from this relation, R NCAL minus 1 minus 1 divided by R minus 1 plus R NCAL minus 1 divided by 2.

Where NCAL is equal to N I minus 1 divided by 2 and where R is your expansion factor, expansion factor for grid step length, grid step length in x direction. So, if you use this then you can see that non-uniform grid you are getting and finer mesh you will get near to this x region and near two x equal to L region.

So, here generally these expansion factor R you can use 1 to 1.2. You see this grid so, it is a square cavity we have taken, where x length is 1 and y is 1 and we have used this expression to generate non-uniform mesh and if these 4 surfaces are solid region, then finer mesh are generated near to the solid region, and this expression you can use. Now, let us see the finite difference approximation of any derivative.

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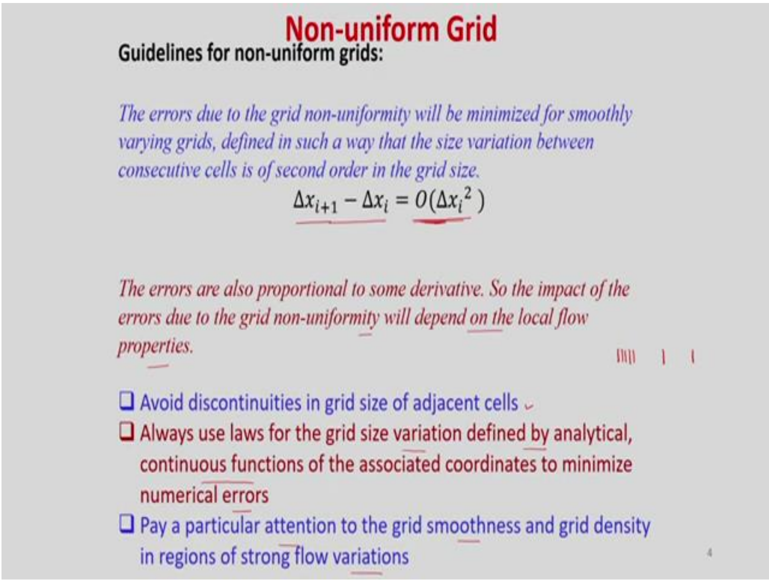

Now we will discuss about the guidelines of non-uniform grid. So, you can see the errors due to the grid non uniformity will be minimized for smoothly varying grids defined in such a way that the size variation between the consecutive cells is of second order of the grid size. So, whatever way you are actually using the expansion factor the delta x i plus 1 minus delta x i should order of delta x i square, then you will get errors will be minimized.

The errors are also proportional to some derivative, so the impact of the errors due to the grid non-uniformity will depend on the local flow properties that we have already discussed. So, when you are generating non-uniform grid please take care about these points, avoid discontinuities in grid size of adjacent cells. So, it should progressively increase or decrease, you should not have a abrupt change in the grid size. Like say you are using very fine mesh near solid region, then suddenly you have taking some grid like this, so you can see that there is a abrupt change in the step size, so you should avoid this type of grid.

Always use laws for the grid size variation defined by analytical, continuous function of the associated coordinates to minimize numerical errors. And pay particular attention to the grid smoothness and grid density in the region of strong flow variations. So, it may not be only near to the solid region that you have very high gradients of the flow, but in some other regions also it may be there due to its flow physics, so that you have to identify and generate the non-uniform grid accordingly.

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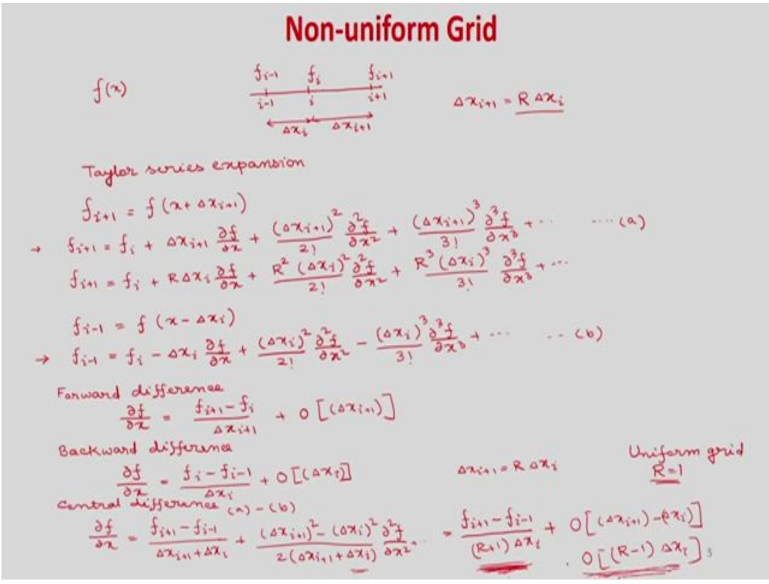

