**Computational Fluid Dynamics for Incompressible Flows**
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**Module 2: Classification of PDEs**
**Lecture 1: System of second-order PDEs**

Hello, everyone. So, today we will study the classification of partial differential equations. So, many important processes in nature are governed by partial differential equation. So, it is very important to understand the physical behavior of this model, governed by these partial differential equations.

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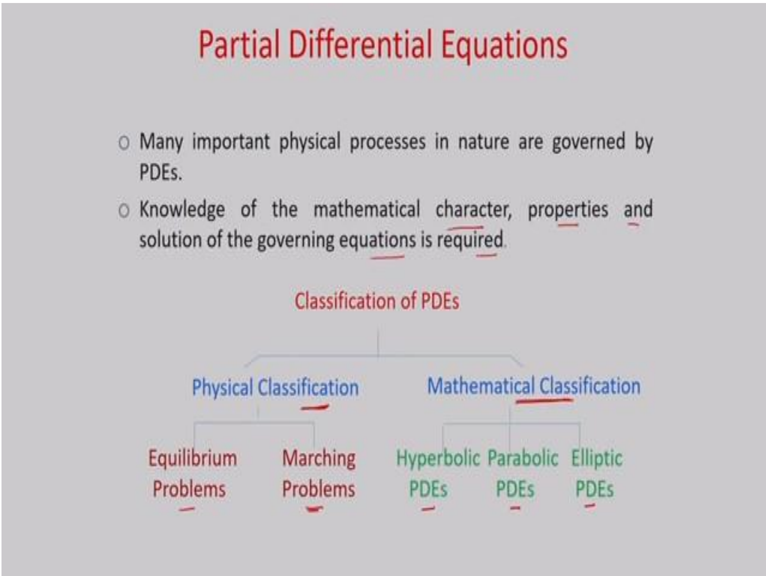


Also, this knowledge of the mathematical character, properties and solutions of the governing equations is required. So, we can classify these partial differential equation in two categories, one is physical classification, and other is mathematical classification. In physical classification, we can classify as equilibrium problems and marching problems or initial value problem. And in mathematical classification, we can classify the partial differential equation as hyperbolic PDEs, parabolic PDEs, and elliptic PDEs.

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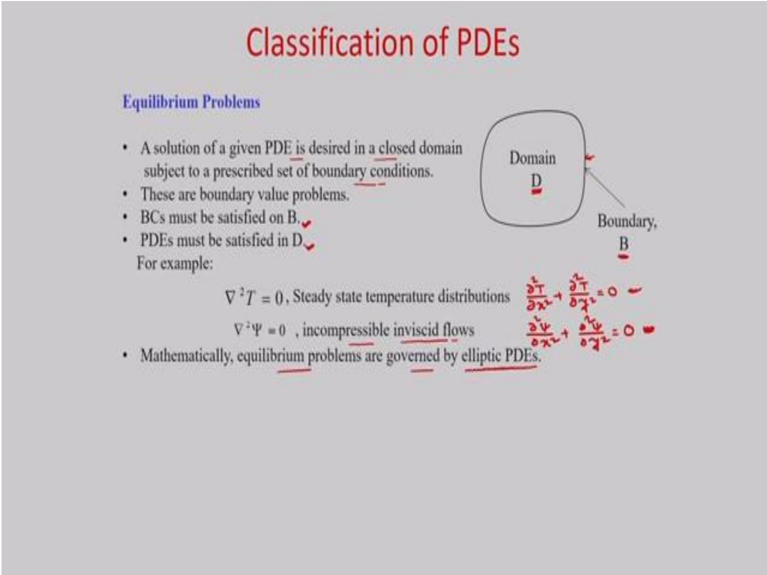


So, first, let us see the physical classification. So, equilibrium problems. So, you have a domain, which is represented by D and it is having the boundary, it is boundary and this represented by B. So, a solution of given PDE is desired in a closed domain subject to a prescribed set of boundary conditions are known as equilibrium problems. So, in equilibrium problems, you need to give the boundary conditions and the solutions and desired in these close domain, D.

So, these are boundary value problems, because you are specifying only boundary conditions. And boundary condition must be satisfied on B and PDEs must be satisfied in the domain D. So, you can see that steady state heat conduction equation, steady state heat conduction equation. So, if you consider 2 Dimension, then it is del 2T by del x square plus del 2T by del y square is equal to 0.

So, here, in this equation you can see that you need to solve these equation in the domain D and you need to specify the boundary condition. So, this is equilibrium problem. Similarly, incompressible inviscid flows, so del 2 psi by del x square plus del 2 psi by del Y square is equal to 0. It is incompressible inviscid flow equation, and this is also equilibrium problem. So mathematically, equilibrium problems are governed by elliptic PDEs. We will show later that mathematically if you see, so you can show that these equations are elliptic in nature, so equilibrium problems are governed by generally elliptic PDEs.

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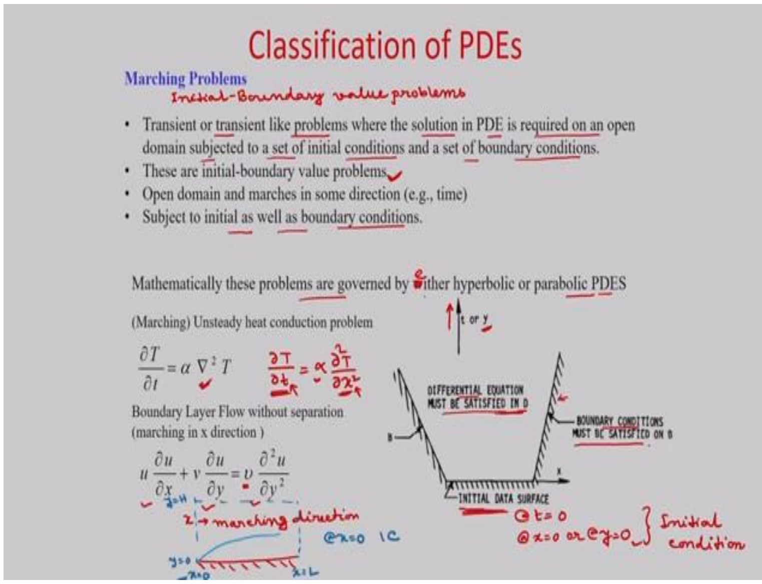


Next, marching problems. So, marching problems are, these are known also as initial value problem, initial, these are known as also initial boundary value problems. So, marching problems are transient or transient like problems where the solution in PDE is required on an open domain subject to a set of initial conditions and a set of boundary conditions.

