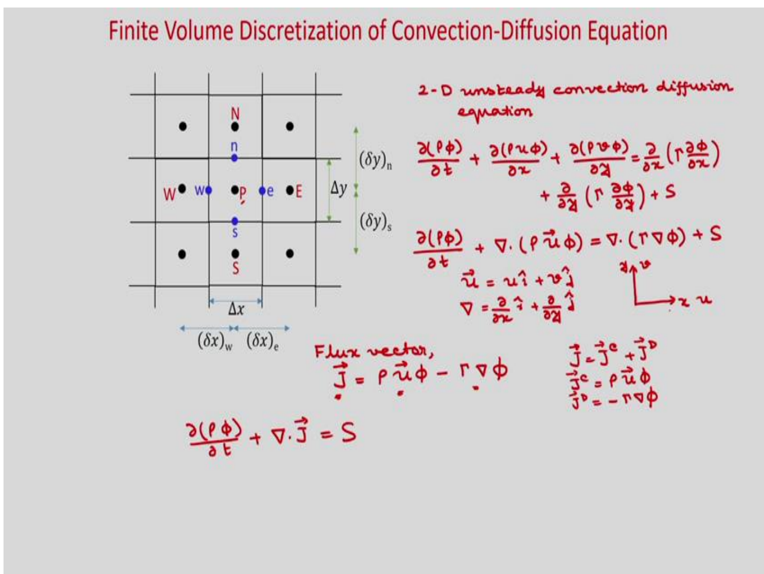
Computational Fluid Dynamics for Incompressible Flows
Professor Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati
Module 11: Finite Volume Method- 2
Lecture 2
Finite Volume Discretization of Unsteady Convection- Diffusion Equation

Hello everyone, so today we will discretize Unsteady Convection Diffusion Equation using finite volume method. So, in last lecture, we discretize steady convection diffusion equation using finite volume method, same procedure will use here only we need to integrate the governing equation over time as well. So, first let us write the two dimensional convection diffusion equation.

(Refer Slide Time: 1:04)



So, 2D unsteady convection diffusion equation. So, in two dimension you can write in conservative form del rho phi by del t plus del rho u phi by del x plus delta rho b five by del y is equal to del of del x gamma del phi by del x plus del of del y gamma del phi by del y plus S. Here, gamma is diffusion coefficient and we have written this convection diffusion equation for any general variable phi and S is the source term and rho is fluid density.

So, in vector form if you write, then this equation you will get like divergence of rho u phi is equal to divergence of gamma grad phi plus S. So, where u is the velocity vector and this is known, so it is linear equation because u is known. So, ui plus vj where you have in Cartesian coordinate. If this is your x and this is your y direction, then velocity u is in the x direction and v in y direction and i is the unit vector in the x direction and j is the unit vector in y direction and grad is del of del xi plus del of del yj. So, this governing equation now will integrate over the main cell P.

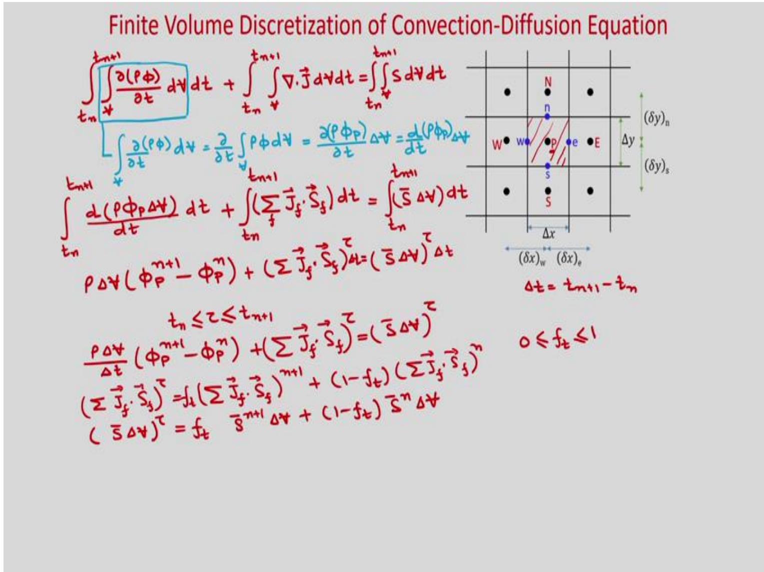
So, you can see this is your main cell. So, in this main cell P we will integrate and these are the face centers, small e, small n, small w and small s and these are the neighbors. So, our variables are available, the value of the variable is available only at this node capital P, capital E, capital W, capital N and capital S but we need to interpolate the value if required at the face center.