So, now let us consider a non-uniform grid. So, this is your i, this is your i minus 1 and this year i plus 1. So, this is your let us say the step size here it is delta x i and this is your delta x i plus 1 So, you are considering any function f, which is function of x and this is your f i, this is your f i plus 1 and this is your f i minus 1 and we can use the delta x i plus 1 is a constant factor R into delta x i.

So now use Taylor series expansion, use Taylor series expansion, to expand f i plus 1. So f i plus 1, so we can write f x plus delta x i plus 1, we are expanding f i plus 1 about point i. So, you can write f i plus 1 is equal to f i plus delta x i plus 1, del f by del x plus del x i plus 1 square by factorial 2, del 2 f by del x square plus del x i plus 1 whole cube by factorial 3 del cube f by del x cube and higher order terms.

So, if you write in terms of R delta x i, so you can write f i plus 1, is equal to f i plus R delta x i del f by del x, here R square delta x i square by factorial 2 by into del 2 f by del x square plus R cube delta x i cube by factorial 3 del cube f by del x cube and other higher order terms.

So, now you expand f i minus 1. So, you can write f i minus 1 as f x minus delta x I, so we are expanding about point i. So, you can write f i minus 1 is equal to f i minus delta x i del f by del x plus delta x i square by factorial 2 del 2 f by del x square minus del x i cube by factorial 3 del cube f by del x cube plus higher order terms.

Now, let us write the for a difference formulation of del f by del x, so forward difference, of the first derivative del f by del x. So, from this equation directly you can write, from this equation directly you can write del f by del x, is equal to a f i plus 1 minus f i divided by delta x i plus 1, and what will be the order of accuracy? Obviously it will be delta x i plus 1.

Similarly, if you use the backward difference then you will get backward difference then you use this expression, so it will be del f by del x is equal to f i minus f i minus 1 divided by del x i, and order of approximation is delta x I, it is first order accurate.

Now, if you take the difference and if you use central difference approximation then central difference, and what you will get? So, it will be say if you do the if this equation is let us say a and this equation is b, then you write a minus b, a minus b and rearrange it, so, you will get del f by del x, you can write f i plus 1 minus f i minus 1 divided by delta x i plus 1 plus delta x i, plus delta x i plus 1 square minus del x i square divided by 2 into delta x i plus 1 plus delta x i del 2 f by del x square and other terms.

So, you can see from this equation, if you substitute delta x i plus 1 is equal to R into delta x i and rearrange, you will get f i plus 1 minus f i minus 1 divided by, so you can see you can write R plus 1 into delta x i plus. So, you can see here, so it will be delta x i plus 1 plus delta x in the denominator, so 1 will be cancelled out, so you can write order of approximation is delta x i plus 1 minus del x i, or it is you can write R R minus 1 into delta x i.

So, you can see that if you use non-uniform grid and using central difference approximation of the first derivative, you are getting order of accuracies is first ordered, because it is of delta x i, you can see it is order of delta x i, with the factor R minus 1, but it is a delta x i. So, you are getting first order accurate if it is a non-uniform grid.

But if you use uniform grid, then delta x i plus 1 is equal to delta x i and R will become 1. So, for uniform greed, R will be 1. So, you can see that these first term will get 0 and the next term in the truncation will be delta x i square. So, we have already shown earlier lecture that the central difference approximation of the first derivative is second order accurate. But when you are using non-uniform grid, then you are getting first order accurate scheme.

So, you can see here obviously R if it is 1 it you will get 2 delta x i in a uniform mesh, this R plus 1 will be 2 delta x i f i plus 1 minus a f i minus 1 divided by 2 delta x on a uniform grid, but in non-uniform grid this is R plus 1 into delta x i and for uniform grid this term will become 0 then next term will give you second order accuracy.