So, look into this picture. So, this is your some initial data surface, where T is equal to 0 and you are marching in time, T, or so if it is transient problem or transient like problem, so you can have some special direction x or y and you need to satisfy at each time instances these boundary conditions. So, boundary conditions must be satisfied on this B. And, differential equation must be satisfied in domain D. So, this is your domain.

So, differential equation must be satisfied in D, boundary condition must be satisfied at the boundary, and you need to specify some initial condition. Initial condition, if it is a transient problem then that T is equal to 0 you need to specify the initial condition or if it is transient like problem, then at x equal to 0 or y is equal to 0 you need to specify the initial condition. This is your initial condition.

So, these are initial boundary value problem and open domain and marches in some direction time or, or some special direction. And, it is subject to initial as well as boundary conditions. So, later we will show, that mathematically these problems are governed by either hyperbolic or parabolic equations, either hyperbolic or parabolic PDEs.

So, let us consider one problem, 1 Dimensional unsteady heat conduction problem. So, that you can write this equation: del T by del t is equal to alpha del2 T by del x square. If it is in 3 Dimension, you can write in general like this, where alpha is the thermal diffusivity and you can see that you need to march in time t, and you need to specify the boundary conditions as well.

So, you need a solution, which must be satisfied in domain D, you need to specify the initial condition because it is a time marching problem you can see here. And, also you need to specify the boundary conditions. Similarly, transient like problems, we have one example, that is boundary layer equation. So, boundary layer equation you can write, u del u by del x plus v del u by del y is equal to nu del2 u by del y square, where nu is your kinematic viscosity.

So, these boundary layer equation is also parabolic in nature. If you mathematically see, this equation will be parabolic in nature and this you need to march in the direction x. So, in here, x is the marching direction, marching direction. So, if you see you have say, flow over flat plate and you have these boundary layer let us say and you need to solve this in a domain, let us say this is a domain then this is your x equal to 0, this is x equal to L, y is equal to 0 and y is equal to H. Then, at x equal to 0, you need to specify the initial condition. So, at x equal to 0, you give the initial condition and in Y direction you give the boundary conditions. You need to give the boundary condition in Y direction.

So, you can see, although time derivative is, it is not there but it is a initial boundary value problem or marching problem and the marching direction is x. So, it is a transient like problem and if you have a time derivative, obviously it is a time marching problem in the direction time.

So, we have discussed about the physical classification, equilibrium problems and marching problems.

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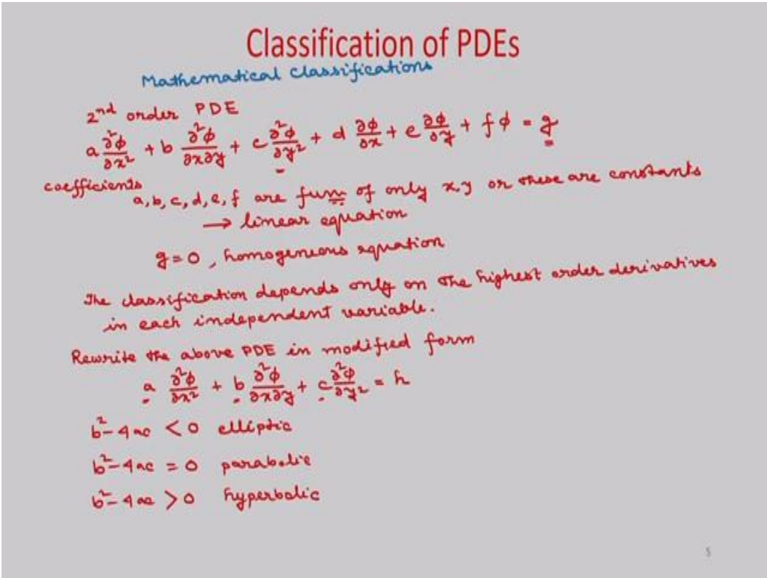


Now, let us discuss about the mathematical classification, mathematical classifications. So, first we will consider in general, a second order partial differential equation. Let us take in general, as a del2 phi by del x square plus b del2 phi by de lx del y plus c del2 phi by del y square plus d del phi by del x plus e del phi by del y plus f phi is equal to g.

So, this is in general we have written one second order partial differential equation and you know, when it is called linear and nonlinear and when it is homogeneous and nonhomogeneous. So, you can see the coefficients, a, b, c, d, e, f, these are the coefficients. If these coefficients are functions of only xy or these are constants, then this is known as linear equation, then it will be linear equation. So, the coefficients are constants or only function of xy. Then this will be called as linear equation, otherwise it is nonlinear. And if g is 0, then it is homogeneous otherwise it is non homogeneous.

So, if this g, g is equal to 0, then this is known as homogeneous equation, otherwise it is nonhomogeneous equation. So, generally the mathematical classification of this PDE will depend on the highest order derivative in each independent variables. So, you can see in this governing equation, highest order is second order, so we can classify based on this second order derivative.

