Computational Fluid Dynamics for Incompressible Flows
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Lecture 3
Convection Schemes

Hello everyone, so in last two lectures, we discretize steady convection diffusion equation and unsteady convection diffusion equation using finite volume method. While discretizing we used the convection scheme central difference. So, today we will discuss some other convection schemes and we will also write those convection schemes for different condition where Fe greater than 0 and Fe less than 0. Before that, let us discuss about the Scarborough criteria for the iterative solvers.

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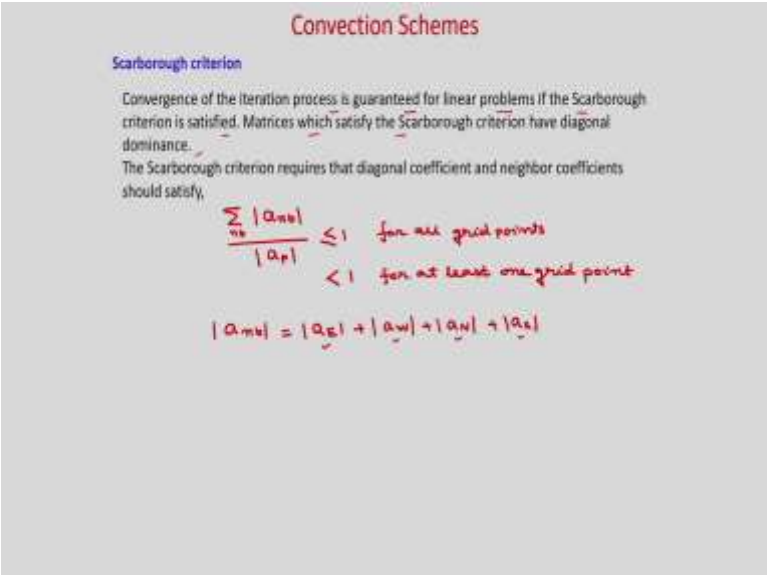

So, Scarborough criteria is that, convergence of the iteration process is guaranteed for linear problems if the Scarborough criteria is satisfied. Matrix which satisfy the Scarborough criteria, have diagonal dominance. The Scarborough criteria requires the diagonal coefficient and neighbor coefficient should satisfy these conditions. So, the condition is that summation of neighbor, all the neighbor coefficients divided by the diagonal coefficient should be less than equal to 1 for all grid points and this should be less than equal to 1 for at least 1 grid point, okay.

So, modulus of anb is equal to mod ae plus mod aw plus mod an plus mod as for a two dimensional grid, okay. So a is the coefficient of phi, aw is the coefficient of phi w, an is the coefficient of phi n and as is coefficient of phi s.

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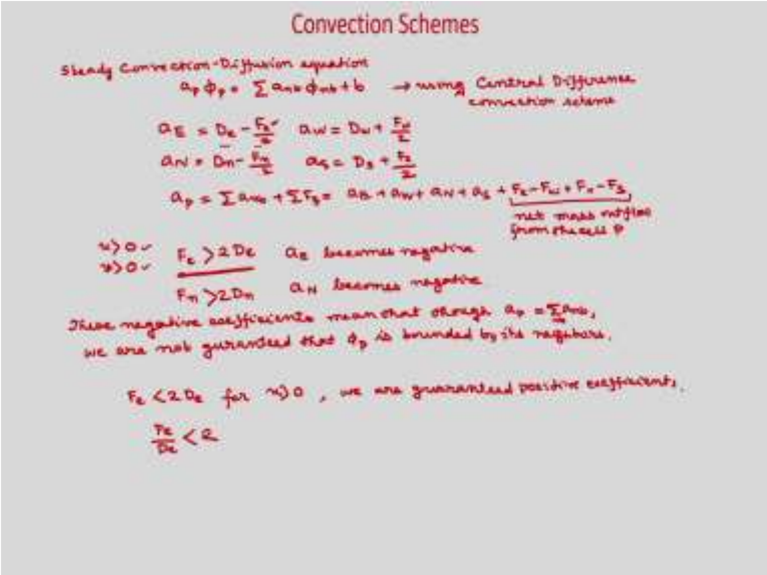

So, now, if you see that in last class using the central difference scheme, whatever we have discretized the equation and written the algebraic equation that is a p phi p is equal to summation of anb phi nb plus b for steady convection diffusion equation, okay. And this algebraic equation we have written using central difference convection scheme where the coefficients are aE is equal to De minus Fe by 2, okay.