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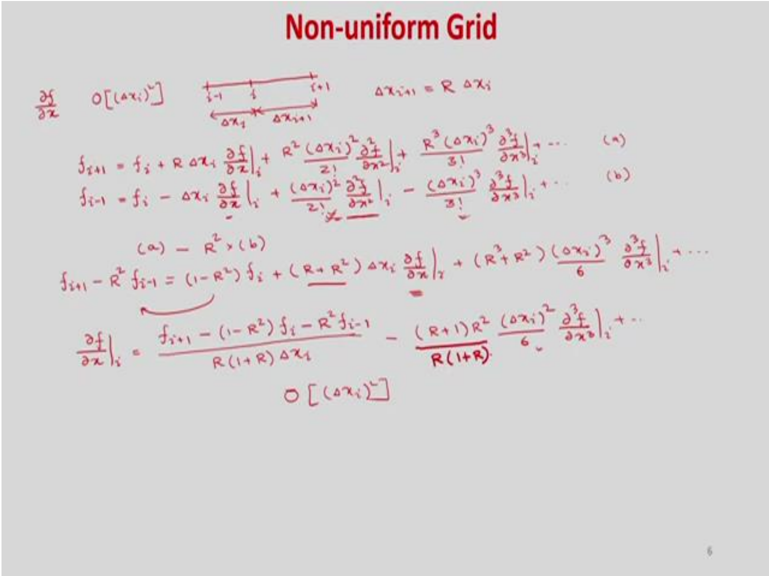

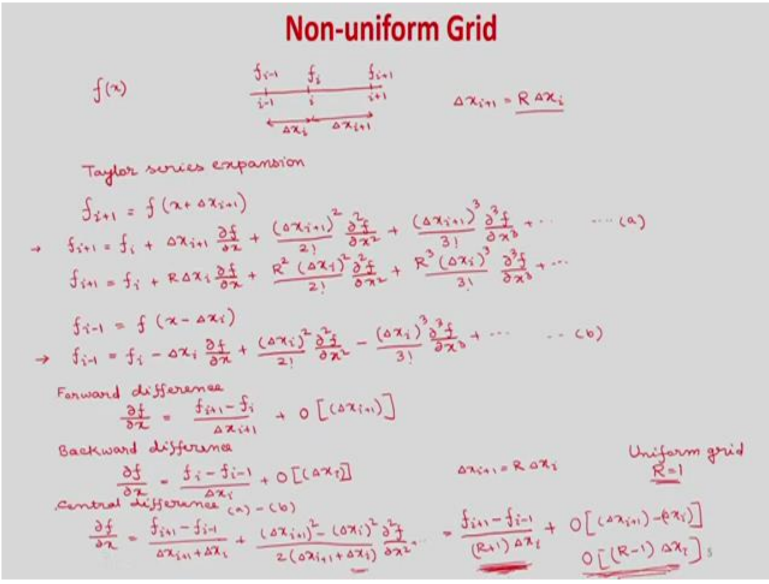

So, now, let us find the second order accurate finite difference approximation for this first derivative. So, the same grid we will use with the constant factor R, so this is your i minus 1 this is your i and this is your i plus 1, non-uniform grid, this is your delta x i plus 1 and this is your delta x i. We want to find del f by del x, which is order of accuracy will be second order. So, we want to find this finite difference approximation, for this first derivative.

So, delta x i plus 1 is your a constant factor R into delta x i. So, now if you see already we have written, if i plus 1 in terms of R in earlier slide you can see, so this we have written, so let me rewrite it again and this we will rewrite it again. So, f i plus 1 will be a f i plus R into delta x i, del f by del x plus R square, delta x i square by factorial 2, del 2 f by del x square plus R cube del x i cube divided by factorial 3, so it will be 6, or factorial 3, let us write, del cube f by del x cube and other terms. And all these derivatives obviously, about i.

Similarly, f i minus 1 already would have done. f i minus 1 as f i minus delta x i del f of del x i plus delta x i square by factorial 2, del 2 f by del x square minus delta x i cube by factorial 3 del cube f by del x cube and higher order terms. So, now to get the second order accuracy you can see these term, del 2 f by del x square, we have to eliminate. So, that we will get the del f by del x and we will divide it by delta x i and we will get a second order accurate approximation.