So, we can write this equation. So, the classification depends on, depends only on the highest order derivative, highest order derivatives in each independent variable. So, now we can write this equation in a modified form. So, rewrite the above PDE in modified form, which contains only the highest derivative modified form, so we can write, a del2 phi by del x square plus b del2 phi by del x del y plus c del2 phi by del y square is equal to h. So, now mathematically we can classify as if this coefficient you know, a, b, c, these are the coefficient. If b square minus 4ac less than 0, then it is elliptic, if b square minus 4ac equal to 0, then it is parabolic, and if b square minus 4ac greater than 0, then it is hyperbolic.

So, if you consider a second order PDE and considering the coefficient of highest derivative, if b square minus 4ac is less than 0, then it is elliptic, if b square minus 4ac is equal to 0, then it is parabolic; and if b square minus 4ac is greater than 0, then it is hyperbolic in nature.

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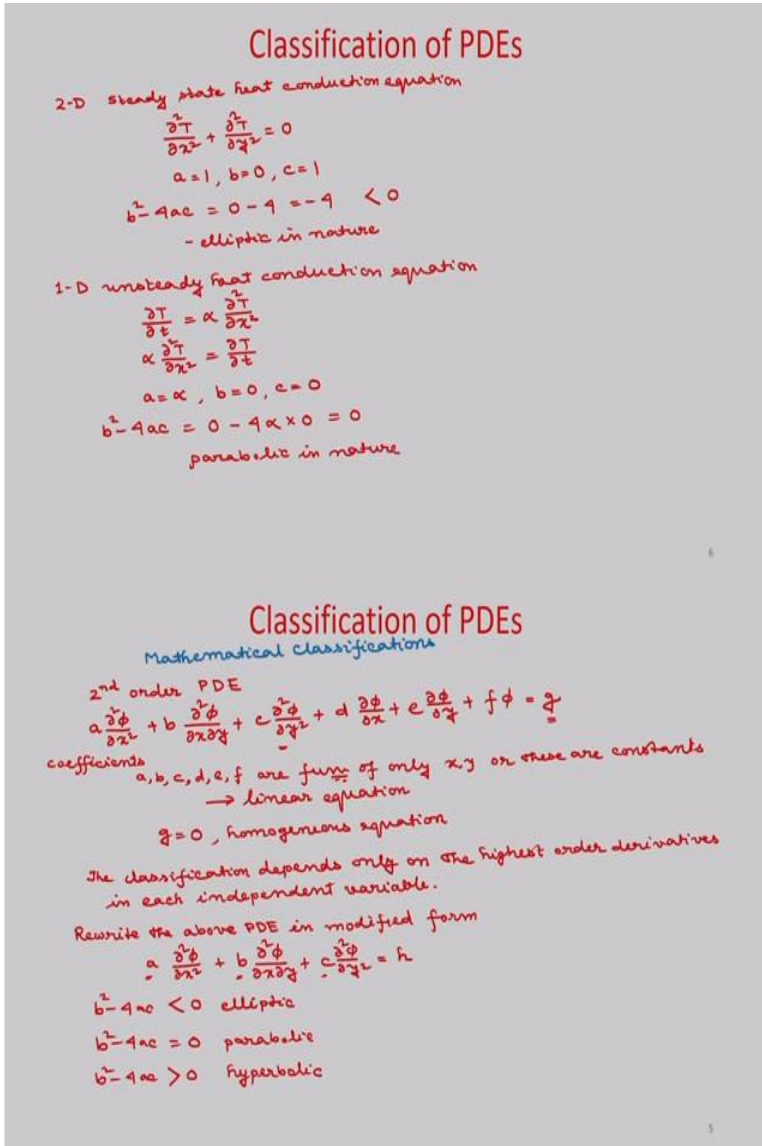


Let us take one example, simple example. So, that is your steady state heat conduction equation. Let us consider, 2D. So, what is that equation, you know, del2 T by del x square plus del2 T by del y square is equal to 0. Physical classification you know, what is the physical classification? So, you know that it is a equilibrium problems. Now, let us see the coefficients. So, if you compare this with this equation whatever we have written, then what you can get.

So, we can write the coefficient is equal to 1. The mix derivative it is 0 coefficient because mix derivative does not appear here, so b should be 0 and c is equal to 1. So, we could write the coefficient, there is no mix derivative so b is equal to 0, and a is equal to 1, and c is equal to 1. So, now find b square minus 4ac. What is b square minus 4ac, b square minus 4ac is equal to 0 minus 4, that means minus 4, which is less than 0. So, based on our definition you can say that this is elliptic in nature. So, this is elliptic in nature.

So, mathematically now you can see this 2D steady state heat conduction equation is elliptic in nature and physically it is equilibrium problems.

Now let us consider, 1 Dimensional unsteady heat conduction equation. So, 1D unsteady heat conduction equation, so 1D unsteady heat conduction equation, so del T by del t is equal to alpha, which is your thermal diffusivity, del2 T by del x square. So, physically this is a marching problems. Now, let us see what is the mathematical character of this equation.

If you see the coefficients, then it is actually alpha del2 T by del x square is equal to del T by del t, so you can see the coefficients, a is alpha, there is no mix derivative, b is equal to 0, and there is no del2 phi by del y square, so that means c is equal to also 0. So, these are the coefficients. And if you find, b square minus 4ac, so it will be 0 and minus 4ac so it is 4alpha into 0 that means it is 0. And we have learned that in b square minus 4ac is equal to 0, then it is a parabolic equation, so it is a parabolic equation.

So mathematically, this equation is parabolic in nature and physically it is a marching problem.

