Computational Fluid Dynamics for Incompressible Flows
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Lecture 2
Module 3 Finite Difference Method
Finite Difference by General Approximation and Polynomials

Hello everyone, so in last lecture of this module, we learned how to discretize the first derivative and second derivative using finite difference method and for that we use Taylor series expansion. Now today will have been lecture two, Finite Difference by General Approximation and Polynomials. So, here we will use some other methods to find the finite difference approximation of the derivatives. So, let us first use the general approximation.

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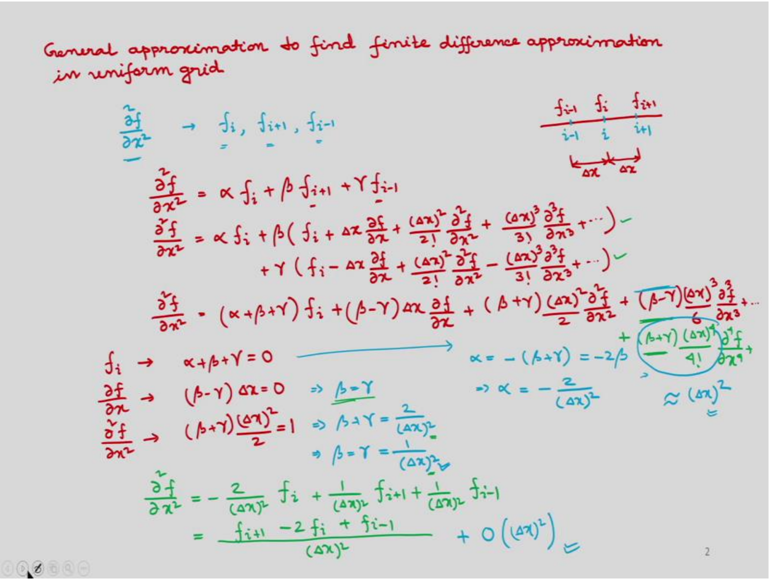

So, General approximation to find finite difference approximation using or in uniform grid. So, in last class also we have used the uniform grid in that sense the spacing, grid spacing is same delta x is same, so here also we will considered, let us say this is your grid and these are the points i i is the index these are the discrete points in X direction and this your i minus 1.

So, in general approximation what is the advantage if you want to find some approximation of derivatives first derivative or second derivative using some discrete points, so that you can do easily. Let us say that we want to find the derivative del 2 f by del x square, using three points f i f i plus 1 and f i minus 1. So, the values at i point is f i i plus 1 point f i plus 1 and f i minus 1. So, we want to find this finite difference approximation of del 2 f by del x square using the values of f i f i plus 1 and f i minus 1.

In last class what we did we use some algebra and did some subtraction or addition and we found the finite difference approximation. Now, in general approximation, we are telling that we want to find the finite difference approximation of any derivative using some discrete values at i i plus 1 and i minus 1 in this case.

So, what we will do for this, first we will write del 2 f by delta x square is equal to some coefficient alpha will get, plus beta, f i plus 1 plus gamma f i minus 1. So, this is the finite difference approximation now we need to find the values of alpha, beta and gamma. So, to do that, what we will do, we will expand this f i plus 1 and f i minus 1 using Taylor series expansion and the coefficients in the right hand side will equate with the coefficient in the left hand side.

So, first let us expand it, so, we can write del 2 f by del x square is equal to alpha f i last beta. Now we will expand it using Taylor series expansion, so that you can write f i plus del x or the distance between these discrete points is delta x, and this also delta x because it is uniform grid.

We have the value here f i here f i plus 1 and f i minus 1. So, now we will write del x del f by del x plus del x square by factorial 2 del 2 f by del x square plus del x cube by factorial 3 del cube f by del x cube and other higher order terms. Similarly, let us expand this f i minus 1. So, plus gamma f i minus delta x del f by del x plus delta x square by factorial 2 del 2 f by del x square minus del x cube, because it is f i minus 1, so it will be minus del cube f by del x cube plus higher other terms.

So, now we can now simplify it del 2 f by del x square, so all the f i coefficient you write together, so you can write alpha plus beta plus gamma. So, this is your f i coefficient similarly, del f by del x whatever coefficient you have that you write, so del f by del x you have beta minus gamma delta x del f by del x.