So to do that, you can see that if you multiply the second equation, this is your a and this is your b, so the B equation you multiply with R square and subtract it, so what you will get, so what you do? You use equation a minus R square into equation b, Then this term will get cancelled, so you can see. So, you can write f i plus 1 minus R square, f i minus 1, second equation you are multiplying with R square then we are subtracting from the first equation a.

Then you are going to get, so it will be 1 minus R square f i plus R plus R square into delta x i, del f by del x this term will get cancelled so we will not write this. So now, this term will be there and you will get R cube minus, so it will be plus R square. So it will be R cube, so it will be plus R square into delta x i cube by factorial 3 means 6, del cube f by del x cube and other higher order terms.

So, now you write del f by del x. So, if you write del f by del x and you take this in this side, then what you will get? del f by del x about i it will be f i plus 1 minus, so this if you are taking minus, it will 1 minus R square, f i minus R square into a f i minus 1 and you are dividing by you are dividing by this, so it will be R into 1 plus R into delta x i and the term will be here, now, it will be minus, so minus R plus 1 into R square delta x I square by 6, del cube f by del x cube and other higher order terms.

So, you can see that with these finite difference approximation you will get the second order accuracy of del f by del x, so this is delta x i square. So, it is order of accuracy is delta x i square.

So, now if we want to find, what will be the approximation of the second derivative? del 2 f by delta x, where using this 3 points, f i, f i minus 1 and f i plus 1, let us find. So, we will use to find this approximation, this polynomial method. So, here in the last term, actually there will be division by R into 1 plus R, because this we are dividing, so this will be so last term will be divided by R into 1 plus R.

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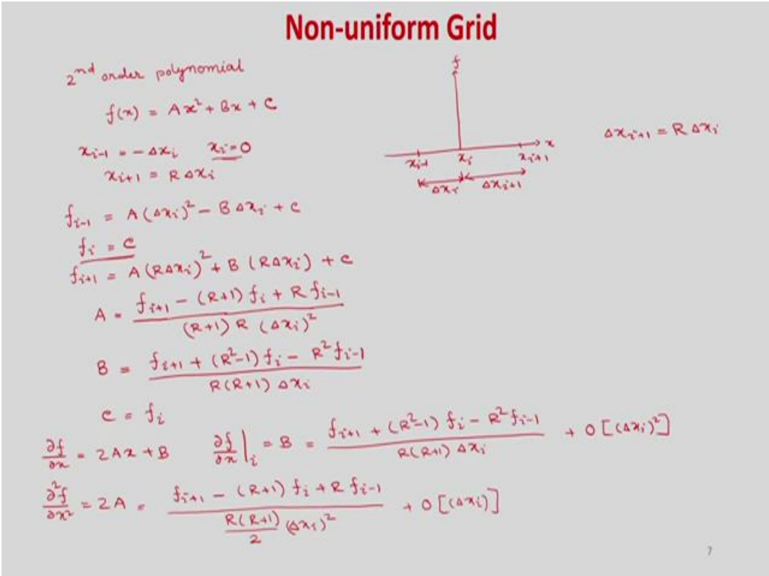