And aw is Dw plus Fw by 2, a N is equal to Dn minus Fn by 2 and as is equal to Ds plus Fs by 2. Now you see these coefficients, okay. And what is a ap? ap is summation of a n b plus summation of Ff, that means you have a e plus a w plus a N plus a s plus F e minus F w plus F n minus F s okay. So this is your net mass flow, net mass outflow from the cell P, Okay.

Now, let us consider that you have a positive velocity at that particular phase u and b are greater than 0 then in that case your coefficients will become so, let us say, u greater than 0 and v greater than 0, what will be your in ae, so, in ae if you see, so, you have De and Fe by 2, so, if u greater than 0 and v greater than 0, okay. So, you will have and for the particular case let us say if u greater than 0, v greater than 0 and F e greater than twice De okay. So, your these mass flow rate is greater than 2 into De, okay.

So, for a particular case let us assume, then what will happens if F e greater than 2 De then, a e becomes Negative, okay. Similarly, if Fn greater than 2 Dn .So, you see Fn greater than 2 D n then what will happen your a n becomes negative, okay. So, if these coefficients become negative then, Scarborough criteria will not be satisfied and there will be problem in convergence using iterative solvers.

So, these negative coefficients mean that though ap is equal to summation of a nb, okay, we are not guaranteed with, we are not guaranteed that phi p is bounded by its neighbors, okay. So, if Fe is less than twice D then only this your phi p will be bounded by the neighbor coefficients and the Scarborough criteria will be satisfied, okay. So, if Fe less than 2 De, okay, for u greater than 0 then, we are guaranteed positive coefficients, okay. So, in that case your Fe by De should be less than 2, okay.

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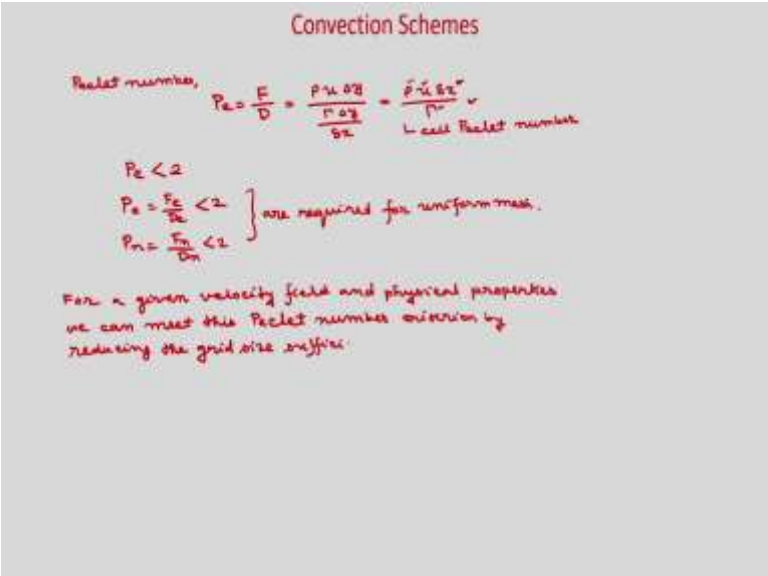

So, now we will define Peclet number as your F the mass flow rate divided by the diffusion coefficient. So it will be, you can see Fe is equal to Rho u okay delta y and of the order of, so gamma, D is gamma area is delta y divided by the distance, so delta x. So, it will be Rho u delta x by gamma, okay. So the Peclet number is defined as Rho u delta x by gamma and if you considered the delta x, okay, so then obviously you are considering the, this the cell size okay. So, this is known as some time cell Peclet number, okay.

So, you can see to satisfy the Scarborough criteria you need Peclet number is less than 2, okay. So Peclet number is less than 2, because F by D we have shown that it is Peclet number less than 2, okay. So, obviously P e you can see that Fe by De should be less than 2 and Pn is Fn by Dn, okay, less than 2 are required for uniform mesh, okay. So, now for a given flow field say if you have Fe greater than 2 D that means Peclet number greater than 2, then how you can make it Peclet number less than 2.

So, you can see from this equation, okay. So, you have the velocities, okay, and also the properties, okay, Rho and the gamma. So, obviously, this you cannot change for a flow field, okay, but delta x is in your hand, So, delta x, you decrease such a way that, Peclet number will be less than or equal to 2 then if you use some iterative solvers, it will guarantee you that convergence will happen, okay.

So, for a given velocity field and physical properties, we can meet this Peclet number criterion by reducing the grid size sufficiently. So, now let us discuss about other convection schemes. So, we have discussed only for central difference convection scheme, but all these schemes whatever we will describe now mostly will derive for the uniform mesh but it can be also derived for the non-uniform mesh.