Now you write for del 2 f by del x square. So, now if you find this you will get, so beta, so it will be beta plus gamma because it is plus so it will delta x square by 2, factorial 2 is 2, del 2 f by del x square. After that you will have plus, so you have a beta minus gamma beta minus gamma. So, you have del x cube by 6 factorial 3 6 del cube f by del x cube and other terms.

So, you can see that now you compare the coefficients. So, f i coefficient in the right hand side is alpha plus beta plus gamma in left hand side there is no term associated with f i so, alpha plus beta plus gamma will be 0. So, you can write, so for f i I am just writing the coefficients if you compare so, it will be alpha plus beta plus gamma will be 0.

Similarly compared the coefficients of the del f by del x. So, in the left hand side no term is there associated with del f by del x. So, it will be 0, so del f by del x if you compare the coefficients, so it will be beta minus gamma into delta x will be 0 and if you compare del 2 f by del x square. So you will get beta plus gamma delta x square by 2 will be now, you see, we have left hand side del 2 f by delta x square whose coefficient is 1. So, you can write this equal to 1.

So, to find the three unknown alpha, beta, gamma from these three equations will be able to find. So, from the second one you can write, beta is equal to gamma. So if beta is equal to gamma from here you can write, you can write, so it will be beta plus gamma is equal to 2 by delta x square.

So, that means you will get beta is equal to gamma. So, beta will be gamma is equal to 1 by delta x square, because it is 2 beta. So, 2 2 will cancel out. So, now, from this you can now write alpha plus beta plus gamma is equal to 0. So, from here you can write alpha is equal to minus beta plus gamma, which is nothing but minus 2 beta.

So, you can write alpha is equal to. So, minus 2 by delta x whole square. So, we found the coefficients alpha beta gamma just comparing left hand side to right hand side the coefficients of f i del f by del x and del 2 f by delta x square. So, now you know alpha beta gamma So, you can write the approximation of del 2 f by delta x square as del 2 f by delta x square is equal to. So, alpha is minus 2 by delta x square f i then beta. So, beta is your plus 1 by delta x square, this your f i plus 1 and gamma is also same 1 by delta x square f i minus 1.

So, if you rearrange it, so you will get delta x whole square f i plus 1 minus twice f i plus f i minus 1. So, you know that already we have derived it. So, it is a central difference approximation of the second derivative del 2 f by delta x square and we are getting the same finite difference approximation. Now, what will be the order of accuracy, so you can see order of accuracy is beta is equal to gamma so beta equal to gamma.

So, if you see in this term, so beta is equal to gamma. So, this term will become 0. So, this term will become 0. So, then another term will be there associated with these. So, if you write it, so it will be beta plus gamma del x to the power 4 by factorial 4 del 4 f by del x 4. So, if you see it, so, it will be beta plus gamma.

So, another show here if you add another term here and this if you add it you will get this. So, now, beta plus gamma. So, these you have delta x to the power 4 and beta and gamma contents divided by delta x square. So, that means, that will be divided so your, you are you can see that beta and gamma contents 1 by delta x square. So, if you divide it beta plus gamma. So, here, so this term will be, so this whole term will be order of delta x whole square because beta and gamma is 1 by delta x square you can see here.

So, delta x to the power 4 divided by delta x square so it will be order of delta x square. So, it is order of accuracy of will be delta x square. So, because this time is becoming 0, because beta minus gamma is 0 already we have seen, so, this term will become 0. So, the other term will contribute. So, it is the leading term in that truncation error. So it will be order of delta x square. So, similarly, let us find the approximation of del f by del x using three points f i f i plus 1 and f i plus 2. So, it is one sided forward difference approximation of del f by del x.

