Computational Fluid Dynamics for Incompressible Flows
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Module 2: Classification of PDEs
Lecture 2: System of first-order PDEs

Hello, everyone. So, in last class, we considered second order PDE and seen the mathematical characteristic of those equations. Today, we will consider system of first order PDEs and we will find the mathematical characteristics of those equations.

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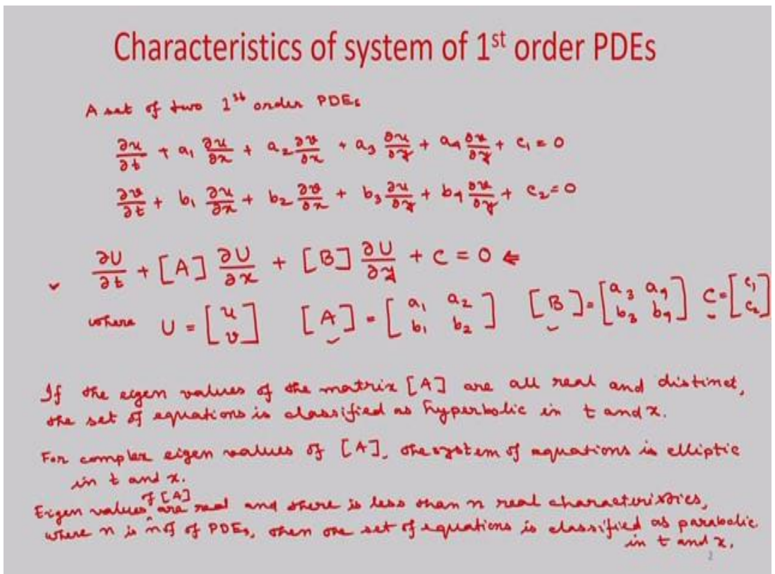

So first, let us consider this system of equations. So, we are considering this a set of 2 first order PDEs. So, we can write these equations like del u by del t plus a1 del u by del x plus a2 del v del x plus a3 del u by del y plus a4 del v by del y plus c1 is equal to 0. So, this is 1 first order PDE. And another PDE we will write as, del v by del t plus b1 del u by del x plus b2 del v by del x plus b3 del u by del y plus b4 del v by del y plus c2 is equal to 0. So, we have considered 2 sets of first order PDEs.

In general, we can write this equation as, del u by del t plus matrix A del u by del x plus B matrix del u by del y plus C is equal to 0, where you have u is equal to this vector u, v. So, what is you are A, so A you can write as, you can see these are the coefficients, so from here you can see that this will be a1, a2, b1, b2. And similarly, you can see that your B will be a3, a4, b3, b4 and C will be your vector, C will be c1, c2.

So, we have represented this 2 partial differential equations, first order partial differential equation in general manner. So, where u is represented by the vector, u, v and correspondingly, A, B matrix and C vector of representative.

Now, we will classify this equation as if, the eigen values of the matrix A are all real and distinct, the set of equations is classified as hyperbolic in t and x. So, we have considered the A matrix and if the eigen values of this matrix are real and distinct, then the equations will be classified as hyperbolic in t and x. And similarly, we are not writing, but similarly you can see the eigen values of B are real and distinct, then the equations will be hyperbolic in nature in t and y. So, that we are not writing, but similarly you can say that.

Now, for complex eigen values of A, the system of equations is elliptic. So, if you have a complex eigen values, then it will be elliptic in t and x. Similarly, for the B matrix if the eigen values are complex, then the equations will be elliptic in t and y. Finally, the eigen values are real and there is less than in real characteristics, where n is number of PDEs, then equations, then the set of equations is classified as parabolic.

So, if you have a system of equations and if you can write in this form, whatever we have written in this equation then depending on the eigen values of A and B matrix, you can say whether the equations are hyperbolic, elliptic or parabolic, and is parabolic in t and x. Similarly, for, if eigen values of A, and similarly for eigen values of B are real and there is less than n number of in real characteristics, then the set of equations will be classified as parabolic in t and y.

