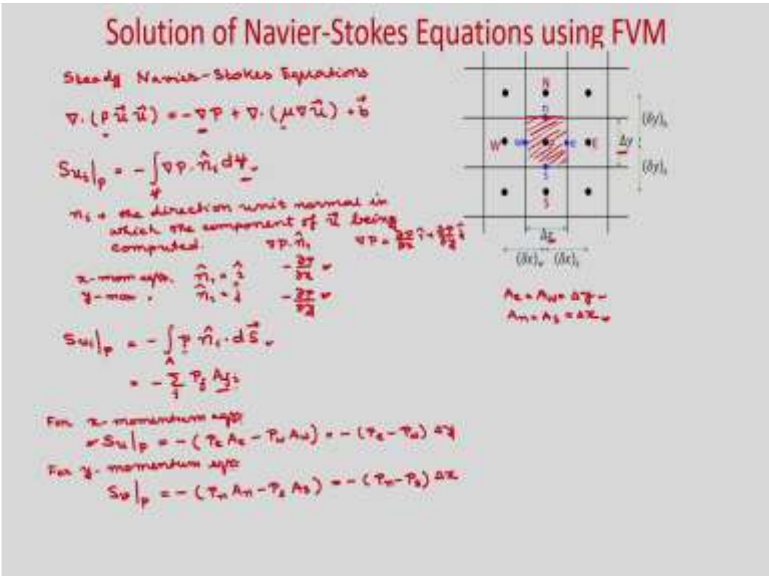
Computational Fluid Dynamics for Incompressible Flows
Professor Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati
Lecture 01
Solution of Navier-Stokes Equations using FVM

Hello everyone, so, till now, we have discretized the convection diffusion equation using finite volume method. So, now you are in a position to discretize the Navier-Stokes equation using Finite Volume Method. So, in the Navier-Stoke equation pressure gradient term appears. First let us discretize this pressure gradient term using finite volume method.

(Refer Slide Time 00:56)



So, consider first the co-located grid where u, b, w, pressure and all the scalars are defined at the cell centre p. So, that is known as co-located grid, you have learned while learning the Mac algorithm. So, first let us write the Navier-Stokes equation. So, we will write for steady Navier-Stokes equations. So, in vector form if you write then it is divergence of rho u, u is equal to minus grad p plus divergence of mu grad u.

So you can see this is your convection term, this is your diffusion term. So, if you do not have this pressure gradient term, then this represents the steady convection diffusion equation. So, you know how to discretize this convection diffusion equation. Now, here pressure gradient term appears in the Navier-Stokes equation. So, we will integrate this pressure gradient term using finite volume method. Here also you can write the general source term b.

So, now this pressure gradient term let us discretize using finite volume method over this main control volume p. So, at this cell centre p, we are defining u, v and pressure p. So, if you write Sui, i is the direction in which you are taking the momentum equation, whether x momentum or y momentum equation, so it will come u or v for the pressure.

So this is the source term. So you can integrate over the control volume v grad p dot ni dV. So, what is ni? Ni is the direction unit normal in which the component of u being computed. So if you consider u momentum equations, so this will be only I, okay. So, if you consider u momentum equation or x momentum equation, okay, so, you will consider only u.

So, for that you will get ni as i okay and for y momentum equation you will get ni as j. So what you are going to get? So grad p dot ni, so if you take this obviously, you are going to get minus del p by del x. So, if you consider only grad p dot ni, so, what you are going to get in the x momentum equation? So, you are going to make the dot product with grad p with i. So, grad p is for two dimensional case it is del p by del x i plus del p by del y j.

So, now if you make the dot product with grad p with the unit normal in the x direction I then you are going to get minus del p by del x. And similarly, when you are making dot product with grad P and j, j is the unit normal in the y direction then you are going to get minus del p by del y and those are the pressure gradient terms in x momentum and y momentum respectively.

So, it is del p by del x and this is your minus del p by del y and this is the pressure gradient term right, in x momentum equation and this is y momentum equation. So, now, this let us integrate over this control volume now use Gauss divergence theorem, then convert it to surface integral okay. So, if it is surface integral then you can write it as p ni dot ds.

So using Gauss divergence theorem this volume integral we are converting to the surface integral and you know p is the pressure, pressure is the scalar. So, now this you can write minus summation of Pf, all the faces Afi. Again Afi okay, Afi is the area, magnitude of the area in particular direction.

So, if you are considering this two dimensional grid, so, if you are considering the x momentum equation then in the i direction what is the area? So, you can see this is delta y okay. So, this is your delta y. So, in this case Ae is equal to Aw is equal to delta y okay. So, i is in the direction which you are actually considering the momentum equation.