So, let us introduce the flux vector. So, your flux vector will be J. So, we can write rho u phi minus gamma grad phi. So, this flux vector now is having two components, one is your convection term, one is diffusion term so, J is like Jc convection term plus Jd where Jc is your rho u phi and Jd is your minus gamma grad phi.

So, this flux vector if you put it here, then what will be the governing equation? Equation will be del row phi by delta t plus divergence of J is equal to S. So, first term represents the temporal term and the second term obviously flux, we have two fluxes one is conviction flux and diffusion flux and right hand side term is your source term.

So, in the last class already we have discretize excluding the temporal term. So now, let us integrate this equation over the control volume and over the time.

(Refer Slide Time: 5:30)



So, let us write here del rho t by del t. Now you integrate over time tn to tn plus 1 where delta t is your time step, tn plus 1 minus tn and integral volume, so this is your in this main cell P we are integrating it. So, this is the main cell, this is the main cell. So, here we are integrating dV dt. The second term, similarly you can write tn to tn plus 1 volume integral and you have divergence of J dV dt and in the right hand side again the source term you integrate tn plus 1 volume integral S dV dt.

So, first let us see the temporal term. So, this temporal term if you integrate this term over this control volume. So, what you will get? So, if volume is not changing with time so non deforming control volume if you assume, then you will get del of t rho phi. These volume integral dV you can write del of del t volume integral rho phi dV and this you can write as del rho phi P, so you are now taking so, you are averaging it out over the control volume, so you are taking the value of phi at the cell center P. So, this is your del rho phi P by delta t and the volume of the cell P. So, these now you can write d of dt rho phi P delta V.

So, now this we need to integrate over the time. So, if you write it so, you will get tn to tn plus 1. So, you will get d of rho phi P and that delta V is not changing with time, so you can write delta V so this is your integral, first integral, so these dt and there is a dt so, this will get or you can write dt, dt. So, obviously integral will give just rho phi P delta V from tn to tn plus 1 plus.

Now, the second time so tn to tn plus 1. So, now you can see the volume integral, volume integral of this flux term, you can convert it into surface integral using Gauss divergence theorem and after making the assumption that J varies linearly and the average value of J lies at the face center.

So, that if you make the assumptions, then you can write the flux term as summation of over the faces Jf dot Sf over all the faces and dt, so this is the second term and similarly the last term also so, you are taking tn to tn plus 1 and you are now averaging over the cell and sale value you are taking at the cell center. So this is your S bar, at the cell center P you are taking and delta V and dt.