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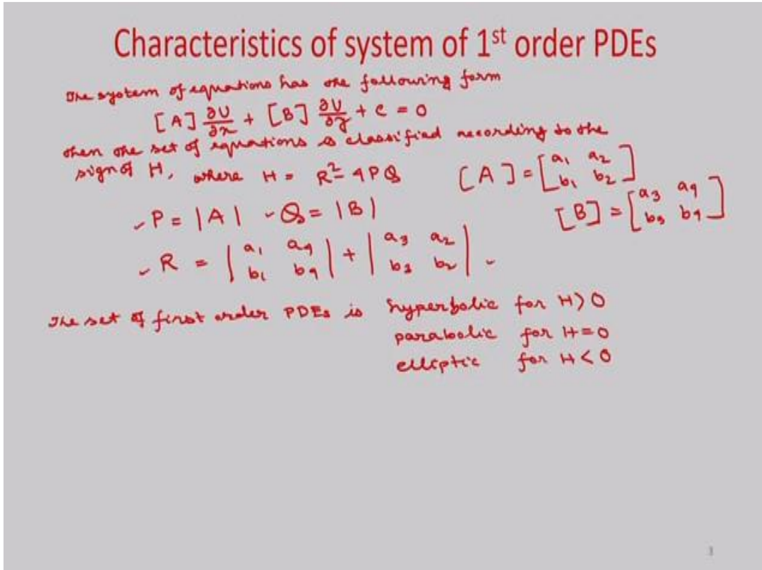

Now let us consider, another set of partial differential equations, first order partial differential equations. So, the system of equations has the following form for steady state, let us say, then we can write as A del u by del x plus B del u by del y plus C is equal to 0. Then the set of equations is classified according to the sign of H, where H is equal to R square by 4PQ and P is just determinant of A, Q is determinant of B, and R is for this set of equations a1, a4, b1, b4 plus a3, b3, a2, b2. So, where A is your a1, a2, b1, b2 and B is your a3, a4, b3, b4.

So, depending on the value of H, where H is equal to R square minus 4PQ and P, Q and R are given as this and if the value of H, so the set of first order PDEs, is hyperbolic for H greater than 0, it is parabolic for H is equal to 0, and elliptic for H less than 0. So, this is another way to find the mathematical characteristics of a system of PDEs.

The next, we will learn another method, which is based on the characteristics. So, you need to find the characteristic, normal characteristic, then depending on the characteristic's values, you can find the mathematical character of those equations.

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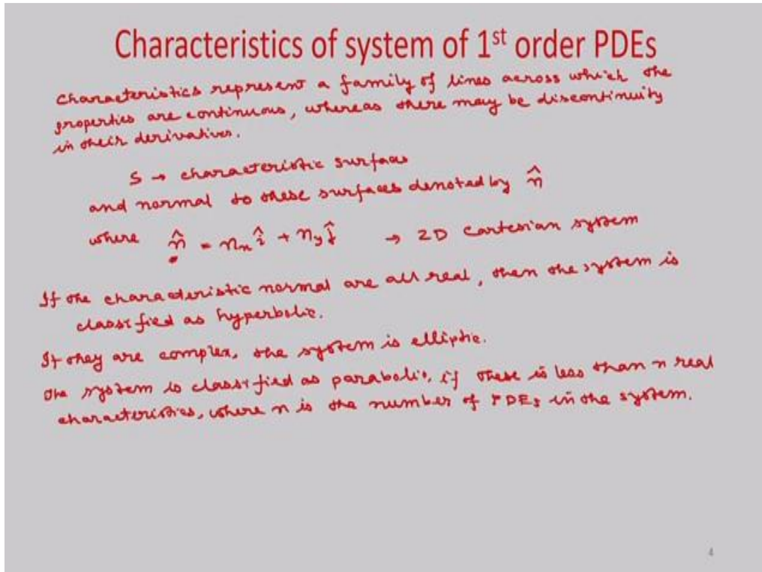

Now, we have already learn what is characteristics. So, characteristics represent a family of lines across which the properties are continuous, whereas there may be discontinuity in their derivatives. Now if you define S, which is the characteristic surface then we will find the normal to that characteristic surface. So, let us say that S is your characteristic surfaces and normal to this surfaces denoted by n, where n is equal to nxi plus nyj for 2D Cartesian system. So, if the system surface is different than S than the normal, we are defining n as nxi plus nyj for a 2D Cartesian system.