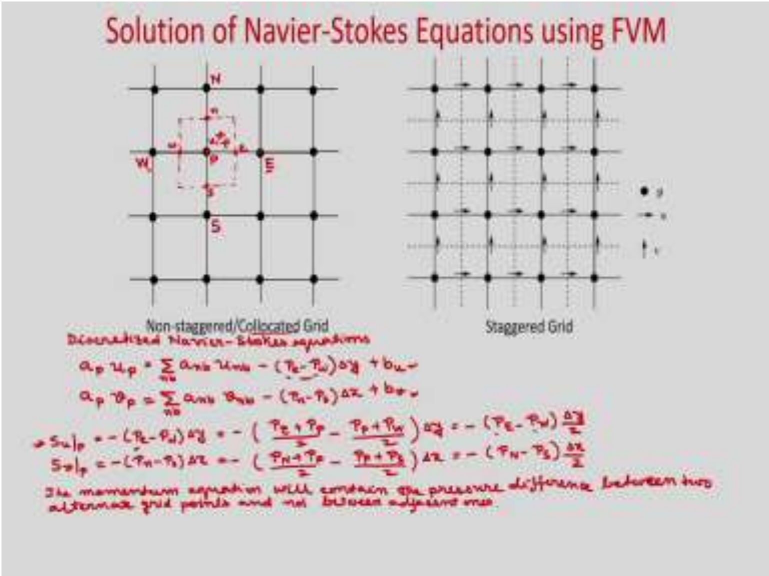
Similarly if you consider b momentum equation, then your, this normal will be in the j direction and you can see this is your surface area. So that will be delta x. So, e n is equal to As is equal to delta x. So, if you write these ways pressure gradient source term for the u momentum equation, let us write what you are going to get.

So, now we are going to write this source term if you are considering the u momentum equation or x momentum equation. So, for x momentum equation, this is u okay the pressure gradient term. So how many faces are there? So it is two. So you are going to get minus Pe Ae minus Pw Aw. So, this Pe Ae minus Pw Aw.

So, if you are considering a uniform mesh, then A is equal to AW is equal to delta y. So, you can write minus Pe Ae minus Pw into delta y. So, this is the pressure gradient term after integrating over this control volume P. Similarly, for y momentum equation this is V, so V, Y momentum equation, so, we are writing v, so minus PnAn minus PsAs. So, what you are going to get?

So, if we will consider uniform mesh, then An is equal to As is equal to delta x. So, this will get minus Pn minus Ps into delta x. So, we have integrated this pressure gradient term over the control volume on a co-located grid or non-staggered grid. So, now, you know the discretization of convection diffusion equation using finite volume method in Ap phi p form. So, that let us write the discretized Navier-Stokes equation on this using this control volume, using this finite volume method.

(Refer Slide Time: 09:39)



Okay so, what we are going to write? ap up, okay ap phi p form we are writing. So discretized Navier-Stokes equation. So x momentum equation, this is ap up summation of all the neighbor anb unb. Now we have the pressure gradient term. So, now we will write as Pe minus Pw, so Pe minus Pw.

So, this is your I forgot to mention here, so PE, so face at East and this is the west okay, this is the face centre. So, here we are writing Pe minus Pw into delta y and the source term associated with x momentum equation bu. Similarly, you write b momentum equation ap vp is equal to summation of all the neighbor anb vnb minus now the pressure gradient term in y momentum equation, that is pn minus ps into delta x plus the source term associated with y momentum equation.

So, this looks like the earlier discretized equation for convection diffusion equation for general variable phi. Only additional terms are for this pressure gradient to come. So, now, you see these are the discretized equation, but the pressure at East face and pressure at West face, if you see the co-located grid, so, we have discretized this equation in co-located grid.

So, if you have, this is your u, v and pressure, all you have defined at the cell centre p and this is your p and this is your small e, this is your small W and this is your small n and this is your small S and it is the neighbor in East side, N, W and South. Now, you see when you are solving this x momentum and y momentum equation in this control volume p then you need to find the pressure at this East face, West face, North face and South face.

Now, if you see that pressure, you have pressure is available only in the main control volume p. So, obviously you have to use some interpolation scheme to find the value of pressure at the face centre. So, if you use the linear interpolation assuming the uniform mesh, then what we are going to get, let us see. So, if you see this term that is your Su P is minus Pe minus Pw into delta y.