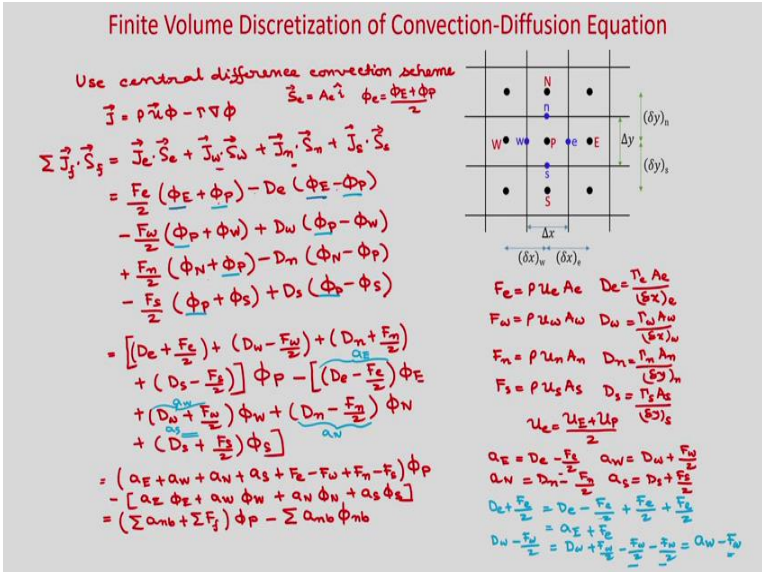
So now, you have to integrate over the time, but we will take some, at some time tau and we will take some average between these tn and tn plus 1 because values are not available in between, only values are available at time tn and tn plus 1. So, this will write as, so this will get rho delta V phi P n plus 1 minus phi P n and this term you will get plus, we are taking Jf dot Sf at some time level tau, which is in between tn and tn plus 1 and the last term also S bar at the cell center P, SP, so, S bar delta V at time level tau delta t, here delta t will be there because you are integrating over this, so this is del by delta t.

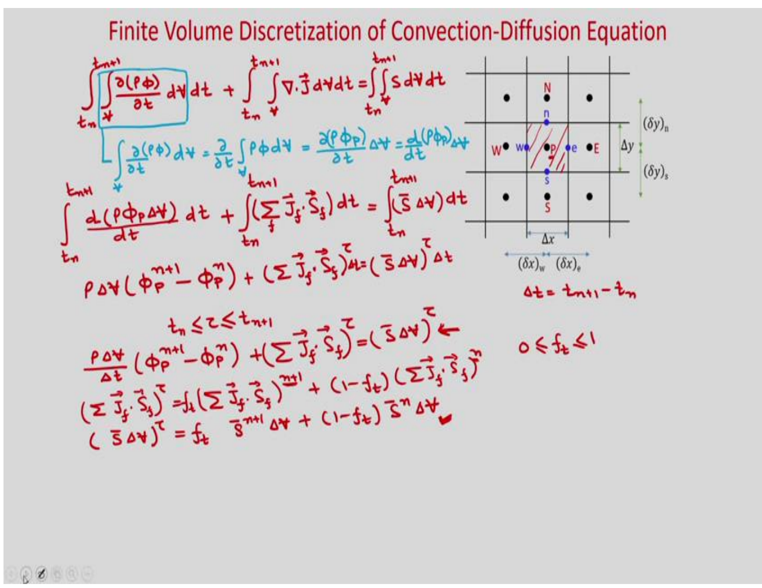
So now, you integrate now, you divide both sides by delta t, so what you will get rho delta V by delta t phi P n plus 1 minus phi P n plus summation of Jf dot Sf at tau is equal to S bar delta V at tau. So, now this value of fluxes at time tau will take as summation of Jf dot Sf at time level tau will take as summation of Jf dot Sf with a factor ft at level n plus 1 and one minus ft Jf dot Sf at time level n.

So ft lies between 0 and 1, so if ft is 0 what will we this scheme only at any time level you will give get? So you will get explicit scheme, similarly the source term S bar delta V at tau you will take as ft S bar at n plus 1 delta V plus 1 minus ft S bar n delta V.

Now, let us write the flux term so, already these flux terms using central difference convection scheme you have discretize it in last class, so, let us write down here.

(Refer Slide Time: 13:44)





So use central difference convection scheme. So, for that now what is J? J is rho u phi minus gamma grad phi, and now you find Jf dot Sf, Jf dot Sf if you find then you can write Je dot Se plus Jw dot Sw plus Jn dot Sn plus Js dot Ss. So, all the faces, four faces, so these now you find. So, what is Se? Se is your Aei, so what you will get?

So, Je dot Se so, Je dot Se now you are going to get using central difference so, phi e will take, phi e you will take average of phi E plus phi P divided by 2, because at this East place you are finding, so using uniform grid phi E plus phi P divided by 2 so, that if you put so, then you will get rho so, you will get Fe by 2 phi E plus phi P this is your convection term and the diffusion term, you will get minus De phi E minus phi P. What is Fe? So, Fe we have written as rho u e Ae and De is your gamma e Ae by delta x and A is your delta y in this two dimensional case.