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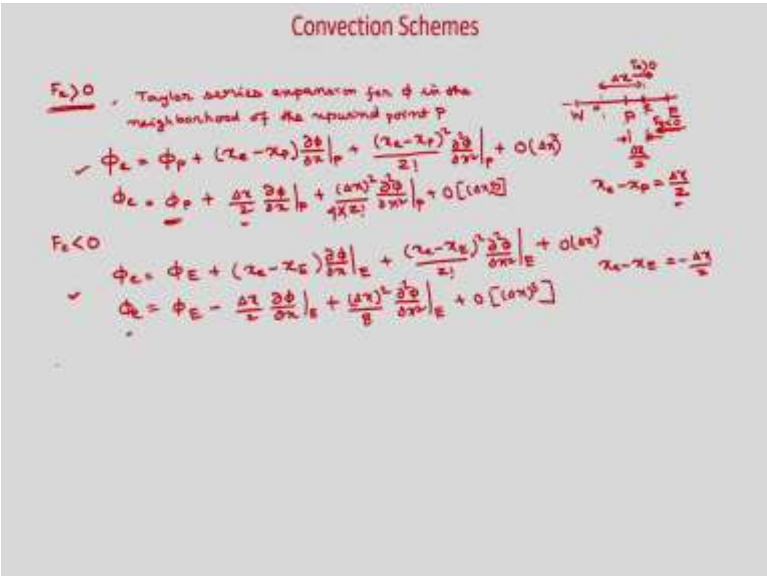

Now if Fe greater than 0, let us write the Taylor series expansion, for Phi in the neighborhood of the upwind point P, okay. So, if we write Phi e okay. So, about the neighborhood point P would expand then it will be Phi p plus x e minus x p del Phi by del x p plus x c minus x p whole square by factorial 2 del 2 Phi by del x square at p plus order of delta x cube, okay.

So, if you see this is your P, okay, and this is your volume, so it is small e and you have capital E and this is your small w and this is your capital W. So, this is your main control volume and here P is the self-centered, e is the East face and capital is the neighbor.

Now, we are expanding Phi e, so, we need to find the values at the face center Phi, so, that Phi we are expanding about this upwind point P. So, now, this distance is obviously your or on uniform gate is delta x by 2. So, xc minus xp is equal to delta x by 2, okay. And this is your delta x, okay. And now if you find these so it will be Phi p plus xe minus xp, so, you can write, delta x by 2 del Phi by del x at P plus delta x squared by factorial 2 del 2 Phi by del x square p, okay.

Similarly, now you use Fe less than 0, okay. So if Fe less than 0, so for that we will expand it about the upwind point. So, what is upwind point? Say, if Fe is greater than 0 then, the flow is taking place from left to right, okay, left to right. So, the value of Phi e you can see that the upwind point of this small e will be capital P. So, the point, upwind point of this small e will be capital P.

But when Fe less than 0, so, if Fe less than 0, then your flow will takes place from right to left. So, if it is like these for then the upwind point for Phi e will be capital Phi E. So now we will expand Fe less than 0 Phi e about the upwind point e. So, it will be Phi E, okay, plus now it will be xe minus xE del Phi by del x at east point plus xe minus xE whole square by factorial 2 del 2 Phi by del x square at point E.

Because we are expanding it, okay, about capital E, okay, plus order of delta x square, sorry delta x cube. Now, you can see what is xe minus x E? So, it is minus del x by 2, okay, because from here if you see, so, xe minus xE so obviously, it will be minus delta x by 2. So, you can write Phi e is equal to Phi E minus delta x by 2 delta Phi by del x at east point plus, so, it will be delta x square by 2, okay, delta x squared divided by sorry here it will be factorial 2.

And it will be another, this is delta x by 2, right? So it will be 4, 4 into this, so, it will be 4 into factorial 2 that means 8, okay, del 2 Phi by del x square about point E and plus order of delta x cube, this is your delta x cube, okay.

So, depending on the direction of the mass flow rate, we are choosing the upwind point and about that we have expanded Phi e, using Taylor series, okay. So, for Fe greater than 0 this way and Fe less than 0 this way. Now, these are known as upwind scheme. So, if we use say if Fe greater than 0 the first point, Phi is equal to Phi p okay and neglect other terms then obviously you will get the order of accuracy as delta x.

And if Fe less than 0 Phi e, the value of Phi e you take from the upwind point Phi e, okay. So, then the order of approximation will be order of delta x. So, now this is known as first-order upwind, okay.

So, as you have seen that for the central difference that if Fe greater than 2 D then the Scarborough criterion will not be satisfied. But if you use this First-order upwind scheme, then obviously, there will be no problem because you will have Fe, Phi is equal to Phi P and diagonal coefficient whatever you will get, so, the Scarborough criteria will be satisfied.