And this Pe is not available at this face centre E. So we will take the linear average between these two points, East and P where the value of pressure is available. So we will average it out like Pe plus Pp divided by 2. And in the West face, we will take Pw plus Pp divided by 2. Now you see this Pp by 2, it is plus and this is Pp by 2 minus, so these will get cancelled and you are going to get minus Pe minus Pw into delta y by 2.

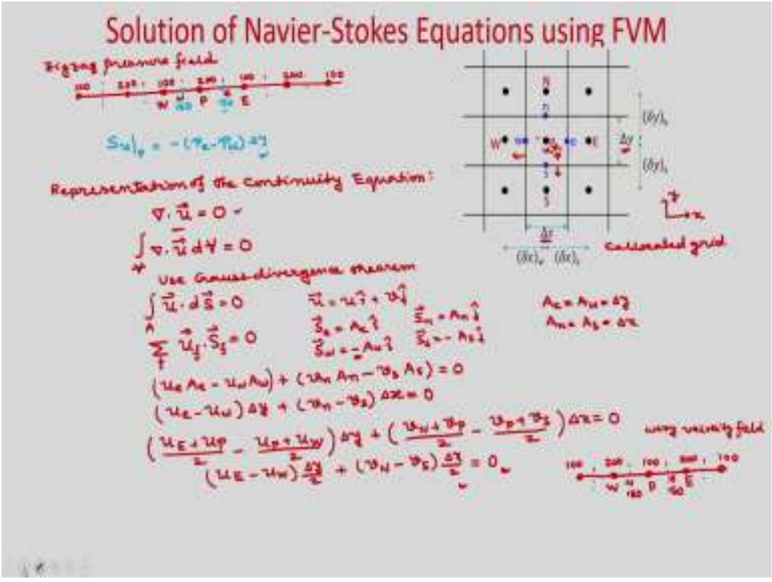
Similarly for Sv p, you can write minus Pn minus Ps into delta x. So, at this point, North and at this face centre South, the pressure values are not available. So, you are going to interpolate it from the values from north and P for this Pn. So, it is Pn plus Pp divided by 2 minus for the South face enter, it is Ps plus Pp divided by 2 into delta x.

Now, you can see, so this is Pp by 2 and this is minus Pp by 2 so, it will get cancel and you are going to get Pn minus Ps into delta x by 2. Now look it carefully. When we are calculating this pressure gradient term for the u momentum equation you are taking the pressure gradient from the alternate grid points because you are solving the u momentum equation for this point p, but you are taking the pressure from the alternate point, capital E and capital W which is far away from this point P.

So, the momentum equation will contain the pressure difference between two alternate grid points and not between adjacent one. So, now you can see that you are taking from alternate grid points. That means, when you are calculating the pressure gradient term for the u momentum equation you are taking the pressure values far away from the main control volume p at East and West control volume.

Now, you can see that when you are calculating this pressure gradient term, you are calculating in a kosher mesh. You have the grid size of delta x and delta y, but you are calculating the pressure gradient term which is 2 delta x apart. So, that means you are calculating the pressure gradient term from the kosher mesh. So, obviously it is not desirable because you are actually calculating the pressure gradient term in a kosher mesh and obviously, this would diminish the accuracy of the solution.

(Refer Slide Time: 17:22)



Now, another implication is that if you have somehow the wave pressure field, then what will happen? Let us say that for the x direction, the pressure field you have zigzag pressure field. So what will happen? So, these are the main control volume okay and this is your face s, if let us say this is your p, this is your small E and this your small W.

One dimensional case we are considering and in the x momentum equation you need the pressure at Pe and Pw but you are taking from the east and west and let us say somehow you have a zigzag pressure field due to the flow field. So, now if you have zigzag like 100, 200, okay 100, 200, 100, okay this is 200 and this is also 100.

So, now you can see that you have a pressure field which is zigzag and it is varying between the grid points as 100, 200, 100, 200, like that. Now, if you want to take the average of this pressure field at the East face, so you want to calculate this East because you have the pressure gradient term, su, you have minus Pe minus Pw into delta y.

So, now Pe minus Pw, so Pe, you will calculate. So, it will be average of these two, so it is 150, okay 100 plus 200 divided by 2, it is 150 and the west face where you have 100 plus 200 divided by 2, it is 150. So, now you can see that you have Pe is equal to 150 and Pw is equal to 150 and Pe minus Pw essentially will become 0.

So, that means in every cell if you calculate this pressure gradient term, it will become 0. So it will feel like a uniform pressure field. So uniform pressure field that means there will be no pressure gradient. But for the flow to take place you need the pressure gradient. So, this is the another problem using co-located grid.