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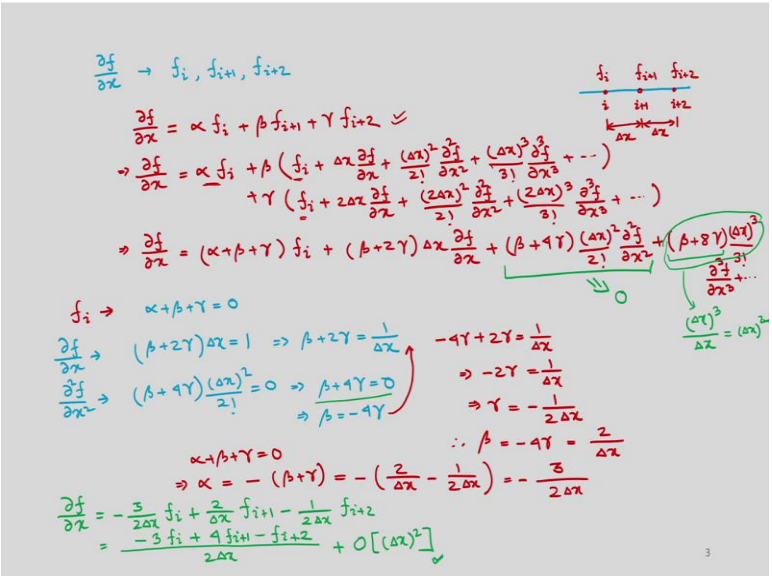

So, we want to find the finite difference approximation of del f by del x, using three points discrete values at three points f i f i plus 1 and f i plus 2. So, similarly now we will take the discrete points. So, let us take three discrete points i i plus 1 and i plus 2 and separate by separated by uniform distance so this is your delta x and this is also delta x.

So, the value associated with i point is f i, this is your f i plus 1 and this is your f i plus 2. Now, you want to find the first derivative del f by del x using the values f i f i plus 1 and f i plus 2. So, let us write similar way del f by del x. So, we are using the general approximation, so, it will be alpha f i plus beta f i plus 1 plus gamma f i plus 2.

So, similarly we will use the Taylor series expansion for f i plus 1 and f i plus 2, then we will compare the coefficients with the left hand side and right hand side. So, you can write del f by del x is equal to alpha f i plus beta, now you write the f i plus 1 as f i plus delta x del f by del x plus delta x square by factorial 2 del 2 f by del x square plus del x cube by factorial 3 del cube f by del x cube and other higher order terms and gamma now you write for f i plus 2.

So, f i plus 2 means f i means x plus 2 delta x. So, it will be f i plus 2 delta x del f by del x plus 2 delta x square by factorial 2 del 2 f by del x square plus 2 delta x whole cube factorial 3 del cube f by del x cube and other higher order terms. So, now, you rearrange it so, del f by del x first you write all the coefficients for f i, so, together we are writing alpha plus beta plus gamma because this is your alpha this is your beta and this is your gamma.

So, this is into f i. Now write the coefficients for del f by del x So, this is your del f del x So, you can write beta plus 2 gamma, this is 2 and this is your gamma into delta x del f by del x then you write this term, so it will be plus.

So it will be beta plus, so it will be beta plus 4 gamma into delta x whole square by factorial 2 del 2 f by del x square and the other term will be, say it will be beta plus 8 gamma del x whole cube by factorial 3 del cube f by del x cube and other higher order terms. Now, you compare the coefficients, so first you compare the coefficients of f i.

So, in left hand side it is coefficients of f i is 0. So, you can write alpha plus beta plus gamma is equal to 0. Similarly f del f by del x coefficient, left hand side you have 1 and right hand side you have beta plus 2 gamma into delta x is equal to 1. Similarly you write, del 2 f by x square coefficient.

So, you can see here beta plus 4 gamma del x whole square by factorial 2 is equal to 0. So, factorial 2 means it is 2 only. So, now you find so you can write from here beta plus 2 gamma is 1 by delta x and from here you can write beta plus 4 gamma is equal to 0. So, from here you can write beta is equal to minus 4 gamma. So, from here now, if you put this value here, so you will get beta value if you put then you will get minus 4 gamma plus 2 gamma is equal to 1 by delta x.

So, that means you will get minus 2 gamma is equal to 1 by delta x. So, you will get gamma is equal to minus 1 by 2 delta x. So, gamma you found, so now beta is equal to minus 4 gamma. So, beta will be minus 4 gamma so, it will be 2 by delta x 1 by 2 delta x it will be 2 by delta x.

Now, you have alpha plus beta plus gamma is equal to 0. So, you put the values, so alpha will be minus beta plus gamma, so that means minus beta is 2 by delta x and gamma is minus 1 by 2 delta x. So, what you will get? So, you will get minus 2 delta x it will be 4 so it will be 3.