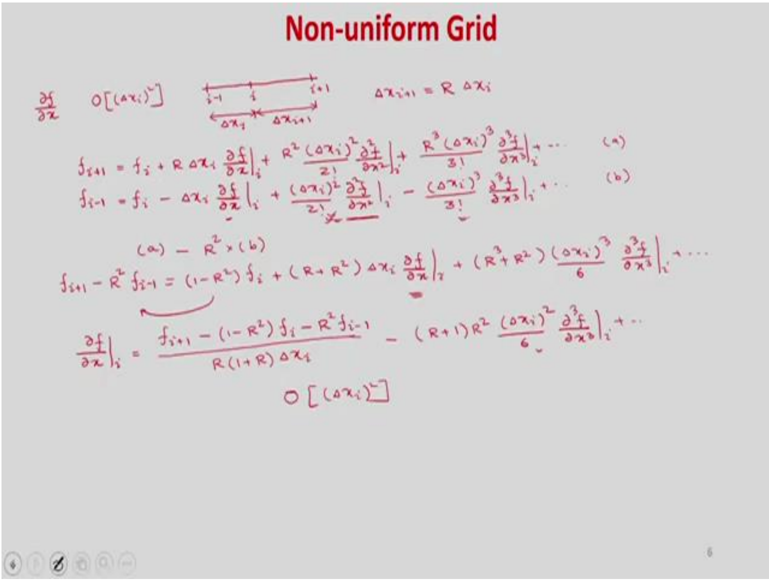

So, we will use the second order polynomial, second order polynomial. So, already we have used it for the uniform grid, so we will use f x is equal to A x square plus B x plus C, where this f x has passed through the point i, i minus 1 and i plus 1. So, obviously, like earlier this is your x, this is your f, so this is your x i, this is your x i plus 1 and this is your x i minus 1, we are using non-uniform grid, this is your delta x i and this is your delta x i plus 1 and delta x i plus 1 is R into delta x i, So, you can see that at x equal to 0, you have x i is equal to 0 here and this is your delta x i plus 1 and this is your minus del x i.

So, delta x i minus 1 is your minus delta x, x i is equal to 0 and x i plus 1, this is x minus 1, this is minus delta x i, x i is equal to 0 and x i plus 1 is your R into delta x i. So, now, you put at f i minus 1, so, if you write a f i minus 1 then it will be A delta x i square, minus B into delta x i plus C. Now, what is a f i? So, f i is f I, and what is a f i plus 1? f i plus 1 is A into R delta x i square plus B into R delta x i plus C, so we are using this second order polynomial.

Now you find the constant ABC if you find it, you will get A is equal to f i plus 1 minus R plus 1 into f i plus R into f i minus 1 divided by R plus 1 into R into delta x i square. B you can find as f i plus 1 plus R square minus 1 f i minus R square f i minus 1 divided by R into R plus 1 into delta x i. And C you will get obviously f i, this is your C f i is equal to C, because x i is 0, x i is 0, so f i will be C.

So, now, if you put in the expression del f by del x, so, what you will get? Del f by del R1. If you take the derivative, so, it will be twice A x plus B and del del x at x i or i, so, you will get x i is equal to 0, so it will be B. So you will get this expression f i plus 1 plus R square minus 1, f i minus R square, f i minus 1 divided by R into R plus 1 into del x i.

So, what is the order of this approximation? Second order, because already we have derived in last slide, you can see, this is the same expression here minus 1 minus R square, so, just R square minus 1, so it will be plus, so this is written.

Now, you can find what will be the second derivative, del 2 f by del x square, it will be just twice and A you know, so you can write as f i plus 1 minus R plus 1 f i and plus R f i minus 1 divided by R into R plus 1 divided by 2. So, it is 2 is there, so it will be divided by 2 into delta x i square and it can be shown that its order of accuracy will be delta x i.

So, this we have used 3 points i, i plus 1 and i minus 1. Now, let us use 3 forward points i, i plus 1 and i plus 2 and let us find the finite difference approximation of del f by del x and del 2 f by delta x square using non-uniform grid and using this polynomial method.

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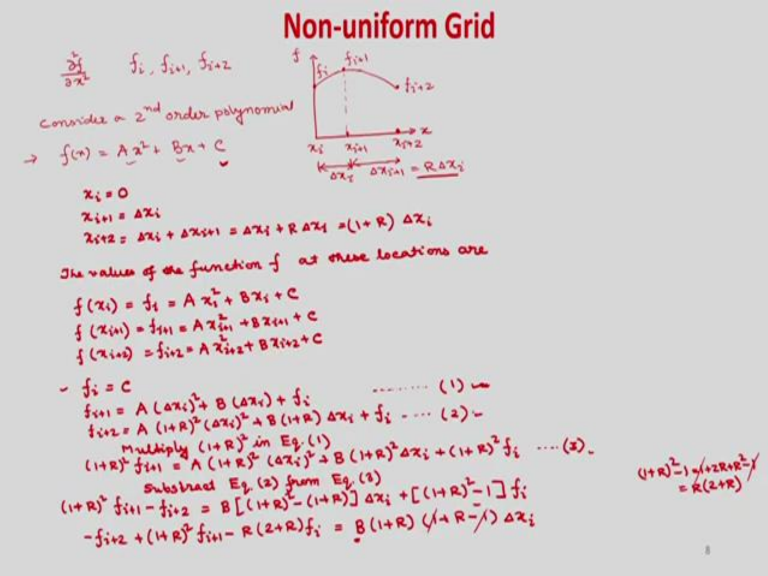