So, one problem is that you are calculating the pressure gradient term in the kosher mesh which will actually diminish the accuracy of the solution and also you can see that if we have a zigzag pressure field then it will feel like a uniform pressure field where there will be no pressure gradient.

So, now, let us see, if you consider the continuity equation and satisfy in the co-located grid in the main control volume u, then what are the implication we will have? So, now let us say that it is your co-located grid and u, v and p, you are solving in this main control volume p and this is your east face, north, west and south.

So, what is your continuity equation? Representation of the continuity equation, so, if you consider incompressible flow then it is just divergence of u, density if you consider constants in then you can take it outside and divergence of u is equal to 0. So for these now if you integrate this continuity equation over this main control volume p, so what you are going to get?

You are integrating over this control volume dV is equal to 0. Now you convert it to surface integral. So, use Gauss divergence theorem. So we are converting this volume integral to surface integral, u dot ds is equal to 0, where u is ui plus vj for the two dimensional case.

So now you can write it, summation of f Uf dot Sf is equal to 0. So, we are going to get Ue Ae, and now Uf Ue dot Se, so, Se is your Aei, and Sw is what is that? Minus Awi because you can see, the normal is in the outside normal is, sorry outward normal is in the negative x direction. So, we are considering x this direction and y this way. So, minus, similarly Sn is your Anj and SS is your minus Asj, minus is because the outward normal in the negative y direction.

So, you are going to get Ue Ae. So, we will write now minus sign because it is minus Aw. So, Uw Aw plus Vn An minus Vs As because the velocity at North is Vn is equal to 0. And you know that Ae is equal to Aw is equal to delta y, that is the surface area and An is equal to As is equal to delta x.

So, if you write on the uniform grid, then you can write u e minus Uw into delta y plus Vn minus Vs into delta y is equal to 0. So, when we are actually satisfying the continuity equation in the main control volume p, so, the velocities we have written in terms of the face velocity, east, west, north and south.

But you can see that we are evaluating the velocities at the cell centre p or at the cell centres, we have the velocities available, not at the face centre. So, now we need to interpolate the velocity at the face centre and on the uniform grid if you use the linear interpolation, then at East face, Ue, you can write as Ue plus Up divided by 2.

So, similarly, you can write Ue is equal to Ue plus Up divided by 2 minus now Uw. So at this west face, you can see Up plus Uw divided by 2, Up plus Uw divided by 2 into delta y. Now, you can see Vn minus v s, so, you can write it as Vn. So here at North face you want to calculate the Vn. So it will be, Vn plus Vp divided by 2 minus at the south face, so Vs will be Vs plus Vp divided by 2.

So it will be Vp plus Vs divided by 2 into delta x is equal to 0. Here it will be x. So now, you see this is Up by 2 and this is minus Up by 2. So you are going to get Ue minus Uw delta y by 2 plus, it will be Vn minus Vs into delta x by 2 is equal to 0. So this is the discretized equation for this continuity equation.

Now you can see, where do we need to satisfy the continuity equation? We need to satisfy the continuity equation in the main control volume P and the velocities it should contain at least the velocity at the cell centre p, but while discretizing this continuity equation using finite volume method in the main control volume p on co-located grid, what we got?

We got the velocities from the neighbor cell. So, you can see that when we are satisfying the continuity equation at the cell centre p, the discretized continuity equation does not contain the velocity at cell centre p, rather it contains from the neighbor velocities like Ue, Uw, Un and Us.

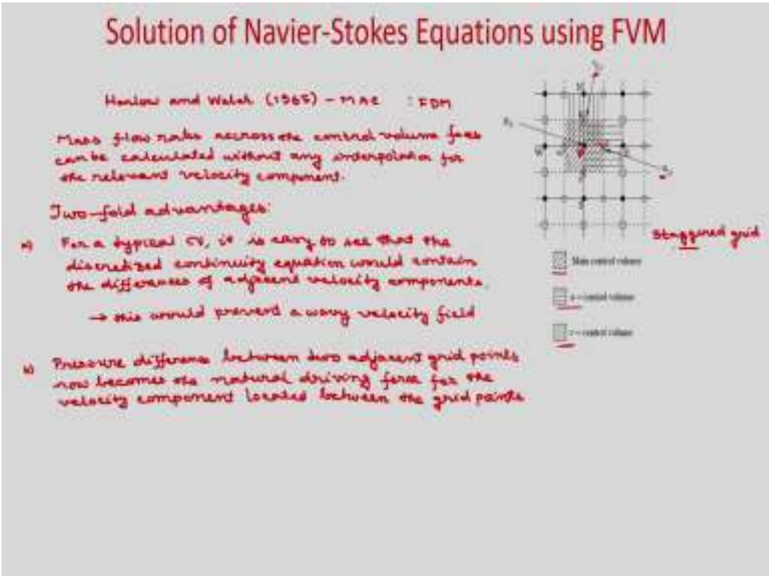
So, obviously you can see that it is not desirable because when are satisfying the continuity equation it should contain the velocity at pressure, at the main control volume p. So, now to and also similarly if you have the wavy velocity field, similarly if you have wavy velocity field then again what is the problem, let us discuss.