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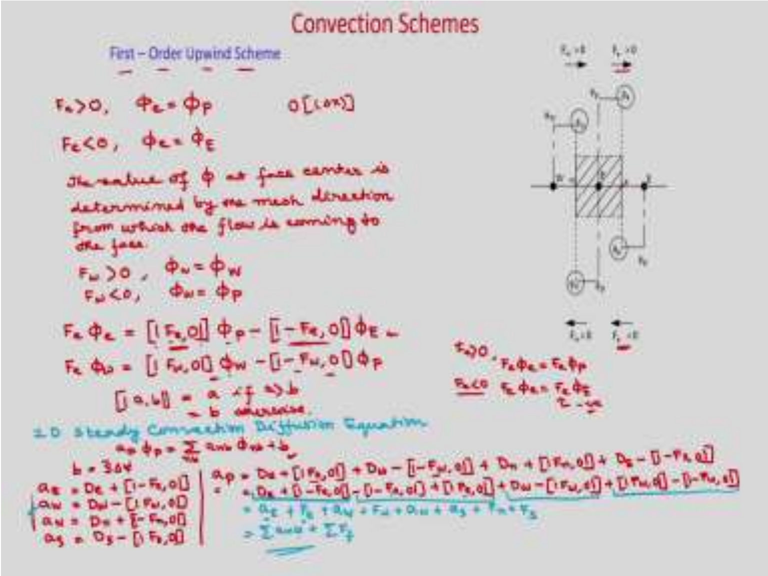

So, for this now if we use the First-order upwind scheme. So, Fe greater then 0, okay, if Fe greater than 0, the value of Phi e you take from the upwind point. So, what is the upwind point for this e? Obviously p if Fe greater than 0 because flow is taking place from the left to right okay, left to right. So, Phi e, I will write as Phi p, so, we are taking the upwind point and what is the order of accuracy? It is delta x.

Similarly, if Fe less than 0 then flow is taking place from the right to left, okay. Then for this east space you can see that the value of phi e we can take the value from the upwind point capital E. So, if F e less than 0 we will use Phi e is equal to Phi capital E, okay.

So, as we are using upwind point and only one upwind point we are taking and we have shown that it is order of accuracy is delta x First-order accurate, okay. So that is why it is known as First-order upwind scheme, okay. So now what we are doing actually we are taking the value of phi at face center okay is determined by the mesh direction from which the flow is coming to the face, okay. So, that value we are taking.

Similarly if you take Phi w, so, if Fw greater than 0, so what will be the upwind point for this west face? So, it will be capital W and if F w less than 0 then phi w will be Phi p, okay. So, if Fw is greater than 0, then Phi w will be Phi capital W and if W is less than 0 then Phi w will be Phi p, okay. So, this in general we can represent using this formula.

So, you can write Fe Phi e is equal to the maximum of these two values, okay, so, these denotes maximum of these two values Phi p minus, minus Fe 0 Phi E okay, for F e Phi w, so, we are writing as Fw 0 Phi w minus, minus Fw 0 Phi p where these denotes is equal to a, if a greater than b and is equal to b otherwise, okay.

So, this symbol actually returns the maximum weight. So, if a greater than b then, a if b greater than a then, B. So, now you can see what is happening, if F e Phi e, now if Fe greater than 0, okay, see this equation if Fe greater than 0, first term what it will return? Obviously, Fe because it is greater than 0, then you will get Phi p, okay.

And from the next you can see minus Fe, so, if Fe greater than 0 it will be negative, so, it will return 0, so, you will get 0. Now, if you see F e less than 0, okay. So, for F e greater than 0 we have written Fe Phi e is equal to Fe Phi p, okay.

Now, if Fe less than 0 what will happen? So if Fe less than 0 so, it will be negative. So, minus Fe is negative, so, it will return 0 so, it will be 0. And the next term if you see, so, minus Fe. So, Fe is negative, so, negative-negative will become positive and this positive will return Fe and this Fe with a negative sign obviously then, what about positive value will be returned from these with negative sign it will become negative, okay, then Fe Phi e will become Fe Phi E, okay.

So, this Fe is negative, okay. So, because this will return positive value, this will return positive value of Fe but Fe itself negative, okay, Fe less than 0. So, this minus will be multiplied with the positive Fe and it will become Fe which is your negative itself, okay. So, this way you can write it and now this you can use in your convective scheme and for steady and unsteady convection diffusion equation you can find the final algebraic equation and find the all the coefficients, okay.