So, now based on this normal, we can find the mathematical characteristics. So, if the characteristic normal are all real, then the system is classified as hyperbolic, if they are complex, then the system is elliptic, and the system is classified as parabolic, if there is less than n real characteristics or repeated characteristics, where n is the number of PDEs in the system.

So, now this characteristics of this normal if you find, so if these are real and distinct then it will be hyperbolic, if it is complex then it is elliptic, and if it is real and repeated then it is called parabolic.

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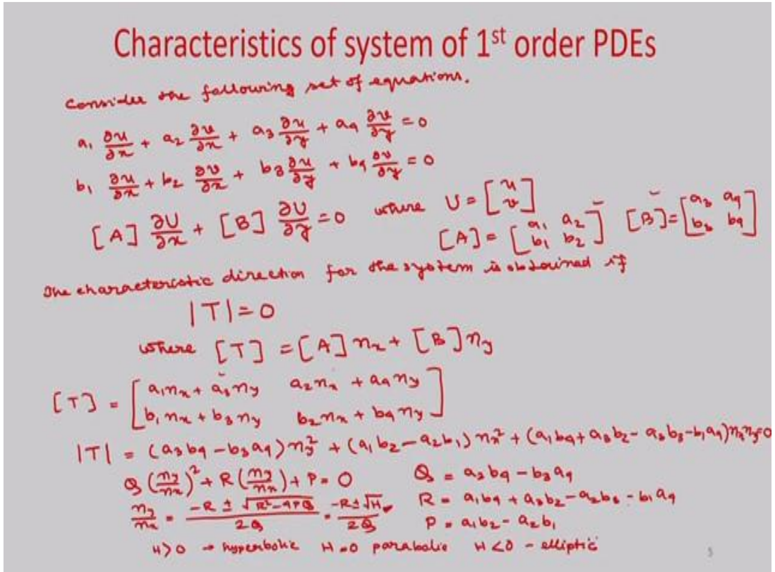

So, now let us consider this set of equations. Consider the following set of equations, a1 del u by del x plus a2 del b by del x plus a3 del u by del y plus a4 del v by del y is equal to 0, and b1 is del u by del x plus b2 del v by del x plus b3 del u by del y plus b4 del v by del y is equal to 0. So, these are the set of 2 first order PDEs.

Now let us write, in general form, A del u by del x plus B del u by del y is equal to 0, where you have U is equal to vector u, v, and obviously A and B will be a1, a2, b1, b2, and B is a3, a4, b3, b4. So, now the characteristic direction for the system may be obtained or is obtained, if determinant T is equal to 0, where T is A nx plus B ny.

So A, B, we know, so these are the matrix. Now, T is defined as A nx plus B ny. So T, you can write as a1 nx, so you can see a is nothing but this, so a1 nx plus a3 ny, then b1 nx plus b3 ny, a2 nx plus a4 ny, and b2 nx plus b4 ny. So, this is your T matrix and if you find the determinant mod T, you are going to get a3 b4 minus b3 a4 ny square plus a1 b2 minus a2 b1 nx square plus a1 b4 plus a3 b2 minus a3 b3 minus b1 a4 nx ny is equal to 0. So, determinant should be 0. So, from here you can write as Q ny by nx square, so if you rearrange it, you will be able to write in this form, ny by nx plus P is equal to 0, where Q is equal to a3 b4 minus b3 a4, then R is a1 b4 plus a3 b2 minus a2 b3 plus a1 minus b1 a4 and P is your a1 b2 minus a2 b1.