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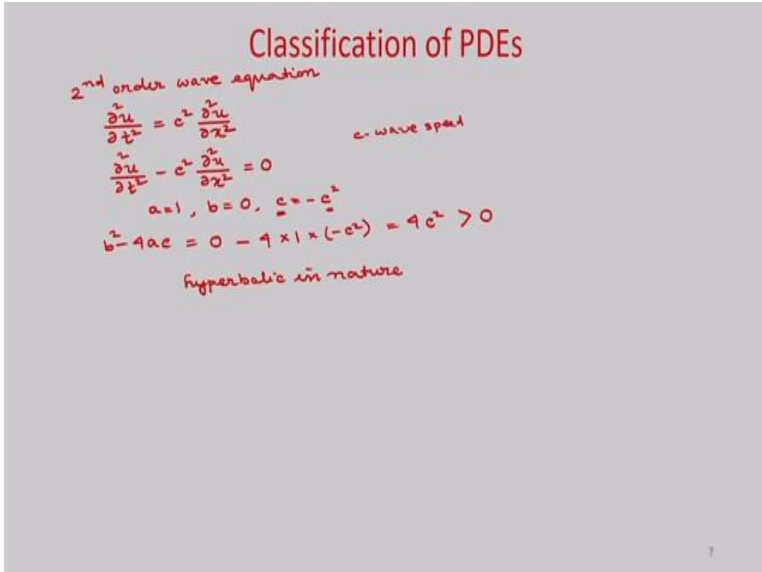


Next, let us consider wave equation, second order wave equation. So, we are considering second order partial differential equations, because we are classifying based on this second order partial differential equation what we discussed in earlier slides. So, we are considering del2 u by del t square is equal to c square del2 u by del x square. So, this is the second order wave equation. Now, you can write it as del2 u by del t square minus c square del2 u by del x square is equal to 0.

So, now see the coefficients. You can write a is equal to 1, there is no mix derivative so b is equal to 0 and c is equal to minus c square, minus c square. C is the wave speed. So, this c is coefficient and this c is your wave speed. So, now you can write b square minus 4ac is equal to 0 minus 4 into 1 into minus c square that means it is 4c square which is greater than 0. So, b square minus 4ac is greater than 0 then obviously it is hyperbolic in nature. So, mathematically it is hyperbolic in nature. And physically, it is marching problem.

So, now we consider 3 different second order PDEs; one is 2 Dimensional heat conduction equation, which is elliptic equation; then we considered 1 Dimensional unsteady heat conduction equation, that is parabolic equation and then we considered second order wave equation and it is hyperbolic equation.

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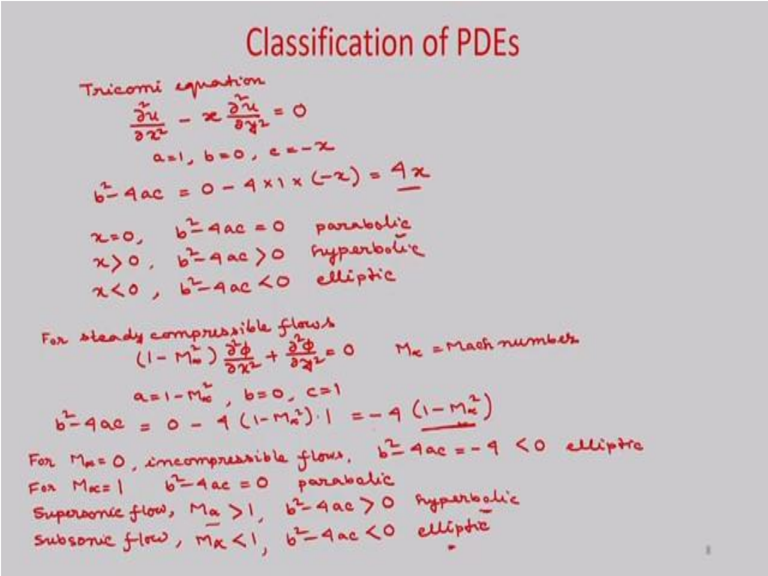


Now, let us consider another equation, which is known as Tricomi equation. It is represented as, del2 u by del x square minus x del2 u by del y square is equal to 0. So, now find the coefficients, so a is equal to 1, there is no mix derivative, so b is equal to 0, and c is equal to minus x. So, now you can write, b square minus 4ac is equal to 0 minus 4 into 1 into minus x, so it is 4x, minus minus plus so it will be 4x.

So, now you can see that the value of b square minus 4ac will depend on the value of x. So, x can be 0, x can be positive or x can be negative and depending on the value of x you will get different mathematical character of this equation. So, you can see if it is 4x, if x equal to 0 what you are going to get, you are going to get b square minus 4ac is equal to 0. So, that means it is parabolic in nature, so it is parabolic. Then if x greater than 0, if x greater than 0 then you will get b square minus 4ac as greater than 0, and it will be hyperbolic. And if x less than 0, then b square minus 4ac also will be less than 0 and you will get elliptic.

So, you can see this Tricomi equation, the mathematical behavior will depend on the value of x and it will change its behavior depending on the value of x so it may be elliptic, parabolic or hyperbolic.

Let us consider another equation, so that is known as for steady compressible flows. So, you can write 1 minus M infinity square, where M infinity is your Mach number, del2 phi by del x square plus del2 phi by del y square is equal to 0. M infinity is Mach number.

So, now you find the coefficients. Here there is no mix derivative, so b will be 0; and a will be 1 minus M infinity square; and c will be 1. So, a is minus M infinity square, b is 0, and c is 1. So, you will find b square minus 4ac is equal to 0 minus 4, 1 minus M infinity square into 1. So, it will be minus 4, 1 minus M infinity square.

So, you can see that b square minus 4ac depends, its sign depends on the value of M infinity. So, if M infinity is 0, for M infinity is 0 that means it is incompressible flow, incompressible flows. So, if M infinity is 0, what will be b square minus 4ac, it will be minus 4, which is less than 0. So, it will be your elliptic in nature. And, if, for M infinity is equal to 1, M infinity is 1, then you will get b square minus 4ac is equal to 0. So, it will behave as parabolic.