So for, 2 dimensional steady convection diffusion equation I am going to write the coefficients but you just as a homework, you discretize it using finite volume method with the First-order upwind convection scheme and derive all the coefficients, okay, for both steady and unsteady convection diffusion equation.

So for steady convection diffusion equation for 2 D, steady convection diffusion equation, okay. So, you can write the final algebraic equation as a p Phi p is equal to summation of all the neighbor a n b Phi nb plus b, okay.

So, this is the algebraic equation for 2D steady convection diffusion equation. Now, if you write the coefficient so b will be s by delta b and a E you will get as De, so, you can see De so, it is a E. So, this will go in the right hand side and you will get plus, so we will get plus, minus Fe 0, okay. Similarly a w you can write D w. Now you can see this is the coefficient phi w coefficient is Fw 0 for the convection term. So, now this if you take in the right hand side, then it will become minus, so it will be minus Fw 0, okay.

Similarly, for a N you can write D n plus minus Fn 0 and a s you can write D s minus Fs 0 okay. So, these are the neighbor coefficients now let us write the diagonal coefficient and we will rearrange it. So, you please see it carefully. So now, if you write a p you will get D okay from the diffusion coefficient and now you see for the convection term, so it will be coefficient for Phi p, okay. So it will be plus Fe 0, okay. Similarly Dw, so this is your Phi p so it will be minus-minus Fw 0, okay.

So now for D n, similarly you can write Dn plus Fn 0 plus Ds and you will get minus-minus Fs 0, okay. But, we want to write this a p at summation of neighbor coefficients okay. So, we will write De now we will add plus minus Fe 0 as we have added now, let us subtract, minus Fe 0 and we have this term so it will be plus Fe 0, okay. So, now you can see for the East coefficient, so the first two terms will give you ae, okay, and the last two terms let us simplify it.

So, you can see these two terms together it will give ae and these two terms let us simplify it. So, now you see if Fe greater than 0, okay, so what it will return, Fe greater than 0? So, it will return Fe and what it will return as Fe greater than 0? It will return 0. Now, if Fe less than 0. So, this will return 0 and if Fe less than 0 so it will return negative Fe and Fe itself negative so it will become positive and it will return Fe.

And as Fe is negative, so this negative will be there so it will return Fe. So, you can see these together it will return Fe whether it is positive or negative, it will return always Fe, so we can write it as aE plus a Fe, okay. So, these two terms, now we are writing as ae and these two terms whether it is Fe is positive or negative, it will return always Fe.

So similarly, now let us write the other terms. So, now it will be Dw okay. It will be a minus, so it will be Fw 0 and as we have subtracted let us add another Fw 0 and you have this is minus-minus Fw 0, okay. So, now you see these two, okay, so these two will return aw, okay so this is plus aw, and these two now you see what will it return?

If Fw is greater than 0, okay so it will return Fw, okay and if Fw is less than 0, then it will become positive and it will return 0, sorry Fw so it will become Fw ,and then aN similarly plus as plus Fn plus Fs, so, it will become summation of anb plus summation of Ff okay. So, this you do as a homework, okay.

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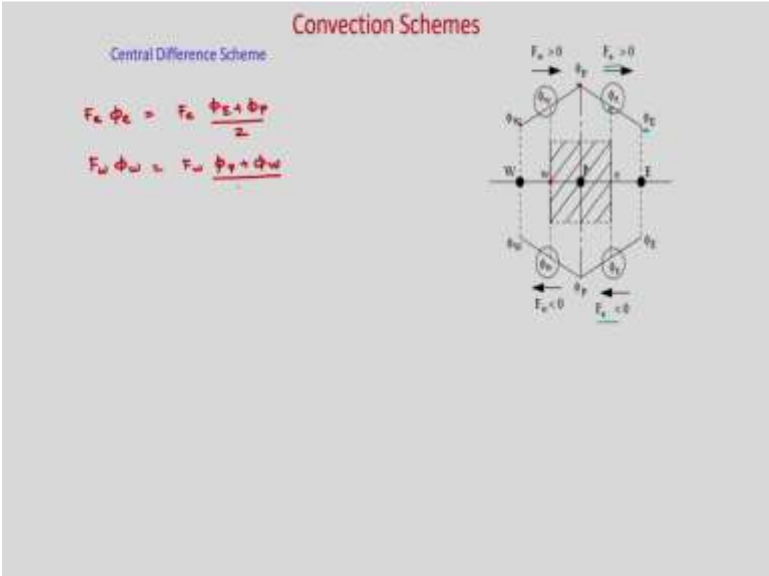

Now, let us discuss other convection schemes. So, already we have derived for the central difference convection scheme, and you can see in this figure, we take the linear average of these two points. So, if you want to find the value of Phi then, we will write as Fe Phi e plus phi p divided by 2, whether it is Fe greater than 0 or Fe less than 0. This is the linear interpolation, okay.