So, 4 minus 1 it will be 3. So, now alpha beta and gamma value you have found now you put the values in this expression, in this expression So, what you will get, so del f by del x. So, what is the value of alpha alpha is minus 3 by 2 delta x into f I, then you will get beta, beta is plus 2 by delta x, f i plus 1 then f i plus 2 is coefficient is gamma. So, gamma is minus 1 by 2 delta x f i plus 2.

So, if you rearrange it, so, you will you are going to get, so 2 delta x, so it will be minus 3 f i. So, you will get, so 2 so it will be 4 f i plus 1 and you will get minus f i plus 2. So, in last class already we derived this directly from the Taylor series expansion and we got the same expression.

So, this is forward difference approximation of del f by del x using three discrete points values. So, f i f i plus 1 and f i plus 2 and what will be the order of accuracy of this scheme. So, you can see the first term in the truncation error is this term but this term you can see that beta plus 4 gamma is 0. So, this term is going to be zero, so, now next term is beta plus 8 gamma.

So, this is the now leading term, so if it is the leading term in the truncation error. So, beta plus 8 gamma will be ordered up on by delta x and here delta x cubes is there so, together these will actually give delta x cube by delta x. So, it will be delta x square. So, the accuracy of this finite difference approximation will be order of delta x square.

So, already we have derived it in last class using Taylor series expansion and we could find that the order of accuracy is delta x square. So, now we found the derivative using known discrete points we wanted to express the derivative using general approximation. Now let us see how we can discretize using polynomial.

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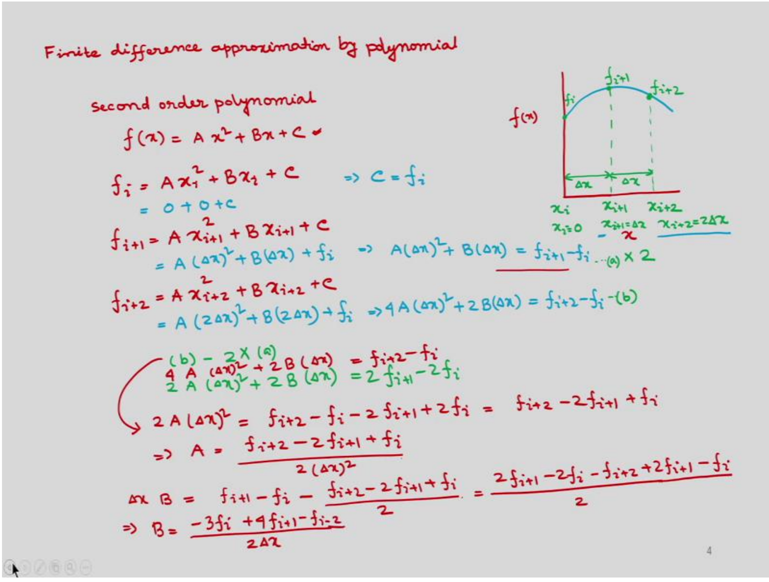

So, we will now learn finite difference approximation by polynomial, so in this case, we will use the value of this function as polynomial, then its derivative will find and using some given data will be able to find the constants of the polynomial. So first, what do we want to find? We will let us consider a second order polynomial, so first you find second order polynomial.

So, we will write f x as A x square plus B x plus C, so it is second order polynomial now A, B, C are constant, so if you know the f x in terms of A B C so all these coefficients if unknown then anyway you can easily find the first derivative approximation as well as the second derivative of f this approximation.

Now to find the constant A B C we will use some data of these dependent variables. So, for that, let us say that the polynomial, so that using three points So, this is let us say x and this is your f x and your polynomial varies like and we are using uniform spacing. So, in this case, let us say your, this is your, so this is your x i, and this is your x i plus 1 and this is your x i plus 2, and we will assume that x i is equal to 0 because we are starting from here and the spacing is delta x.