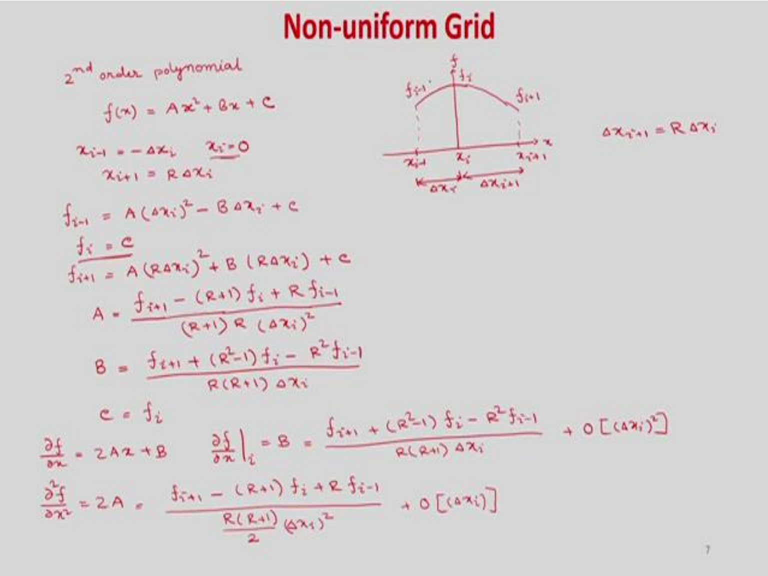

So, we want to find del 2 f by delta x square, on non-uniform grid using 3 forward points f i, f i plus 1 and f i plus 2. So, origin will take here, x is 0 here and this is your f. So, this variation may be something like this you can show, this is your f i, this is your f i minus 1 and this is your f i plus 1. So, the values of at point x i it is value f i, if x i plus 1 f i plus 1 and it is x minus 1, f i minus 1.

And here we will take the origin here, so it will be x i this will be x i plus 1 and this is your x i plus 2. So, we have considered non-uniform grid, this is your delta x i and this is your delta x i plus 1, which is R into delta x I and maybe it is varying like this. So, this is your f i plus 2, this is your f i plus 1 and this is your f i.

So, we will use the second order polynomial. So, consider a second order polynomial. So, it is f x is equal to A x square plus B x plus C. So select the origin x i as 0. So, x i is equal to 0, so x i plus 1 will be delta x i, and x i plus 2 will be delta x i plus delta x i plus 1. So, it will be delta x i plus delta x i plus 1 is R into delta x I, so it will be R into delta x I, so it will be 1 plus R into delta x i.

So, now the values of this function at this location you can write it. So, the values of the function if at these locations are so you can write, if at location x i will be a f i and it will be just from this polynomial if you put you will get A x i square plus B x i plus C. Similarly, if you put x i plus 1, then you will get at this location f i plus 1 is equal to A x i plus 1 square plus B x i plus 1 plus C and similarly, at location x i plus 2 is equal to f i plus 2 is equal to A x i plus 2 square plus B x i plus 2 plus C.

So, now you need to put the values of x i, x i plus 1 and x i plus 2. So, you can see that it you put the value of x i as 0 then f i is equal to C. Then, f i plus 1 if you put, so, x i plus 1 is delta x i, so you are going to get A delta x i square plus B delta x i and C is a f i, so you can write f i.

Because C is equal to f i already we have seen here. Now if you f i plus 2 is equal to A, and what is x i plus 2? x i plus 2 is 1 plus R into delta x i. So, it will be 1 plus R whole square delta x i square plus B into delta, so B into x i plus 2, so it is 1 plus R into delta x i and C is f i, so C is f i. So, now, if you give this equation as equation number 1 and this as equation number 2.