So now, similarly, you write Jw dot Sw and other terms. So, if you see Jw dot Sw, so we will write minus Fw by 2. So, it will be average between this w so, if you take the central difference convection scheme it will be phi P plus phi w divided by 2. So, it will be phi P plus phi w divided by 2 plus the diffusion term will be Dw phi P minus phi w, where Fw we are defining as rho u w A w and Dw is gamma w A w delta x w.

Similarly, now you write for other two terms, so you will get Jn dot Sn if you write so, it will be a Fn by 2 so, phi N plus phi P minus Dn phi N minus phi P and for the last term you will get Fs by 2 phi P plus phi S minus, sorry it is plus Ds phi P minus phi S. So now, rearrange it and write as. So, here Fn is rho u n A n and Dn is gamma n A n by delta yn and An is in this case delta x and Fs is rho us As and DS gamma sAs by delta y s, and what is us? un, uw and ue you will take central difference. So, ue you calculate as ue plus uP divided by 2, here also very using central difference.

So now, if you rearrange it you are going to get, so all the P terms you take. So, for P terms you see, what are the P terms? So, this is your P, P, P, P, P, P and P. So, if you take all the P terms together, then you see it is will be minus minus plus, so it will be De phi P so, first we will write De plus Fe by 2, so for these phi P this is Fe by 2 so, this is Fe by 2. So, this is one coefficient then for this you will get Dw minus Fw by 2, Dw minus Fw by 2, then for this north you will get Dn plus Fn by 2, then you will get plus Ds minus Fs by 2. So, this is the coefficient of phi P.

Now, for the neighbor terms you can see, so this is your East, this is your East so, you can write it. So, you will write taking that minus, so we will get De so this is your minus so, we have taken outside, so De and here minus Fe by 2 because it is plus so minus I have taken outside, so it is minus Fe by 2 phi E plus Dw plus Fw by 2 phi W. Similarly, the other terms Dn minus Fn by 2 phi N all our neighbor terms and Ds plus Fs by 2 phi S.

So now, if you define the coefficients as aE is equal to De minus Fe by 2 aW as Dw plus Fw by 2 aN as Dn minus Fn by 2 and aS as Ds plus Fs by 2. So, you can see, this term you can write as aE, this is your aW, this is your an and this is your aS. And this term, the first term, this you simplify so if you write first term this one, so you can write as you see De plus Fe by 2, I will write as aE, so if you write aE, so what is aE? So, De minus Fe by 2 so, I have subtracted minus Fe by 2, so again you add plus Fe by 2. So, let me write here, so I have De plus Fe by 2.

Now, I want to write in terms of coefficient aE but aE is De minus Fe by 2. So, if you write De minus Fe by 2, I am subtracting one Fe by 2, you add another Fe by 2. So, then there will be no change and one Fe by 2 is already there. So now, these De by Fe by 2 is aE and Fe by 2 plus Fe by 2 it will be Fe.

And similarly, Dw minus Fw by 2 what you can do the second term, you can write Dw plus Fw by 2 and you subtract one Fw by 2 there will be no change then minus Fw by 2. So, this you can write as aW minus Fw because these minus Fu by 2 and minus Fu by 2 will be minus Fw.