So, x i plus 1 will be delta x and x i plus 2 will be 2 delta x and these are the values of f i plus 1, f i plus 2 and this is f i. So, now we have assumed the second order polynomial. So, we can now write f i as A x i square plus B x i plus C. Similarly, we can write f i plus 1 is equal to A x i plus 1 square plus B x i plus 1 plus C and f i plus 2 we can write A x i plus 2 square plus B x i plus 2 plus C because as f x we have assumed like the second order polynomial.

So, we can write for f i f i plus 1 f i plus 2 like this. Now, as we have assumed that x i is equal to zero and x i plus 1 is equal then delta x and x i plus 2 is 2 delta x because we have uniform spacing. So, we can write these as, so x i as 0, so it is 0 plus 0 plus c. So, from here you can write the value of C as f i.

So now similarly now you put x i plus 1 as delta x because here you can see, so you can write A, so this is your delta x square plus B delta x plus C is f i. So, from here, so you can write A delta x square plus B delta x is equal to f i plus 1 minus f i and this now you can write f i plus 2 as x i plus 2 is 2 delta x.

So, you can write A 2 delta x square plus B 2 delta x plus C is f i, so f i you can write. So, from here you can write A so, it will be 4 delta x square plus 2 B delta x is equal to f i plus 2 minus f i. So, now you have these two equations and two unknowns A and B C already we have found, so you find the values of A and B. So, this equation you multiply with 2 and subtract with this.

So, equation let us say this is your A and this is your B, so you subtract multiplying this equation A with 2 with the equation B. So, you can write B minus 2 into A, so what you will get, so you can see. So this you will you are going to get 2 A delta x square plus 2 B delta x is equal 2 f i plus 1 minus twice f i, okay? And you have B, B is 4 A delta x square, plus twice B delta x is equal to f i plus 2 minus f I, so if you subtract it.

So, these term term will get canceled, so you will get twice A delta x square is equal to so here we are writing. So, f i plus 2 minus A f i minus 2 f i plus 1 and plus 2 f i you can write this as so f i plus 2 minus twice f i plus 1 and plus f i . So, from here you can write the value of A as f i plus 2 minus twice f i plus 1 plus f i divided by 2 delta x square.

So, let us find the value of B, so B will get. So, from here you can find. So, you will get f i plus 1 minus f i. So, minus delta x square So, A is this so it will be f i plus 2 minus twice f i plus 1 plus f i divided by, so it will be, 2 and so B, B into delta x let us say so B into delta x is f i plus 1 minus f i minus A delta x square.

So, delta x square and delta x square will cancel out. So, f i plus 2 minus twice f i plus 1 plus f i. So, you will get 2 f i plus 1 minus twice f i minus a plus 2 plus twice f i plus 1 minus f i. So, if you divide by delta x, so you will get B as so now divided with delta x or 2 delta x and if you write all these terms. So, f i f i you take C, so minus three f i, minus 3 f i, then you have f i plus 1 f i plus 1 and so 4 f i plus 1, so plus 4 f i plus 1 and f i 2 so minus f i minus 2.

Now, we could find the values of A, B and C. So now if you put here a f x then in this expression, so you will get the f x so what will be the fx.

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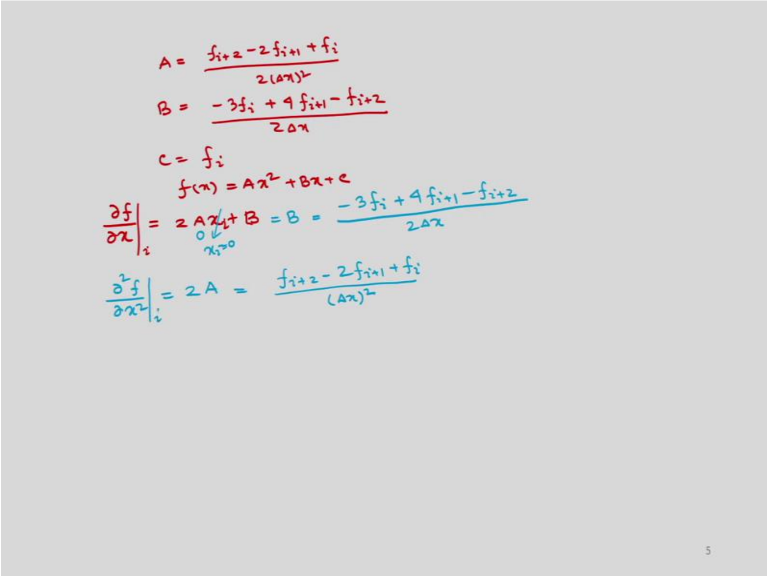