So, now we need to find the values of A and B, A and B we need to find. So what we will do, we will multiply 1 plus R whole square in equation 1. So, multiply 1 plus R whole square in equation 1, so if you multiply in this equation 1, so what you will get? 1 plus R whole square f i plus 1 is equal to, so now you are multiplying 1 plus R whole square, so A into 1 plus R whole square delta x i square plus B into 1 plus R whole square into delta x i plus 1 plus R whole square into f i. Let us say this is equation number 3.

So, now you subtract equation 2 from equation 3. So, subtract equation 2, from equation 3. So, what do you will get? You are going to get 1 plus R whole square f i plus 1 minus f i plus 2, in the left hand side, and in the right hand side you see that first terms will get cancelled. Because both are same, so you will get the second term. So what is second term? Second term is B into 1 plus R whole square minus 1 plus R into delta x i, and this you will get 1 plus R whole square minus 1 into f i.

So, you rearrange it, so what you will get? minus f i plus 2 I am writing first plus 1 plus R whole square into f i plus 1 and you take this term in the left hand side this term. So, 1 plus R square minus 1, what you will get? 1 plus R whole square minus 1, so it will be 1 plus 2 R plus R square minus 1.

So, these 1, 1 will get canceled. So, you will get R into, 2 plus R, 2 plus R, R into 2 plus R. So, this we will write in the left hand side, so you will get so, it is plus it will become minus R into 2 plus R, R into 2 plus R into f i, and in the right hand side what you will get with B? So, B now you see 1 plus R if you take common, then you will get 1 plus R and minus 1. So, this 1, 1 will get cancelled and you will get delta x i. So, what do? Now you find the values, if now you find the values of B, you divide this.

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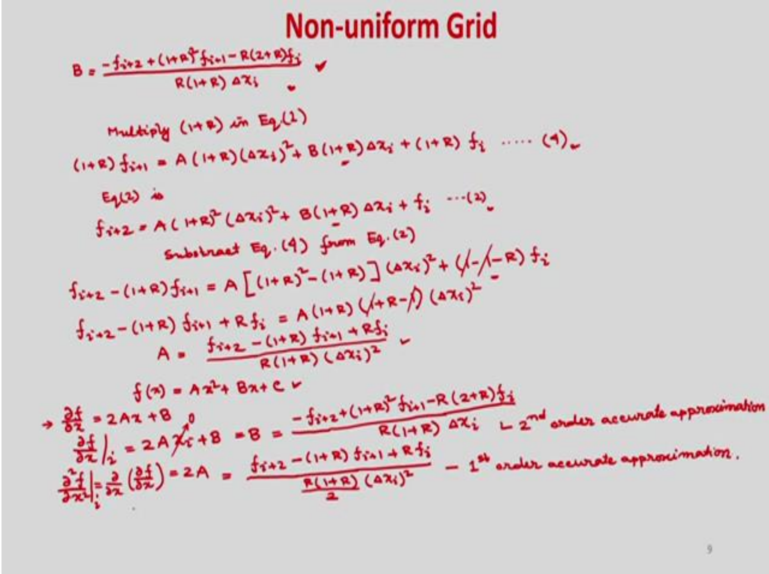

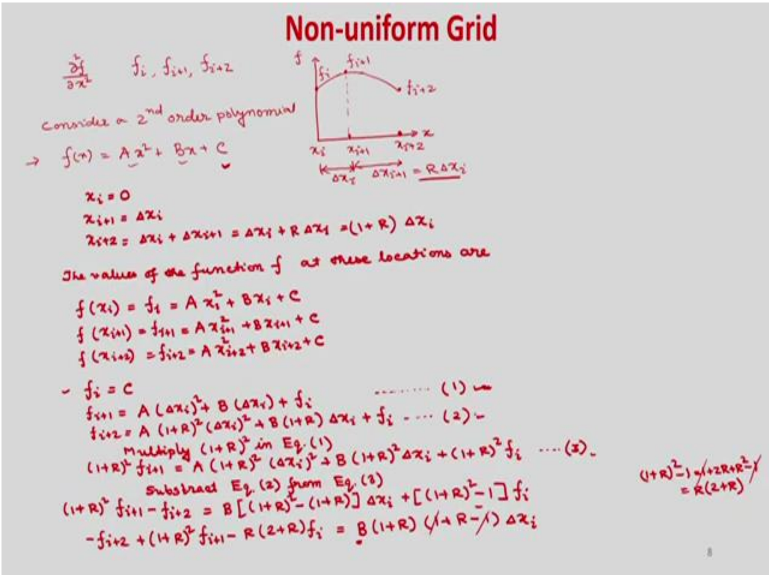

So you will get in the next slide let us write, so we will get B, so you will get B is equal to so if you divide that so you will get minus f i plus 2 plus 1 plus R whole square f i plus 1 minus R into 2 plus R into f i divided by R into 1 plus R into delta x i. So, now we have found the value of B, now we need to find the value of A. C we know, C is f i, so already we have put value.