So, let us consider two different kinds of flow, one is subsonic flow and supersonic flow. So, if we consider supersonic flow, then M infinity will be greater than 1. So, if M infinity greater than 1, so you can see, so it will be negative and this negative, negative will be positive.

So, b square minus 4ac will be positive and it will be greater than 0. So, if it is greater than 0 then obviously it is hyperbolic in nature, so it is hyperbolic. And if you consider, subsonic flow, so it will be M infinity less than 1. So, if M infinity is less than 1, so you can see in this expression if it is, M infinity is less than 1 then this itself will be a positive. 1 minus M infinity square will be positive and b square minus 4ac will be negative. So, that means it is, b square minus 4ac will be less than 0 and it will be elliptic in nature.

So, now you can see, for this particular equation depending on the value of infinity, you are getting deeper in mathematical character of this equation. If M infinity is 1, then b square minus 4ac is 0. So obviously, it is becoming parabolic in nature. If M infinity is 0, then you can see it will be less than 0, then it will be elliptic in nature; but for supersonic and subsonic flow, depending on the value of M infinity, you are getting hyperbolic if M infinity is greater than 1 and if it is, M infinity is less than 1 for subsonic flow then it will become elliptic.

So, now we have seen the second order PDE and depending on the highest order derivative, we have classified this equation mathematically and it can be elliptic, parabolic, and hyperbolic depending on the value of b square minus 4ac. So, can we reduce this PDE to ODE and can we see the characteristic lines or curves of these equations?

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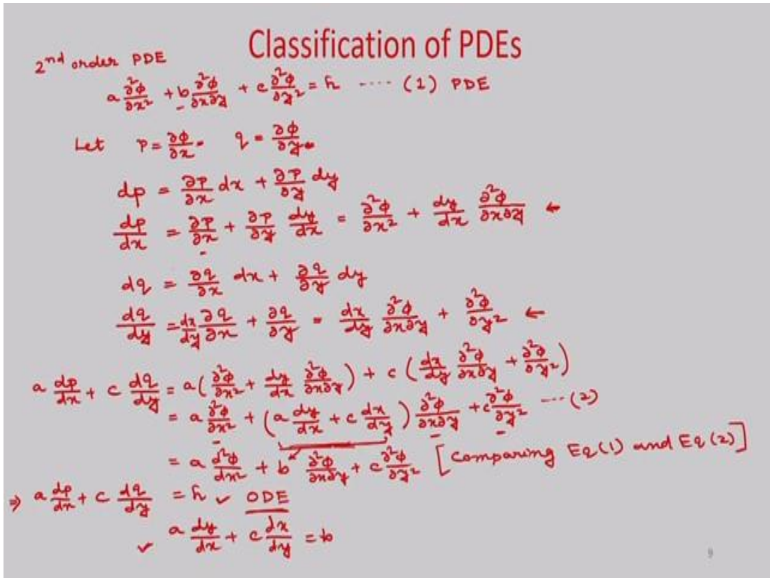


So, we had equation a del2 phi by del x square plus b del2 phi del x del y plus c del2 phi by del y square is equal to h. It is a second order PDE. So, now can we reduce this PDE to ODE, so now to reduce it, let us say this equation is 1. Let us, take p is equal to del phi by del x and q as del phi by dely. So, we can write dp as del p by del x into dx plus del p by del y into dy. So, and also you can write, dp by dx is equal to del p by del x plus del p by del y into dy by dx.

So, you can see del p by del x, what you can write in terms of phi, so p is del phi by del x, so del p by del x will be del2 phi by del x square plus and del phi by del y, so obviously if you take del phi by del y so it will be del2 phi by del x and del y. So, you can write dy by dx, del2 phi by del x del y.

Similarly, if you write dq is equal to del q by del x, dx plus del q by del y, dy. And you can write dq by dy is equal to del q by del x plus dx by dy will be, so it will be dx by dy plus del q by del y. So, q is del phi by del y, so del q by del x if you take the derivative with respect to x of this of this q is equal to del phi by del y, then you will get dx by dy as del2 phi by del x del y. And if you take del q by del y, then you will get del2 phi by del y square.

So, now let us take a dp by dx plus c dq by dy. So, dp by dx, this is the expression, dq by dy, this is the expression you put it, then rearrange, what you will get, so it will be a del2 phi by del x square plus dy by dx, del2 phi by del x del y plus c into del q by del y. So, it will be dx by dy, del2 phi by del xy, del x del y plus del2 phi by del y square. So, if you see the right hand side, so you can write a del2 phi by del x square plus a dy by dx plus c dx by dy into del2 phi by del x del y plus c into del 2 phi by del y square.

So, now you can see, so a del2 phi by del x square plus some coefficient into mix derivative plus into del2 phi by del y square.

So, left hand side we have ODE, so when you can reduce it to this PDE to ODE, so if these a dy by dx plus c dx by dy, if it becomes b then you can write b is equal to a. So, if you can write it as a d 2 phi by dx square plus b, so this time together if you can write as b del2 phi by del x del y plus c del2 phi by del y square. If you can write comparing, so if it is equation 2, so comparing equation 1 and equation 2 we could write this, then it will be equal to h.

So, in a certain condition you can actually reduce this PDE to ODE, provided here, you have a dy by dx plus c dx by dy is equal to b. So, in this condition, you can reduce this PDE to ODE because now you are getting the ODE as a dp by dx plus c dq by dy is equal to h. So, that you are getting. So, this is, you can see this is your ODE and this is your PDE, so this PDE you can write, ODE with the condition a dx plus c dx by dy is equal to.