Let us say for the u velocity you have, for u velocity let us have the velocity distribution like this. So, these are the main control volume and these are the face centre. So, if it is p then it is east, this is your west, this is your capital E and this is your capital W. So, this is your wavy velocity field. And you have now let us say similar way you have some velocities here 100, this is your 200 this is your 100, this is your 200 and this is your 100.

So if you want to calculate the velocity as East face, so, you will get 150 and this is also 150. So, you can see this is your Ue is 150 and Uw is on 50 Okay, but when you are actually satisfying the continuity equation, although it satisfied the continuity equation, you can see 150 minus 150 but there is no pressure field exist because in the pressure gradient term, you can see it is becoming 0, right. So, this is not realistic.

So, to overcome this problem, Harlow and Welch, they first propose the staggered grid where the velocities are computed in a staggered manner. So, that already we discussed when we discussed the Mac algorithm, but let us see again.

(Refer Slide Time: 30:36)



So, to overcome this problem Harlow and Welch in 1965, they proposed the Mac algorithm for finite difference method where the pressure is computed in the main control volume P and all other scalar but the velocities are computed at the staggered manner.

So at the East face okay of this main control volume, here U is computed and if you consider the North Face, the velocity V is computed or you can do also in other ways. You can compute the velocities Uij at west face or V velocity at the south face. So, you can see this is the main control volume, hatched like this, where you are solving the pressure but velocities U, you are computing in this control volume.

So, in this control volume we are solving and the V control volume is this one. So, here it is the V where you are computing the Vij. So, you can see that the velocities are computed in staggered ways, and this grid is known as a staggered grid. And using this staggered grid whatever problem we discussed for the co-located grid that can be overcome.

So, mass flow rates across the control volume faces can be calculated without any interpolation for the relevant velocity component. So, whatever problem we face while solving the pressure gradient term for the u momentum equation it contains Ue minus Uw. So, but you see that you are actually calculating Ue in the staggered grid at the face centres.

Similarly, b at the face centres, north and south. While calculating the pressure gradient terms, so, you can directly use the velocities, u and v without the interpolation okay and when you are satisfying the continuity equation for this main control volume p then you can use the Ue minus UW and Un minus Us which actually for the main control volume p.

So, that way you can overcome those problems using staggered grid. So, there are two fold advantages, let us write down. So, for a typical control volume, it is easy to see that the discretized continuity equation would contain the differences of adjacent velocity components.

So, this would prevent a wavy velocity field and pressure difference between two adjacent grid points now becomes the natural driving force for the velocity component located between the grid points. So, you can see that when you are solving the u momentum equation, so, for solving u momentum equation you need the pressure at the face centre of this u control volume.

That means Pe and Pp, but Pe and Pp now it is available at the cell centre of P and E. So that means you do not need an interpolation for pressure as well. Similarly, when you are solving the V momentum equation, you need the pressure at the North face and this main control volume p. So, obviously, these are already available, so you do not need any interpolation and this is the driving force for the velocity V and this is the driving force for velocity U.

So, in todays lecture, we discretized the pressure gradient term of the Navier-Stokes equation using finite volume method and we have shown that if you use co-located grids, there are some problems because you are actually taking the pressure from a faraway grid points. And when we discretize the continuity equation using finite volume method, there also we have seen that the when you are satisfying the continuity equation in the main control volume, it does not contain the velocity of this main control volume.

So, to overcome those problems, you can use staggered grid where the pressure is solved in the main control volume P and the velocities are solved in staggered way. And you can see that when you are solving the u momentum equation, the natural driving force is Pe minus Pp and those pressures are available at East and P point. Similarly for V momentum equation the pressure driving force is Pn minus Pp and those are also available in the main control volume of pressure.

And when you are satisfying the continuity equation in the main control volume, then you need to get the velocities at the faces like Ue minus Uw and Vn minus Vs and those are actually computed in the staggered grid at those face centre. So, that is the advantage but obviously, there are some disadvantage of the staggered grid because we have already discussed while learning the Mac algorithm that you need different indexing for velocities and pressure. Thank you.