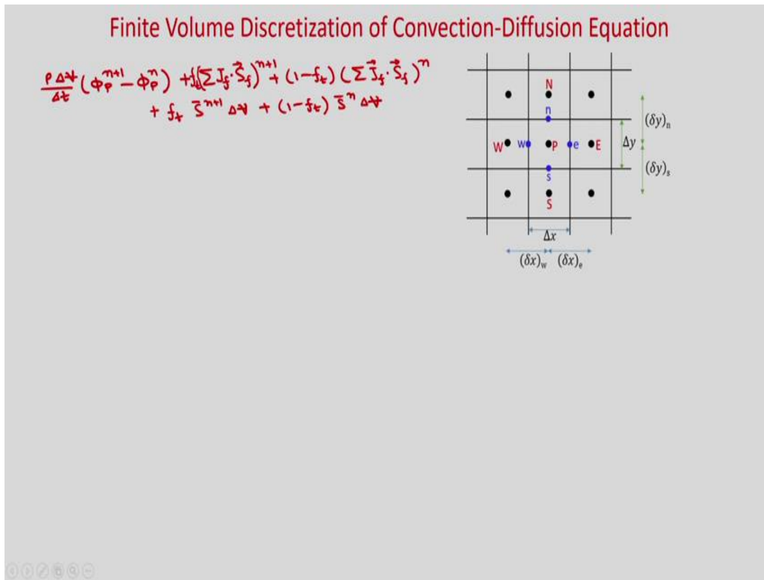
Similarly, for other coefficients you can show, and finally we can write the first term as aE plus aW plus aN plus aS minus, sorry plus Fe minus Fw plus Fn minus Fs. So, this is the coefficient of phi P, and we have aE and we have minus aE phi E plus aW phi W plus aN phi N plus aS phi S.

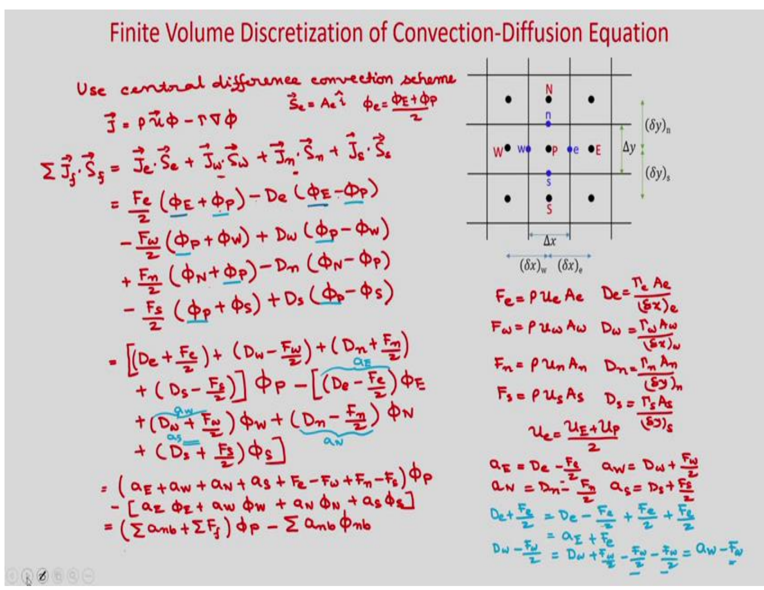
So, now this we can write as, you can see this is your so, this aE plus aW plus aN plus aS we can write summation of anb because all neighbor coefficient summation and this Fe minus Fw plus Fn minus Fs. So, these we can write as summation of Ff, what does it mean by summation Ff? So, summation of Ff is net out flux.

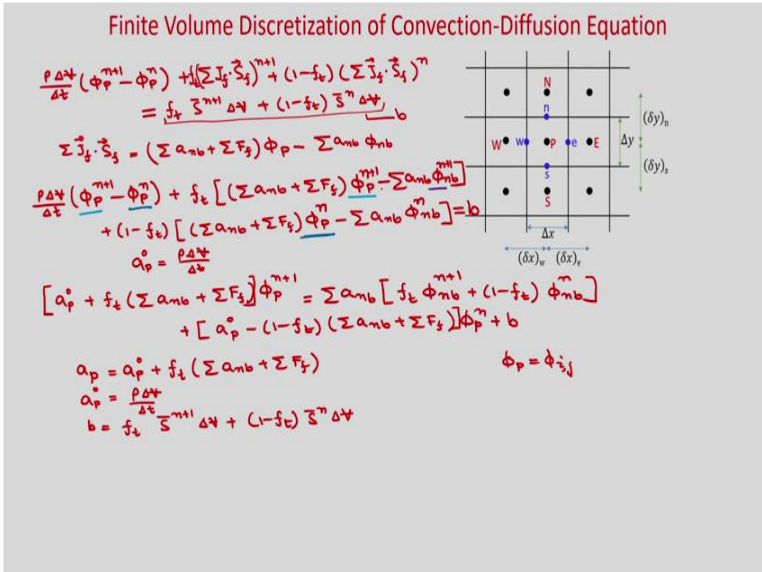
So, this is your mass flux, so, obviously it satisfies the continuity. So, in the beginning of the solution it may be nonzero but at the convergence summation of Ff will be tend to 0 because it will satisfy the continuity equation. So this we have written then this is your phi P and this you can write as minus summation of anb phi nb.