So, we will write Fe Phi e we will write Fe plus Phi e plus Phi p divided by 2, okay. So on uniform mesh. And (Fe) Fw and Phi w will be Fw. So, you can see at this point we want to find, so, we will take the linear average of Phi p plus Phi w divided by 2. So, Phi p plus Phi w by two.

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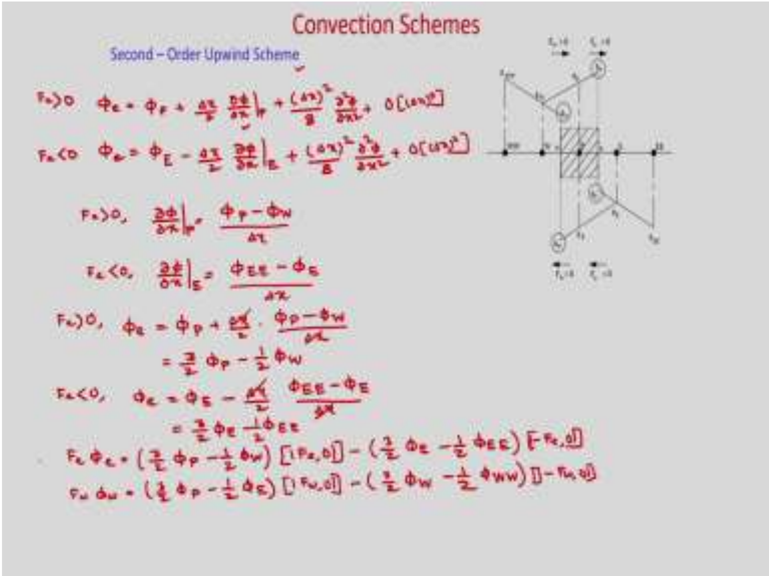

So, now we will use two Second-order upwind points, okay. And it will be Second-order accurate and this is known as Second-order upwind scheme. So, we will use the similar way Phi e we will expand about the point Phi p upwind point, so it will be Phi p plus delta x by 2 del Phi by del x at point p plus delta x squared by factorial 2 and there will be 4, so, it will be 8 del 2 Phi by del x square, okay.

And Phi w will be Phi p, so we will use Phi e about the point Pe upwind point. So, it is for Fe greater than 0 will use and it is Fe less than 0. So, Phi e will be Phi e minus del x by 2 del Phi by del x at point E plus delta x square by 8 del 2 Phi by del x squared plus order of delta x cube, okay. So, now let us find what is the del Phi by del x at p, when they Fe greater than 0, okay.

So, if Fe greater than 0, okay del Phi by del x at p del Phi by del x at p, okay. At this point we will find as Phi p minus phi w divided by delta x, okay. So, at this point we will use Phi p minus Phi w divided by delta x. So an Fe less than 0, okay. So Fe less than 0, so we will use del Phi by del x at east point. So, capital E, so, we will use at this point it will use phi E. So, this east-east minus phi East divided by delta x, so on uniform mesh we are deriving.

So, if it is so, now you put these values here. So, what you will get for Fe greater than 0 you will get Phi e is equal to Phi p, now plus delta x by 2 del Phi by del x at p So, it will be Phi p minus Phi w divided by delta x, okay. So, what you will get? So, we will get Phi p. So Phi p, so this delta x, delta x will get cancelled. So, Phi p plus Phi p by 2, so it will be 3 by 2 Phi p minus half phi W, okay.

And if Fe less than 0, then we will get phi e is equal to Phi e minus delta x by 2 phi east-east is minus Phi E divided by delta x, okay. So we will get Phi e, sorry, we will get so you can see, minus-minus plus so it will be plus 5 by 2 and this is 1 so 1 plus half is 3 by 2 phi E and this is minus phi E-E by 2, okay, (and half)

So, now, this in general you can write as Fe Phi e as 3 by 2 Phi p minus half Phi w Fe 0, okay. So, if it is Fe greater than 0 we will get this and if Fe less than 0 we will get 3 by 2 Phi E minus half Phi E-E minus Fe 0, okay. Similarly, you do for the west face, and you find the phi w and you can write this (())(35:41) phi w general way, Fw Phi w as 3 by 2 Phi p minus half phi e Fw 0 and minus 3 by 2 phi w minus half phi w, w minus Fw 0. So, depending on the value of Fe or Fw you will get the Fe phi.