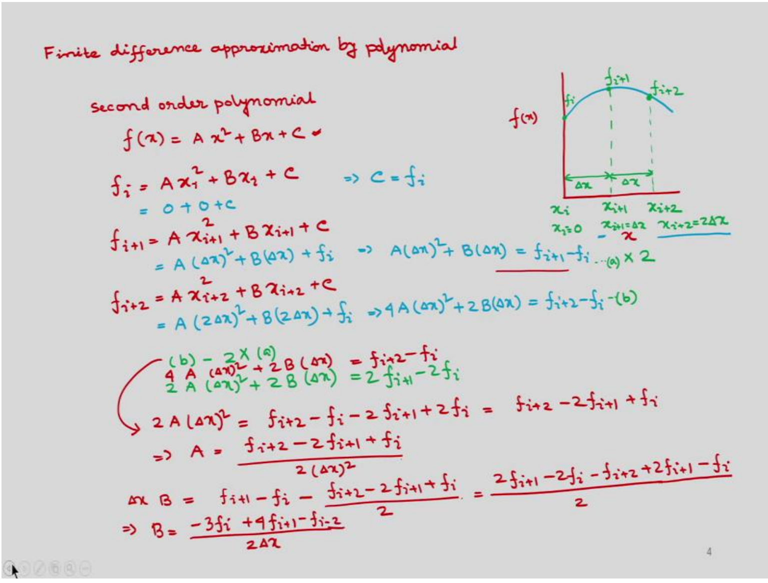

So what do we got? A is equal to f i plus 2 minus twice f i plus 1 plus f i divided by 2 delta x square and B as minus 3 f i plus 4 f i plus 1 minus f i plus 2 divided by 2 delta x and C as f i. So, now if we want to find the values of the first derivative del f by del x, so you know the polynomials, so you can write del f by del x del f by del x now you can write, so you have f x as A x square plus B x plus C. So, del f del x will be twice A x plus B.

So, that means, del f by del x at i, it will be 2 A x i plus B. So now, as x i is 0, so this x i is 0, x i is 0. So, this term will become 0. So, you will get equal to B, so it will be minus 3 f i plus 4 f i plus 1 minus f i plus 2 divided by 2 delta x. Now, similarly if you want to find the second derivative then what do you write del 2 f by del x square will be twice A. So, if it is twice A, so, you can write so, 2 2 will cancel out it will be f i plus 2 minus twice f i plus 1 plus f i divided by delta x is square.

So, at del 2 f by del x square at i. So, this is the way to find the polynomial. So, first you approximate the polynomials second order or third order polynomials, then you can write the coefficients you can find using the data of the dependent variable. So, putting which is in the at the origin that you can put x i is equal to 0 and taking a uniform mix, x i plus 1 you can write as delta x and x i plus 2 you can write 2 delta x and put all the values and find the constants A, B, C and once you know the polynomial, then you can find the derivatives del f by del x and del 2 f by del x square.

So, now let us find the central difference approximation of first and second derivative of any function f x by polynomial technique. So, for that purpose let us consider a second order polynomial f x equal to x square plus Bx plus C which has passed through three points i minus 1, i and i plus 1.

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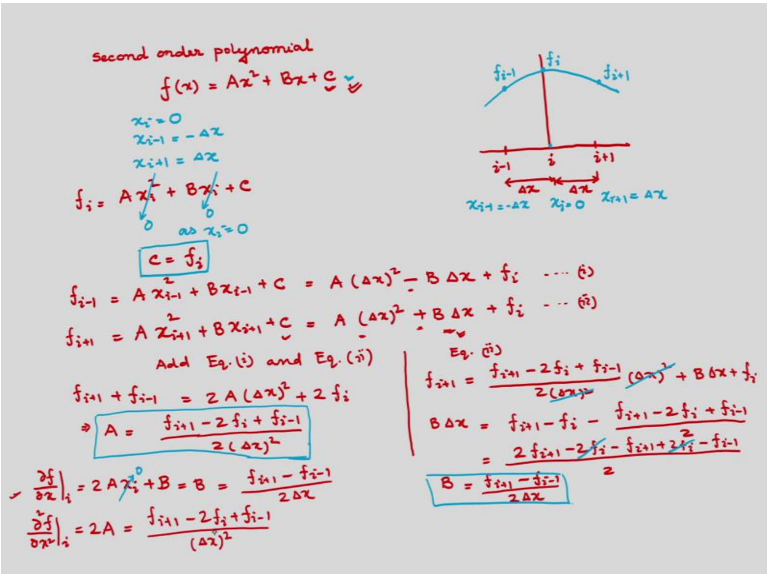