Now we will multiply 1 plus R in equation 1. So, you see equation 1. So this is the equation 1, you multiply 1 plus R in both side. So, we will get 1 plus R into f i plus 1. In right side you will get A into 1 plus R, so delta x i square plus B into 1 plus R into delta x i and you will get 1 plus R into f i. So let us say this equation number is 4.

So now, in equation 2, we had equation 2 is whatever we have written this we are rewriting equation two we are rewriting, f i plus 2 is equal to A into 1 plus R whole square delta x i square plus B into 1 plus R into delta x i plus f I, so this is equation 2.

Now subtract equation 4, equation 4 from equation 2. So, you see, now we are subtracting equation 4 from equation 2. So, what you can write f i plus 2 minus 1 plus R, f i plus 1, in the left hand side. In the right hand side the first term will be A 1 plus R whole square minus 1 plus R, into delta x i square. And the second term in the right hand side you can see these are same, so it will get cancelled, so now you have the last term as 1 plus, 1, so last term we have 1 minus 1 minus R into f i.

So, you can see, so, if you take, so this 1 1 will get canceled and you take this in the left hand side, so you will get f i plus 2 minus 1 plus R f i plus 1. So, this will if you take in the left hand side, it will be plus R f i and in the right side now, you have A one plus R if you take common, then you have 1 plus R minus 1 into delta x i square. So, this 1 1 you cancel, so you will get the value of A as f i plus 2 minus 1 plus R, f i plus 1 plus R f i divided by R into 1 plus R into delta x i square. So, now we have found the values of A, B and C.

Now, let us find the values of first derivative and second derivative. So, what is first derivative? So you have f, which is function of x, A x square plus B x plus C. So, if you find the first derivative, what you will get? Del f by del x is equal to twice A x plus B. So, let us find the value of del f by del x at i. So, what you will get? 2 A into x i plus B.

So, what is the value of x i? x i at the origin we have chosen, so, x i is 0, so equal to B. And what is the value of B? So, this is the value of B. So, that you put, so it will be minus f i plus 2 plus 1 plus R square f i plus 1 minus R into 2 plus R into f i divided by R into 1 plus R into delta x i and if you see this will be second order accurate approximation. So, that we are not showing here, but if you can find the order of approximation you will get second order accurate approximation.

Now let us find the second derivative. So, if you find the second derivative, so del 2 f by del x square. So you see del f by del x is this, so if you take another derivative with respect to x. So, it is del of del x of del f by del x, if you take another derivative of the del f by del x with respect to x you will get twice A. So now you can see that it will be A you know, what is the value?

So, A will be f i plus 2 minus 1 plus R f i plus 1 plus R f i, R into 1 plus R into delta x i square and we have 2, so 2 you can take here, divided by 2. So, this is the approximation of del 2 f by delta x square at point I and if you see the order of approximation is first order first order accurate approximation and similar relations apart backward and central difference approximation may be obtained by this procedure.

Today, we have learned the finite difference approximation of first derivative and second derivative, using non-uniform grid and you can see when we use the central difference of first derivative del f by del x using points i plus 1 and i minus 1 although on the uniform grid it is second order accurate, but in non-uniform grid it becomes first order accurate.

But if close to uniform grid means R is the factor, it is a factor, if R tends to 1, then you will get the order of accuracy towards 2. Because you can see in the del f by del x, del f by del x the first derivative the central difference we have used f i plus 1 minus f i minus 1 divided by the f i x i plus 1 minus x i minus 1. So, when you use the factor R, so R will tends to 1 that means it will close to uniform grid and once it goes to close to uniform grid, then you will get close to second order accurate. Thank you.