So now, these fluxes now you substituted in this expression, so we will substitute in this expression this Jf dot Sf with the term n plus 1 and n.

(Refer Slide Time: 25:34)







So we have rho delta v by delta t phi p n plus 1 minus phi pn. Then we have summation of Jf dot Sf with a factor Ft and at level n plus 1 and we have 1 minus Ft summation of Jf dot Sf at level n and we have the source term ft S bar n plus 1 delta v plus 1 minus ft S bar n delta. So now, we have already written this term. So, the summation of a Jf, Sf we have written in this form, so, this let us substituted there.

So, summation of Jf dot Sf we have written as summation of anb plus summation of Ff phi P plus summation of anb phi nb, here it is minus. So now, if you put it there, so you are going to get rho delta v by delta t phi P n plus 1 minus phi pn plus Ft now these old term at n plus 1 level. So, you have summation of anb plus summation of Ff phi P minus summation of anb phi nb at level n plus 1 and we have 1 minus ft.

Again, we have summation of anb plus summation of Ff phi P minus summation of anb phi nb at n. So, n let us write in this way n plus 1, this is your n plus 1 and this is your n and this is your n and let us write this source term plus b, where b contains the source term. So now, you take the phi P n plus 1 this terms in the left hand side and all other terms you take in the right hand side.

So, let us rearrange this equation. So, phi P n plus 1 you see phi P n plus 1 here we have and phi P n plus 1 here we have. So, this you take in the left hand side so, you can see if you put ap 0 is equal rho delta v by delta t, ap naught, if you write rho delta v by delta t, so now you see ap naught this phi P n plus 1. So this will take here so it will be ft so, ft into this, so plus ft summation of anb plus summation Ff into phi P n plus 1. So, this is the whole bracket, so this is the coefficient of phi P n plus 1.

Now you take the neighbor term in the right hand side. So, this is your neighbor term n plus 1 and this is at n plus 1. So, this you take in the right hand side, so equal to so, you can write summation of anb so, you see this is minus and minus ft and here also minus 1 minus ft, so let us take in the right hand side so it will become plus.

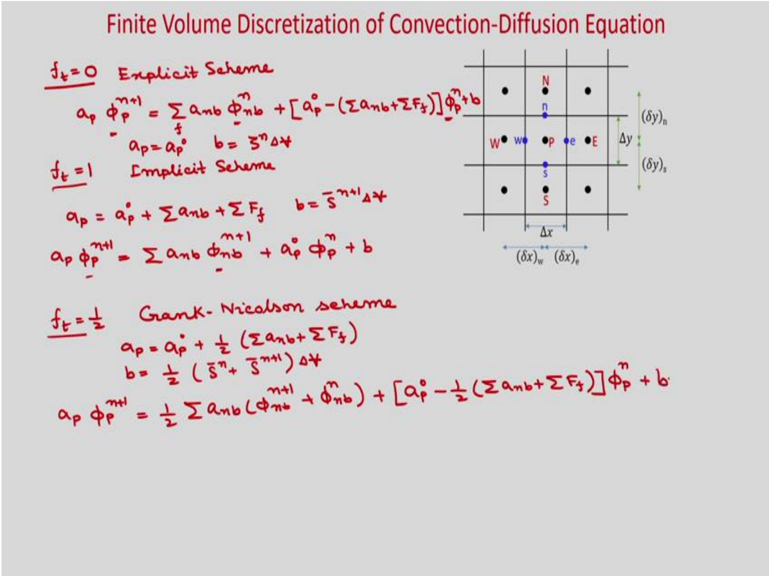
So, it will be a ft summation of, summation already we have taken. So, ft phi nb n plus 1 and this term if you take, so it will be plus 1 minus ft. So, this we are taking in the right hand side so it will be phi nb at n and what is left so, phi P at term level n. So, this is your phi P at time level n and this is your phi P term level n and so that you take in the right hand side, sorry here it will be equal to b.