So, now you can see this is a quadratic equation, from here you can find ny by nx, it will be minus R plus minus root R square minus 4PQ by twice Q,0 where we have defined H as minus R plus minus root H divided by twice Q. So, now you find the ny by nx and you see whether these are desired real and distinct characteristics then it is hyperbolic, and if it is complex, then it is elliptic and if it is real and repeated then it will be parabolic.

So, here you can see that when this ny by nx will be complex or real and distinct, you can see from the value of H. And already we have defined, that if H is greater than 0, then you have hyperbolic, if H is 0 then it is parabolic, and if H less than 0 then you can see it will be, if it is H less than 0 then it will be complex, so it will be elliptic.

So, now today we have learned 3 different ways to, find the mathematical characteristic of system of PDEs. Now we will consider, different set of PDEs and we will find the mathematical characteristic of those equations.

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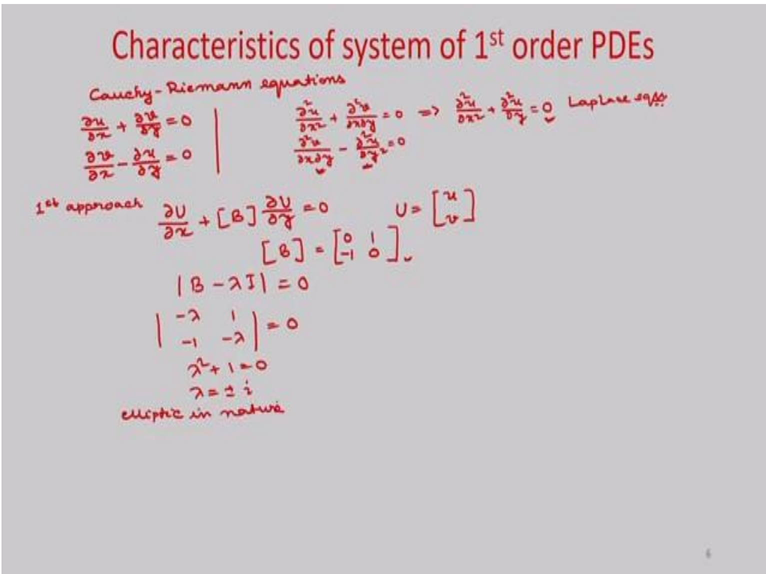

So first, let us consider Cauchy-Riemann equation. So, we consider 2 set of PDEs, one is del u by del x plus del v by del y is equal to 0, and del v by del x minus del u by del y is equal to 0. So, these set of equations are known as Cauchy-Riemann equations. And essentially this boils down to Laplace equation.

So, you can see here del u by del x, so if you take, in the first equation if you take the derivative with respect to x then you will get del2 u by del x square plus del2 v by del x del y is equal to 0. And, if you take the second equation, the derivative with respect to y then you will get del2 v by del2 del x del y minus del 2 u by del y square is equal to 0. And now if you substitute del 2 v, del x del y is equal to del 2 by del y square then from here you will get, del2 u by del x square plus del2 u by del y square is equal to 0. So, from here del2 v by del x, del y is equal to del2 u by del y square. So, if you substitute it, you are going to get a Laplace equation.

Now, let us say that this set of equations we have, how to find the mathematical characteristics of this set of PDEs. So, we will consider this 3 different methods what we discussed and we find, we will find the mathematical characteristic of this set of first order PDEs.

So whatever, first approach whatever we have learned, let us see how to find. So, we have to write in terms of del u by del x plus B del u by del y is equal to 0. So, now if you write in this way, then what will be your B. So now, del u by del x, B del u by del y. So, U is your u, v, this is the vector. So, if you put it del u by del v, then what will be your B, B will be, you can see from here. So, first equation you are getting, you should get del u by del x plus del v by del y. So, if you want to get del v by del y, then you have to see first one will be 0 and this will be 1. So it will be 0, 1.

And, now, second equation if you want to get then you will get del v by del x, so here you should get minus del u by del y so it will be minus 1, and it will be 0. So, this is the B. So you need to find the B matrix from here. So, from here you can see that if you put this, then you are going to get B as this.