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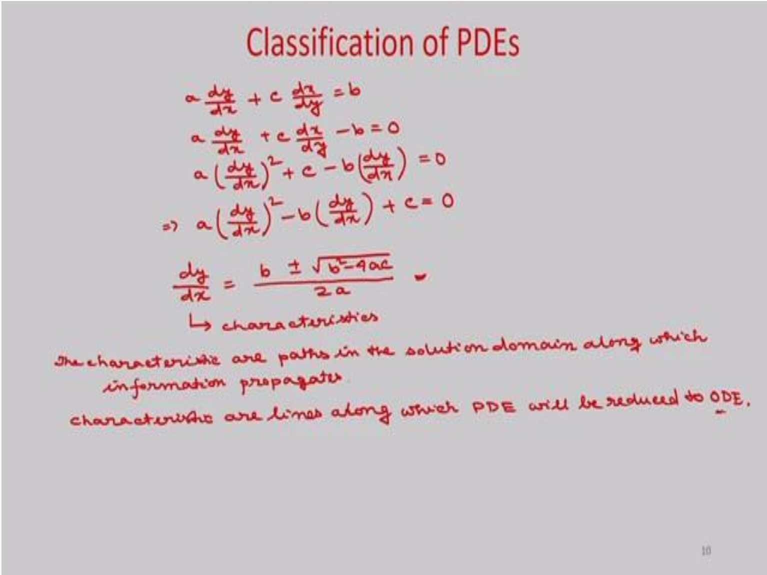


Now, if a dy by dx plus c dx by dy is equal to b, if you rearrange it then you can write, a dy by dx plus c dx by dy minus b is equal to 0. Now, you multiply with dy dx. So, you can write a dy by dx square, so you are multiplying with dy dx so it will be c minus b dy by by dx equal to 0, or you can write it as, a dy by dx square minus b dy by dx plus c is equal to 0 and you can find the value of dy dx, which is the slope of the curve, which is the slope of the curve, you can write dy dx as b plus minus b square minus 4ac divided by 2a. So, now you can find the slope of this curve dy dx as b plus minus b square minus 4ac by 2a and this dy/dx is known as, curves of the characteristics. So, it is characteristics.

The characteristics, are paths in the solution domain along which information propagates. And for this particular situation, these characteristics are lines along which PDE will be reduced to ODE. Because we started with PDE and with this condition dy by dx is equal to b plus minus b square minus 4ac by 2a, you can see that characteristics are lines along which PDE will be reduced to ODE.

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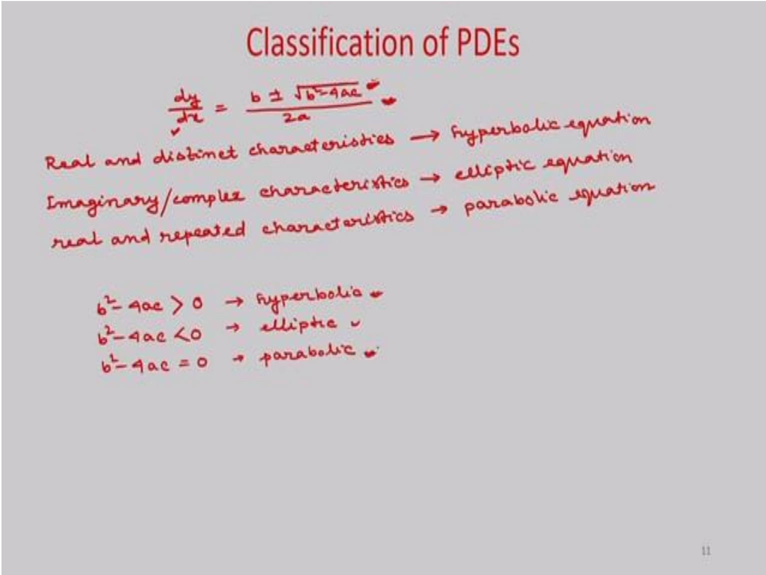


So, now we will see, how we can classify mathematically seeing the characteristics. So, dy by dx is equal to b plus minus root b square minus 4ac divided by 2a, so you have real and distinct characteristics, then this will be mathematically hyperbolic equation. Then if you have imaginary or complex characteristics, then it will be elliptic equation. And, if you have real and repeated characteristics, then you have parabolic equation.

So, first, you have to find the characteristics. So, if you find that you have real and distinct characteristics, then it will be hyperbolic in nature; and if it is imaginary or complex characteristics, then you will get elliptic in nature; and if you have real but repeated, then you will get parabolic in nature.

So now, from here, now you can see when it will become real and distinct. So, this discriminant, b square minus 4ac we will just see. So, if b square minus 4ac, if it is greater than 0, then only you will get a real and distinct characteristics.

So, if b square minus 4ac is greater than 0, then this characteristic will be real and distinct, that means b square minus 4ac for the second order PDE already we have discussed that it will be hyperbolic. And, when it will become imaginary and complex, if b square minus 4ac is less than 0. So, b square minus 4ac, if it is less than 0, then you will get imaginary characteristics. So, it will be elliptic. And when you will get repeated, real and repeated if b square minus 4ac is 0, then you will get b square minus 4ac is 0 then you will get parabolic.

So now, converting this PDE to ODE with a certain condition, we have shown that characteristics dy by dx, we have expressed in terms of the coefficients abc. And this is the relation, b plus minus root b square minus 4ac by 2a. So, dy dx is the characteristic. So, this characteristic you have represented in terms of coefficients a, b, c, of the second order PDE.

So now, we have defined when it will become hyperbolic, parabolic and elliptic, seeing the nature of the characteristics. And you can see now, you can correlate it what we defined in the beginning that b square minus 4ac, if it is greater than 0 then only you will get real and distinct, so you write it is hyperbolic without seeing the characteristics; then b square minus 4ac less than 0, then elliptic; and if b square minus 4ac is 0, then you will get only real and repeated characteristics, then it will parabolic in nature.