So, it is equal to b please correct it. So, it is equal to and this is equal to b. So, this whole term we are considering as the source term b and that will be in the right hand side, equal to will be here and here also will be equal to b, so please correct it. So now, you write all this phi P term at time level n. So, you can see this will come negative, so right hand side plus so, it will be ap naught so, plus ap naught.

Then, we have this if you take in the right hand side so it will be minus 1 minus ft, then we have summation anb plus summation of Ff, summation of Ff. So, phi P n and plus b. So the coefficient of phi P is the diagonal coefficient that we can represent as ap so you can write ap as ap naught plus ft summation of anb plus summation of Ff.

So, this is your diagonal coefficient and ap naught is your rho delta b by delta t and we have b. So b is ft S bar n plus 1 delta b plus 1 minus ft S bar n delta v. So, this is the final discretize equation for this unsteady convection diffusion equation and we have written in terms of the coefficient. So, diagonal coefficient is ap and phi P n plus 1, obviously it represents the value at the cell center and that is your ij. So P, phi P obviously, it is phi ij and similarly, you can see that East will be i plus 1j, i minus 1j and this North will be ij plus 1 and South will be ij minus 1.

(Refer Slide Time: 33:31)



Now, if you consider the special cases, so ft if it is 0, then it will become explicit. So, you can see if ft is 0 special case if you consider ft is 0, then it will become explicit, explicit scheme and obviously, you will get ft is 0 so, you will get ap naught phi P n plus 1 and in the right hand side you will get summation of anb phi nb at n all the faces ft is 0.

So, you will get ap naught minus summation of anb plus summation of Ff at phi P n plus b. So, you can see here right hand side all are at nth time level. So, all are known from the previous time level and left hand side you have phi P n plus 1. So, you are actually calculating at present time level phi P n plus 1.

So, one unknown and right hand side are known, so obviously it is explicit scheme and it is order of accuracy is first order and this you can see if ft is equal to 1, so you will get implicit scheme. So, in this special case it will become ap naught. So, ap will be ap naught plus summation of anb plus summation of Ff.

So, you will get ap phi P n plus 1 is equal to and in this case if you represent ap, so it will be ap is equal to ap naught only and b will be only S bar n delta v and here it will be S bar n plus 1 delta v. So, ap naught phi P n plus 1 and right hand side you will get summation of anb, all the neighbors phi nb at n plus 1.

Then, the next term you will get plus ap naught phi P n and plus b. So, b only S bar n plus 1 delta v and you can see that right hand side all the neighbor terms at n plus 1, left hand side it is n plus 1 then you will get more than one unknown, so it is implicit scheme and it is also first order accurate and if ft is equal to half if you put, then you will get Crank Nicolson. So, in that case you will get ap is equal to ap naught half summation of anb plus summation of Ff and b you will get half of S bar n plus S bar n plus 1 into delta v and the equation you will be getting ap phi P n plus 1, then you will get half summation of anb phi nb n plus 1 plus phi nb n.

So, then you will get ap naught minus half summation of anb plus summation of Ff phi P at n and plus the source term. So, today we consider two dimensional unsteady convection diffusion equation and we integrated this equation over the main cell P and over the time, then we have written the discretize form in general way and we have considered few special cases where if ft is equal to 0 it is explicit, ft is equal to 1 it is implicit and ft is equal to half it is Crank Nicolson and obviously Crank Nicolson is second order time accurate.

While discretizing these equations we considered central difference convection scheme, but you can extend it to faster upwind scheme, so this may be your homework so you just consider first upwind scheme and discretize this equation and find the coefficients. Thank you.