So now, for, we have to find the eigen values, so it will, you can find as B minus lambda is, determinant should be 0. So, that means minus lambda so B is this one, so it will be 1 minus 1 minus lambda, should be 0. So that means lambda square plus 1 should be 0, so lambda should be plus minus i. So now we can see, that the eigen values are complex, so obviously, the set of equations is elliptic in nature. So, elliptic in nature.

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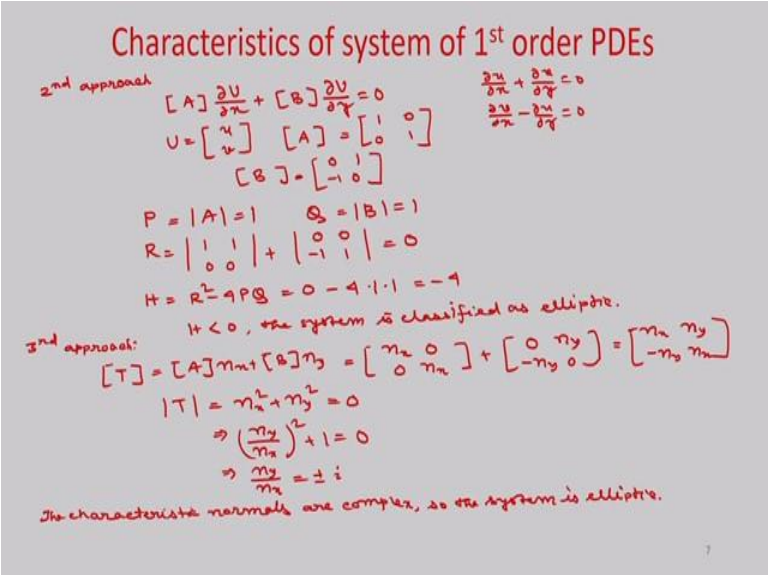

The second method if you use, then you can see how you can find. So, second method or second approach whatever you have learned today, so if you use then you can write this equation in this way, A del u by del x plus B del u by del y should be 0, where U is u, v, vector. So, you can find what is A, so A will be your 1, 0, 0, 1.

So, you put here A, then you can find that whether you are going to get the set of equations whatever we have written, del u by del x plus del v by del y is 0 and del v by del x minus del u by del y. So, from here you find the matrix A and B. So, B will be your 0, 1 and minus 1, 0. So, now find P, Q and R. So, P is your determinant A, which is 1, Q is determinant B, so it will be also 1, and R now with this you can write all this coefficients you have, so it will be 1, 1, 0, 0 plus 0, 0, minus 1, 1. So, this will give you 0. You find the determinant and add it, it will be 0.

So, now you can find H, H is R square minus 4PQ and R square is 0 minus 4, P is 1, Q is 1, so it is minus 4. So, the value of H is less than 0, so obviously it is elliptic in nature. So, as H is less than 0, the system is classified as elliptic.

Now, another method we will use that is your characteristics, we will find the normal characteristics. So third approach, so now you find, A, B you know. So T will be A nx plus B ny. So, from here you can find that it will be nx, 0, 0, nx plus 0, ny, and minus ny, 0 and you will get as nx, ny, minus ny, nx after addition you will get this. Now, you find the determinant of T, so it will be nx square plus ny square and it should be 0. So, you can see that ny by nx whole square plus 1 should be 0 and ny by nx should be plus minus i.

So, you can see the characteristics are complex. So obviously, the system of equations will be elliptic. So, that means the characteristic normals are complex, so the system is elliptic. So, you can see that we have considered Cauchy Riemann equation and we have used 3 different methods which is, which we learned today and applied to find the mathematical characteristics of this Cauchy Riemann equation and we found that this is elliptic in nature. And also, you can see that if you convert it to second order PDE then we are getting one Laplace equation and in last class we have shown that Laplace equation is elliptic in nature.