So, these are the convection schemes and it is a Second order accurate. Similarly, we can derive some third-order upwind scheme, okay.

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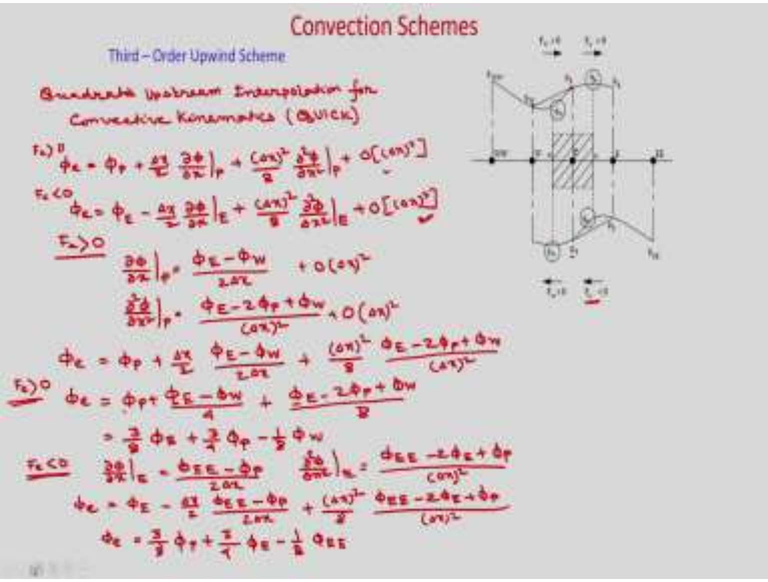

So, this is known as quadratic upstream interpolation for convective Kinematics, okay. So, it is known as Quick and this third-order upwind scheme. So, in this case you can see, to find the value of Phi at the face center e will use two upwind points, if Fe greater than 0 we will use phi p and phi w and one downwind point we will use. So this is your phi e okay it is downwind point.

Now, if it is Fe less than 0, so to find phi, we will use two upwind points. So, now upwind points are a phi e plus phi and hi E-E and for phi e, now it is the upwind point phi p, so we will use these 3 points and we will use this quadratic interpolation and similarly now you can write Fe greater than 0 Phi e is equal to Phi p plus delta x by 2, okay, del Phi by del x at p plus delta x squared. So, by 4 and factorial two, so, it will be 8 del 2 Phi by del x squared at p plus order of delta x cube and if Fe less than 0, you will get Phi e as Phi p minus sorry about East will expand, so, it will be delta x by 2 del Phi by del x about East plus delta x squared by 8. So, we have written these for uniform grid del 2 Phi by del x square at East plus order of delta x cube, okay.

So, now, we will find the first derivative as well as the second derivative, okay. Del 2 Phi by del x square at p and del 2 Phi by del x square at East. So, if you see, so, already we have derived, so del Phi by del x at p, okay. So we will use Phi E minus Phi w divided by 2 delta x, okay. So, del Phi by del x at East point we will use phi e, sorry del Phi by del x at p, okay, del Phi by del x at p we will use Phi e minus Phi w, okay. And the distance between these two points is 2 delta x.

And delta 2 Phi by delta x is squared at p we will use now, we will use Phi E minus twice Phi p plus Phi w divided by delta x square, okay. So, what is the order of approximation? Delta x square, this is also delta x square. So now you put these values here, okay. So, you will get Phi e is equal to Phi p plus delta x by 2 del Phi by del x, so, this Phi E minus Phi w divided by 2 delta x plus delta x squared by 8 and this is your Phi E minus twice Phi p plus Phi w divided by delta x square, okay.

So, if you rearrange it you will get Phi e is equal to Phi p, so, it will be Phi E minus Phi w divided by 4 plus Phi e minus twice Phi p plus phi w divided by 8. So, first let us write the coefficient for Phi E, okay Phi E, this is your Phi E, so, it will be 1 by 4 plus 1 by 8 so it will be 3 by 8, so, it will be 3 by 8 phi east. Now, Phi p, so Phi p it is 1 minus 1 by 4, so it will be 3 by 4.

So, plus 3 by 4 Phi p and w so it is minus 1 by 4 plus 1 by 8 so it will be minus 1 by 8 phi w. So, this is for Fe greater than 0, okay. This is we have writing Fe greater than 0, okay. So, now if you do Fe less than 0, okay so del Phi by del x, okay, at east you find, okay, it will be. So, del Phi by del x at East we are using. So, Phi double E-E minus Phi p divided by 2 delta x.