So, let us consider second order polynomial, second order polynomial f x equals to A x square plus B x plus C and this is passing through three points. So, this is your i this is your i plus 1 and this is your i minus 1 and you have uniform step size delta x and let us assume the polynomial like this which is a second order polynomial.

So, this values is f i minus 1, this values is f i and this values f i plus 1. So, now we let us considered the origin here, where x i is equal to 0. So at this point, so this point it is delta x, so x i plus 1 will be delta x and x i minus 1 will be minus delta x, because this point is the origin. So, this is your minus delta x.

So, you can see that xi is 0 x i minus 1 is minus delta x and x i plus 1 is equal to delta x. So, now we can write f i is equal to A x i square plus B xi plus C. But xi is equal to 0. So, this will be 0, this will be 0 because xi as xi is equal to 0. So, you will get C is equal to f i.

So, similarly let us write f i minus 1. So, you can write A x minus 1 square plus B x i minus 1 plus c because from this expression. Similarly, you can write f i plus 1 is equal to A x i plus 1 square plus B x i plus 1 plus C. So, already we have found the value of C because C is equal to f i. So, let us put it there and put the values of x i minus 1 and x i plus 1 because x i plus 1 is equal to delta x and x i minus 1 is equal to minus delta x.

So, if you write it so you will get so, A delta x square minus B delta x because x i minus 1 is delta x plus f i. Similarly, here you will write A delta x square plus B delta x plus f i, because C value of C is f i. So, we have written this. Now, let us consider this is equation 1 and this is the equation 2.

So now add equation 1 and equation 2, so what you will get if you add it, so you will get f i plus 1 plus f i minus 1 is equal to., so this will be 2 into A delta x square and this term will get cancelled because you are adding so this is your minus this is your plus so it will get cancelled and you will get plus 2 f i.

So, from here you can write the value of A as f i plus 1 minus 2 f i plus f i minus 1 divided by 2 delta x square. So, now you find the value of B so, find the value of B from any equations. So, let us consider equation 2. So, if you consider equation 2, then you can write f i plus 1 is equal to f i plus 1.

So, this is the value of A divided by 2 delta x square into delta x square plus B two delta x plus f i. So, you find out the value of B. So, you can see these delta x square this delta x will get cancelled and you will, can write B delta x, B delta x you can write as f i plus 1 minus f i minus f i plus 1 minus 2 f i plus f i minus 1 divided by 2.

So, then you will get 2 so it will be 2 f i plus 1 minus 2 f i. So, here minus is there, so it will be minus f i plus 1 plus 2 f i and minus f i minus 1. So, you can see that here this 2 f i and 2 f i will get cancelled because it is plus here minus. So, finally, you will get B is equal to, so this is 2 f i plus 1, so it will be f i plus 1 minus f i minus 1 divided by 2 delta x.

So, now, you see we have found the values of A B and C. These constants C is this one B is this and A is this. So, now you can find the derivatives del f by del x and del 2 f by delta x square. So, you can see from here that what will be your del f by del x. So, from this expression you can see del f by del x at point i you can write twice A x i plus B and xi, what is the value of x i? xi is 0.

So, you will get this is equal to B and this is your f i plus 1 minus f i minus 1 divided by 2 delta x. So, this is the central difference approximation of the first derivative and you know the order of accuracy is delta x square.

Similarly, now, you find the second derivative. So, del 2 f by delta x square at i. So, from this expression if you write del 2 f by delta x square is equal to so, you will get twice into A so, you will get twice A means say these 2 2 will get canceled. So, you will get f i plus 1 minus 2 f i plus f i minus 1 divided by delta x square.