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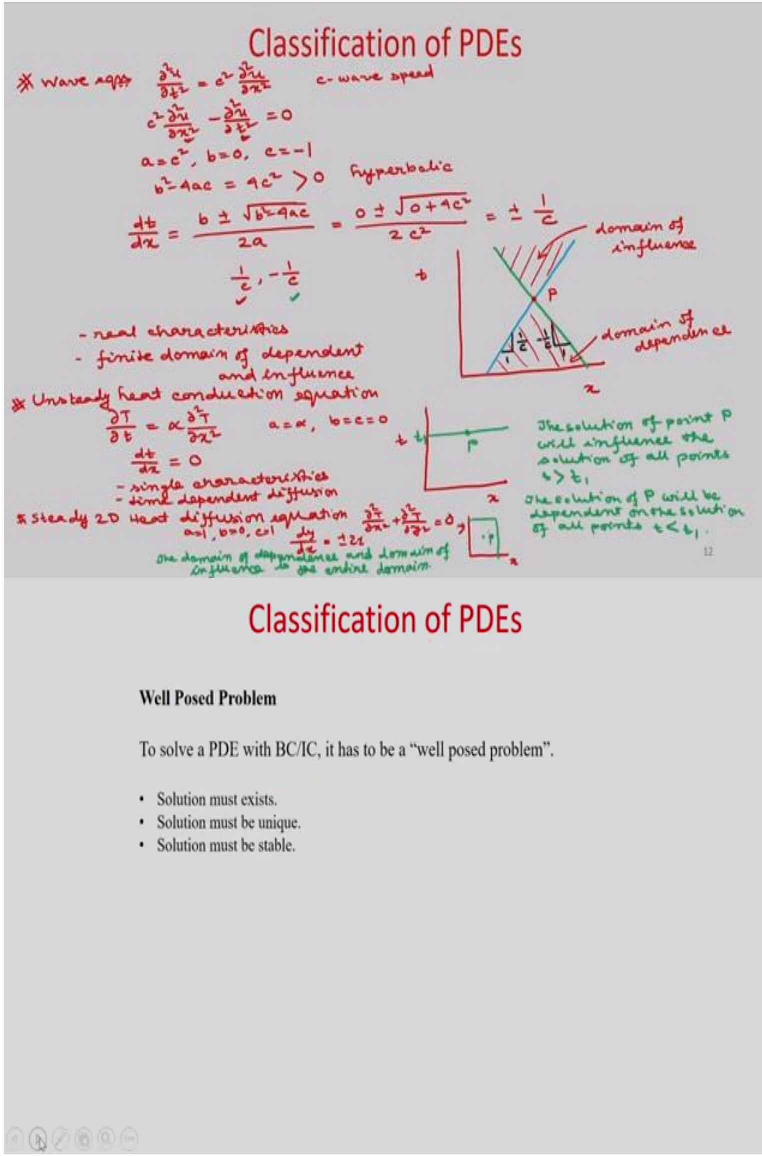



Now, let us consider the second order wave equation. Wave equation, so what is that equation, it is del2 phi by del t square, or del2 u, let us write, del2 u by del t square is equal to c square del2 u by del x square, c is the wave speed. And if you write the coefficients, so let us say c square del2 u by del x square minus del2 u by del t square is equal to 0, then your coefficient a is c square. There is no mixed deri, B is equal to 0 and c is minus 1. We have shown that b square minus 4ac, we have shown b square minus 4ac is, it is your 4c square which is greater than 0 and it is hyperbolic in nature.

So now, let us find the characteristic of this equation. Now here, you have x and t, so you will write dt by dx, dt by dx because earlier we have written dy by dx because it was del2 u by del y square, so now you are writing dt by dx which is the characteristic as b plus minus root b square minus 4ac divided by 2a. So now, what is b, b is 0 plus minus, so it is 0 minus 4ac, so it will be plus 4c square divided by 2a. So, it will be 2a means, 2c square, so it will be, so 2c so it will be plus minus 1 by c.

So, dt by dx is plus minus 1 by c. So, you can see, you have a real characteristics and distinct characteristics, because one you have 1 by c, and another you have minus 1 by c. So, these are the real and distinct characteristics.

Now, if you plot it in, so this is x and this is your t. So, if you see, if you for, 1 by c if you plot, then you will get the curve as like this because you will have the, you will have the slope, so if it is 1 and it is 1 by c then dt by dx will be 1 by c. And similarly, if you see for this minus 1 by c, then it will be like this. And the slope you can see, slope is, this is your 1, this is minus 1 by c.

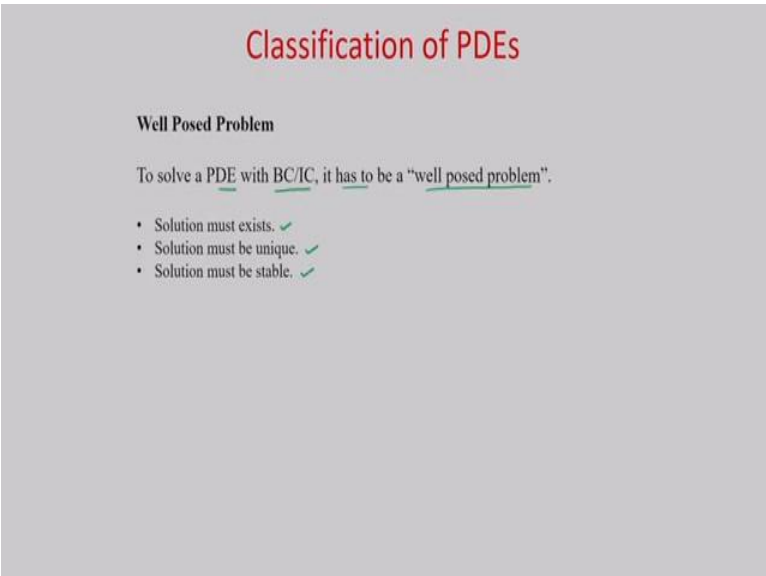
So, 2 characteristics, so here you can see that you will get, this is your right traveling waves and you will get left traveling waves. And at any point p if you consider, so you can have two characteristics one with the positive slope, and another with negative slope. And, the solution at point p is dependent on only solution of a finite region. So, you have real characteristics and you have finite domain of dependent and influence. So, you have domain of influence is this one, because this is a domain of influence and this is your domain of dependence, so this is your domain of dependence.