So, this will be and del 2 Phi by del x square at east if you find, so central difference we use Phi double E-E minus twice Phi E plus Phi p divided by delta x squared, okay. So, now you put all these values in this equation, okay. So you will get Phi e, so, this is (Fe greater than 0) Fe less than 0, okay.

So, Phi e is equal to Phi E minus delta x by 2 del Phi by del x at east so it will be Phi E minus Phi p divided by 2 delta x plus delta x square by 8 and you will get Phi EE minus twice Phi E plus Phi p divided by delta x square, okay. So, if you rearrange it, you will get Phi e as 3 by 8 Phi p plus 3 by 4 Phi E minus 1 by 8 Phi E-E, okay.

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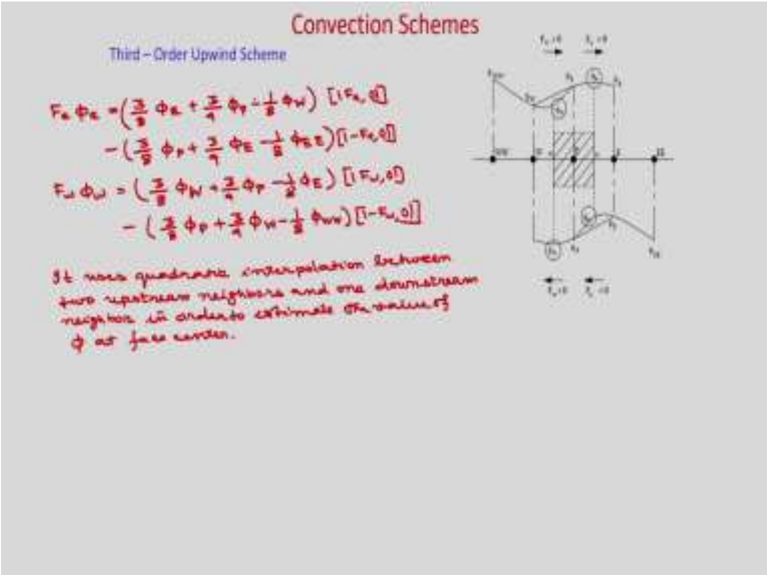

So, in general now if you write this So, you will get as Fe Phi e is equal to 3 by 8 Phi E plus 3 by 4 Phi p minus 1 by 8 Phi w Fe 0 minus 3 by 8 Phi p plus 3 by 4 Phi e minus 1 by 8 phi double E minus Fe 0, okay. And for Fw Phi w it will be 3 by 4 or you can write 3 by 8 Phi w, okay, then plus 3 by 4 Phi p minus 1 by 8 Phi E Fw 0 minus 3 by 8 Phi p plus 3 by 4 Phi w minus 1 by 8 Phi w-w minus Fw 0, okay.

So, you can see that it uses quadratic interpolation between two upstream neighbors and one downstream neighbor, okay downstream neighbor in order to estimate the value of Phi at face center, okay. So, today we have discussed first the Scarborough criteria for iterative scheme. So, to use the say, using the central difference, approximation there are some problems, because you will get if Peclet number greater than 2, then you will find the problem in convergence.

Now, to avoid that, we have introduced the upwind scheme. So, now, first we discuss the first-order upwind scheme, where depending on the value of Fe, which is your mass flow rate or flow direction, you will use the upwind point.

So, only one point is involved, so, it is a First-order accurate. So, if Fe greater than 0 then the face value Phi e we will take as Phi P and if Fe less than 0 then you will take Phi e as Phi capital E because that is the upwind point. Then also we have discussed about the second-order upwind and third-order upwind.

So, in the Second-order upwind, we have found the del Phi by del x at point p when Fe greater than 0 and if Fe less than 0 then we have found the first derivative of Phi del Phi by del x at point capital E. So, with that we have written the second-order upwind convection scheme using two upwind points, okay.

So, two upwind points are involved. So, if you are finding at small e, then two upwind points if Fe greater than 0, then Phi p and Phi W and similarly, if Fe less than 0 then two upwind points with respect to Phi e are Phi capital E and Phi capital EE.

Similarly, we have used also Third-order u up on scheme where we have used a quadratic interpolation, which involves 3 points, one point in downstream direction and two points in upstream direction. So, while finding the value of Phi at face center E if Fe greater than 0 we have used the down point as Phi capital E and the upstream points as Phi P and Phi w okay.

So, using similarly for the Fe less than 0 then obviously two upwind points are Phi capital E and Phi capital double E and one downwind point is Phi p okay and it is third-order accurate. So, although in last two classes, we have used only central difference method to discretize these convection diffusion equation but using at least for the first-order upwind scheme, as a homework you please solve you discretize this equation and find all the coefficients using first-order upwind schemes. Thank you.