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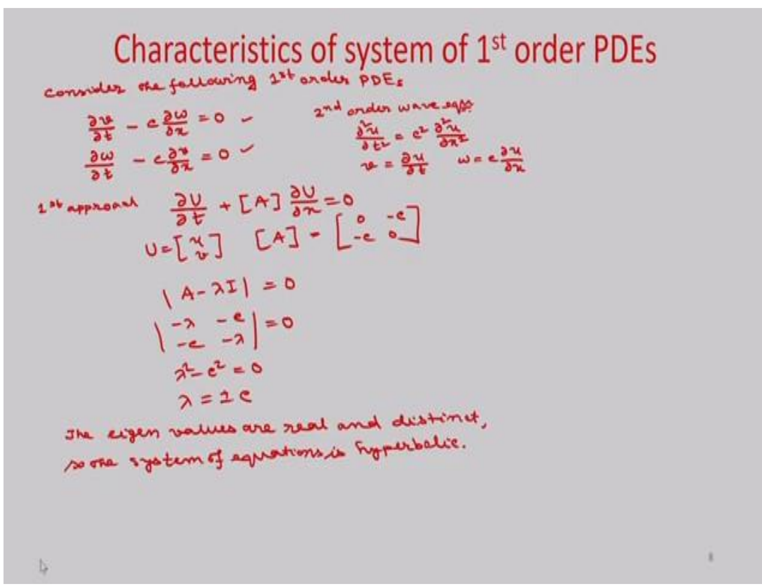

Now, let us consider these 2 sets of PDEs. So, consider the following first order PDEs. So, this we will write as del v by del t minus c del omega by del x, where c is the wave speed and del omega by del t minus c del v by del x is equal to 0. So, these equations actually if you convert it into second order PDE, it will become second order wave equations.

You can see, del v by del t, so this wave equation you can write, second order wave equation. Second order wave equation, so you will get as del 2 u by del t square is equal to c square del 2 u by del x square, where v is del u by del t and w is c del u by del x. So, if you use this transformation you, you are going to get the second order wave equation and we know that second order wave equation, is hyperbolic in nature. But now we have 2 sets of equations, so we will find the mathematical characteristic of these PDEs.

So, first we will use this first approach whatever we have learned. So you can, you have to write this equation in this form del u by del t plus A del u by del x is equal to 0, so where U is the vector u, v and A is, you can find from this set of equations, A will be 0 minus c, minus c, 0. So, c is the wave speed.

So, now the eigen values you can find, so determinant of A minus lambda I should be 0, so it will be, so minus lambda, minus c, minus c, minus lambda should be 0. So, you will get lambda square minus c square is equal to 0, so lambda is equal to plus minus c. So, you can see the eigen values are real and distinct, so the set of these PDEs is hyperbolic in nature. So, the eigen values are real and distinct, so the system of equation is hyperbolic.

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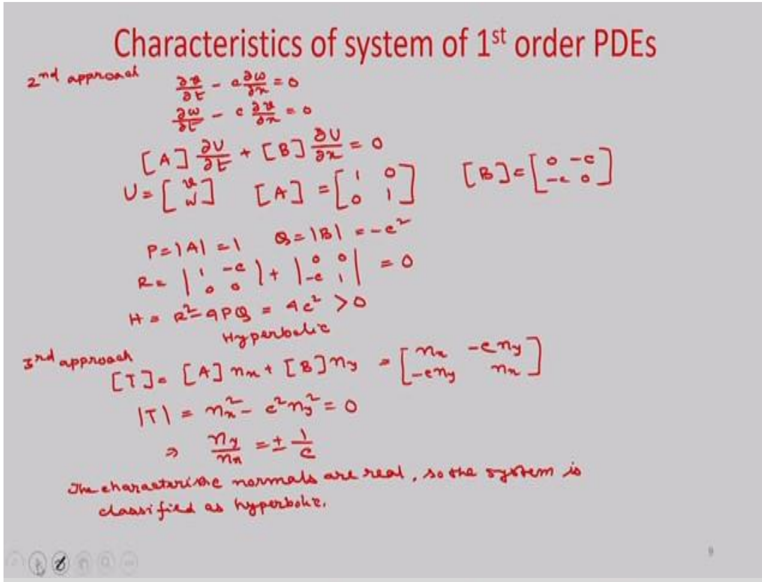