Next, let us consider the heat conduction equation. So, if you consider unsteady heat conduction equation. So, we have del T by del t is equal to alpha del 2 T by del x square. So, if you see so it will be a is equal to alpha and b and c are 0, so what will be your dt by dx, so dt by dx if you put it, it will be 0. So, you have a single characteristic and its value is 0, so single characteristics and you have time dependent diffusion.

If you plot it, in the tx plot, so this is your x, this is your t, so dt by dx is 0, right, dt by dx is 0. So, you will get the curve as this, if this is the point p, then the solution of point p, will influence the solution of all points t greater than t1. If it is t1, so the solution of point p will influence the solution of all points at t greater than t1. And, the solution of p will be dependent on the solution of all points t less than t1. The solution of p will be dependent on the all solution, on the solution of all points, which is t less than t1.

And next, let us consider the elliptic equation, which is your steady 2D equation, heat diffusion equation, 2D heat diffusion equation. So, del2 t by del x square plus del2 t by del y square is 0. So, you will get a is equal to 1, b is equal to 0; c is equal to 1, and if you see the characteristic, dy by dx is your plus minus 2y. So, you can see these are imaginary characteristics, obviously, it is elliptic in nature. So, if it is elliptic in nature, so the domain of, so if you see this is your x and this is your y and you have, this is the domain and if any point p then domain of dependence and domain of influence is the entire domain, entire domain and it is a equilibrium problem.

So here, the domain of dependence and domain of influence is the entire domain. What does it mean, so, if you see the solution of p, if you forget it will go in all direction. So obviously, domain of dependence and domain of influence will be the entire domain. (Refer Slide Time: 52:06)




So, with that now see what is well posed problem. So, to solve a PDE with initial condition and or boundary condition, it has to be a well posed problem. What is well posed problem, so well posed problem, if the solution must exist, solution must be unique and solution must be stable. Then you can say, that it is a well posed problem.

Now in summary, so let us see the comparison of elliptic equation, parabolic equation and hyperbolic equation.

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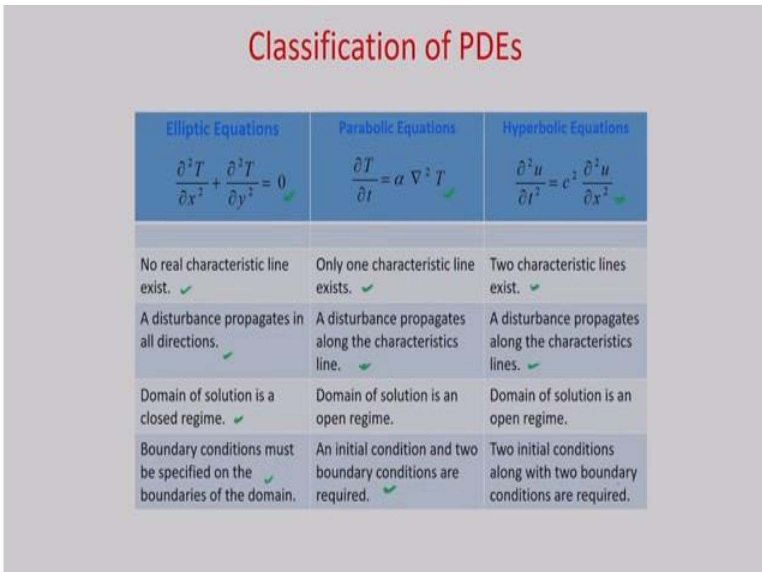


So, we are taking these 2 Dimensional heat conduction equations, steady state heat conduction equation,1 Dimensional unsteady heat conduction equation and also we have considered, second order wave equation, so which is hyperbolic in nature. So, you can see that for the elliptic equation, no real characteristic line exist, because we got imaginary characteristic.

For the parabolic equation, only one characteristic line exists. So, that we have shown. So, the dt by dx is 0 And for the hyperbolic equation, we have shown that two characteristic line exists. So dt by dx is equal to plus minus 1 by c we have shown.

In case of elliptic equation, if you disturbed, so that disturbance propagates in all direction. In case of parabolic equation, a disturbance propagates along the characteristic line. And here, for hyperbolic equation, a disturbance propagates along the characteristic lines. For the elliptic equation, domain of solution is the closed regime but for parabolic and hyperbolic equation, domain of solution is an open regime. And for elliptic equation, boundary condition must be specified on the boundaries of the domain. For parabolic equation and initial condition and two boundary conditions are equal, but for hyperbolic equation two initial conditions along with two boundary conditions are required.

In today's lecture, so we have seen the mathematical and physical classification. So, in physical classification, so we have categorized in two ways, one is elliptic problem, equilibrium problem and marching problem. So, equilibrium problems are generally elliptic in nature and marching problems are parabolic of hyperbolic in nature. And mathematically, we have categorized as elliptic equation, parabolic equation and hyperbolic equation.

First, depending on the coefficient, b square minus 4ac we have seen if it is less than 0 then it is elliptic in nature, if it is b square minus 4ac is 0 then it is parabolic in nature, and if b square minus 4ac if it is greater than 0, then it is hyperbolic in nature. Then we have found, the characteristic lines dy by dx and based on that we have defined if two real and distinct characteristics are there, then the equation is hyperbolic in nature, if you have real and repeated characteristics, then it is parabolic in nature, and if you have real and imaginary characteristics, then it is elliptic in nature. Thank you.