So, that you can see that this is the second order approximation of the second derivative using three points i minus 1, i and i plus 1 and we have use second order polynomial and found the values of constant A, B and C and we could find the derivative first derivative del f by del x and del 2 f by del x square using three points f i, f i minus 1 and f i plus 1.

So, now let us write this forward difference, backward difference and central difference of first derivative, whatever we have derived using Taylor series expansion as well as general approximation and polynomial techniques.

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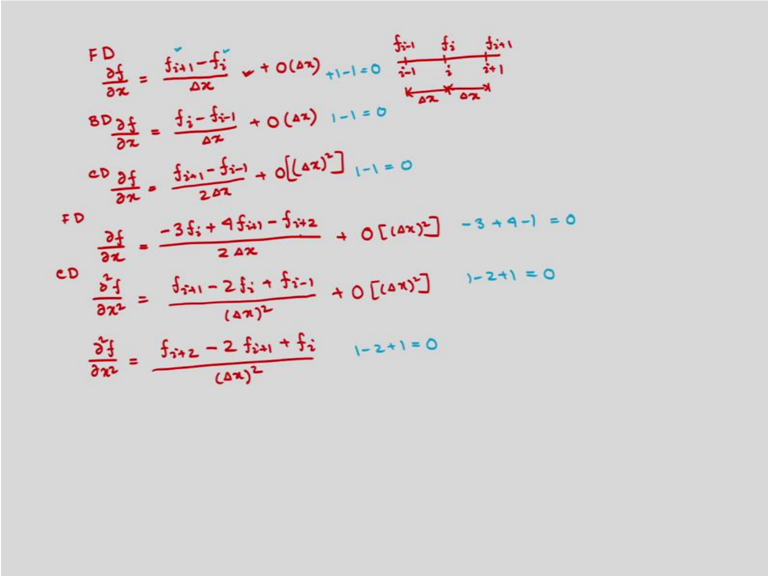

So, if you have seen that for first derivative we do the forward difference approximation what is that? del f by del x is equal to f i plus 1 minus f i divided by delta x and you know that it is the first order accurate. So, that means you have i, i plus 1 i minus 1. so, the value of f i f i plus 1 and f i minus 1 and these are uniform step size.

So, delta x and delta X, so this is the forward difference approximation. Similarly backward difference approximation we have derived del f by del x is equal to f i minus f i minus 1 divided by delta x and you have seen that this is order of delta x and this is the order of delta x and central dependency you have used.

So, del f by del x is equal to f i plus 1 minus f i minus 1 and divided by 2 delta x and what is the order of accuracy it is order delta x square. Similarly, also we have derived using three points forward and backward and of this first derivative and we have shown that the approximation is second order accurate.

So, let us see that we have used del f by del x, is equal to minus 3 f i plus 4 f i plus 1 plus f i minus i plus 2 divided by 2 delta x. So, this is forward difference approximation and order of accuracy is delta x square. Similarly del 2 f by delta x square, we have used f i plus 1 using central difference method minus twice f i plus f i minus 1 divided by delta x square, and what is the order of accuracy? It is order delta x square.

Similarly, del 2 f by delta x square also you found using three points f i plus 2 minus twice f i plus 1 plus f i divided by delta x square. So, now let us see the coefficient of the values of f.

So, if you see the coefficient of f, f i plus 1 and if you make the summation of coefficients it will be always 0.

So, you can see this is your plus 1 minus 1 is equal to 0, here also you can see, 1 minus 1 is equal to 0, this is also 1 minus 1 is equal to 0, here you can see minus 3 plus 4 minus 1 is equal to 0. Here also you can see 1 minus 2 plus 1 is equal to 0, here also 1 minus 2 plus 1 is equal to 0.

So, when you use some finite difference approximation and see the discretized this derivative first derivative second derivative the summation of the coefficient must be 0. So, to cross check whether your approximation is correct or not, you can sum up all the coefficient and check whether that is coming as 0.

So, if it is 0, so most probably your approximation may be correct. So, for any approximation, so, if you use the third derivative or fourth derivative of any derivative function and if you see that the approximate the finite difference approximation of it, the summation of all the coefficient must be 0.