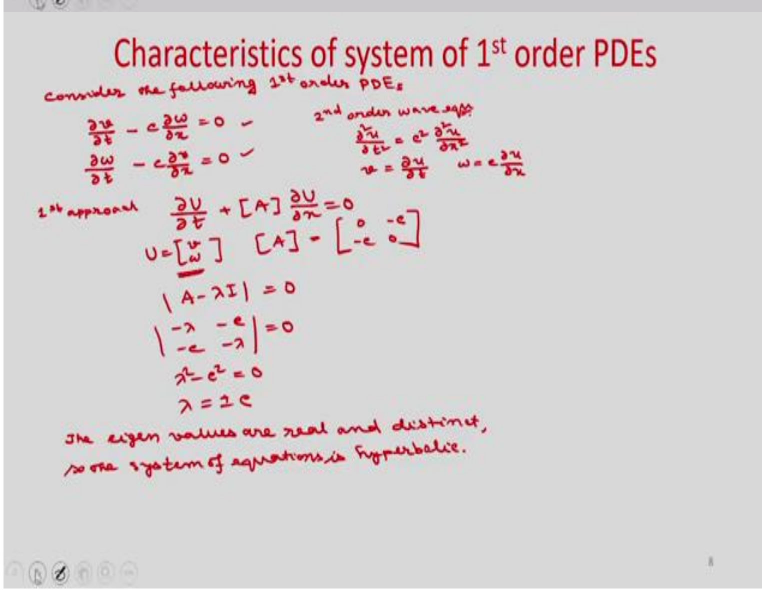

Now, we will use the second method, so we will use the second approach, so whatever we have learned. So, our equation is del v by del t minus c del w by del x is equal to 0 and del w by del t minus c del v by del x is equal to 0. So, we need to write this equation in this form to use the second approach, del u by del t plus B del u by del x. So, we are learning just these 3 different methods. So, we are applying these 3 different methods to find the mathematical characteristics of these set of PDEs. So, this is your 0. So, U is in this case, v, w.

So, here it is not u, it is v, w. So, make this correction, so I wrote u, v, it is only v and w. So U will be v and w. So, this correction you please make.

So v, w and you have A, you find A, A will be 1, 0, 0, 1 and B will be 0, minus c, minus c, 0. So this is your, so P you can find determinant A, 1, Q will be determinant B, so determinant B will be minus c square, and R will be 1, minus c, 0, 0 plus 0, 0, this you just find from these coefficients, so I am not going to write these but you have to find, then it will be 0. So, H will be R square minus 4PQ, so it will be 4 c square, which is greater than 0. So, it is hyperbolic.

Now, let us use the characteristics, normal characteristic method. So, that is your third approach. So you find the T, so it will be T is your A nx plus B ny, so if you, A, B if you add it, you will get nx minus c ny, this is minus c ny, it is nx. So, determinant T will be nx square minus c square ny square is equal to 0, so you will get ny by nx is equal to plus minus 1 by c. So, the characteristics are real and distinct, so obviously the set of equations is hyperbolic. The characteristic normals are real, so the system is classified as hyperbolic.

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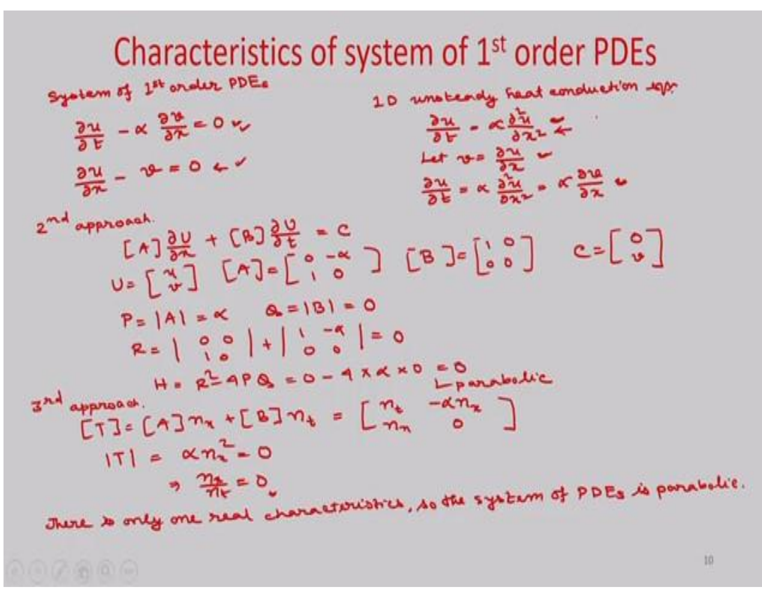

So, next we will consider another set of PDEs. So, we will write these system of PDEs like, del u by del t minus alpha del v by del x is equal to 0 and del u by del x minus v is equal to 0. So, this equation actually represent, if you combine it into second order PDE, then you will get unsteady 1 Dimensional heat diffusion equation. 1D unsteady heat conduction equation, so it is del u by del t is equal to alpha del2 u by del x square. So now, alpha is the thermal diffusivity.

So now, let v is equal to del u by del x v is del u by del x. Then you can see del u by del t so, it is alpha del2 u by del x square. So, you can write alpha del v by del x. So, that means you are getting, one equation is this which is we have written del u by del x minus v is equal to 0, another equation we have written del u by del t minus alpha del v by del x is 0. So, that means these 1 Dimensional unsteady heat conduction equation actually we have written in 2 set of PDEs.

So, now we are considering these 2 sets of PDEs, now let us find the, use the second approach. So, in the second approach you write, A del u by del x plus B del u by del t is equal to c, where U is equal to u, v A is equal to 0, minus alpha, 1, 0 and B is 1, 0, 0, 0 and C will be the vector 0, v.

So, you just find from these 2 equations, you can find this matrix A, B, C. And, now P will be just determinant A, so you will get alpha, Q will be determinant B, so it will be 0, you can see from here and R will be, so 0, 0, 1, 0 plus 1, minus alpha, 0, 0 and it will be 0. So, each will be R square minus 4PQ, R square minus 4Q, so R is 0 and minus 4 into alpha into 0. So, that means it is 0. So, if R is 0, then obviously the set of equations is parabolic. So, it is parabolic in nature.

Similarly, if you use the third approach, so you find the T, T will be your A nx plus B nt, so nt I am writing because there is a time derivative, that’s why. And if you put it, so A and B you put it, you finally will get nt, minus alpha nx, nx and 0. And determinant T if you find, then it will be alpha nx square is equal to 0 that means nx by nt is 0.

So, you can see the characteristic you have, so nx by nt, so this is nx by nt is 0, so it is real and you have only 2 equations but you have 1 characteristic. That means less than n, where n is the number of PDEs so here n is equal to 2, because you have 2 PDEs. You have 2 PDEs, but you are getting one characteristic, 0, nx by nt is equal to 0. So, there is only one real characteristic, so the system is, system of PDEs is parabolic. And also, you can see that you have 1 Dimensional unsteady heat conduction equation, and it is a second order PDE and we have shown in last class that this is also parabolic in nature.

So, today we considered set of first order PDEs and we learnt 3 approaches. And first approach we have found the eigen values, and from the eigen values we have determined whether the system of PDEs are parabolic, elliptic or hyperbolic. Then we have considered this eigen values if these are real and distinct, then it will be hyperbolic, if it is complex, then it is elliptic, and if you have eigen values real and less than n eigen values or n is the number of PDEs, then it is parabolic.

In second approach also, we have found the R, we have found the H, where H is equal to R square minus 4PQ and depending on the value of H, so if H is greater than 0 then we have found that it is hyperbolic, if it is less than 0 then it is elliptic, and if it is 0 then it is parabolic.

And in the last approach, we have found the normal characteristics and if normal characteristics are real and distinct, then it is hyperbolic, if normal characteristics are real and repeated, then it is parabolic, and if normal characteristics are complex, then it is elliptic in nature. Thank you.