Computational Fluid Dynamics for Incompressible Flows
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Module 10: Finite Volume Method- 1
Lecture 3
Finite Volume Discretization of Unsteady Diffusion Equation

Hello everyone, in today's lecture, we will consider two Dimensional Unsteady Diffusion Equation and we will discretize using finite volume method. In last class, you learnt the finite volume discretization of steady state diffusion equation and you have also learned the assumptions made in different places.

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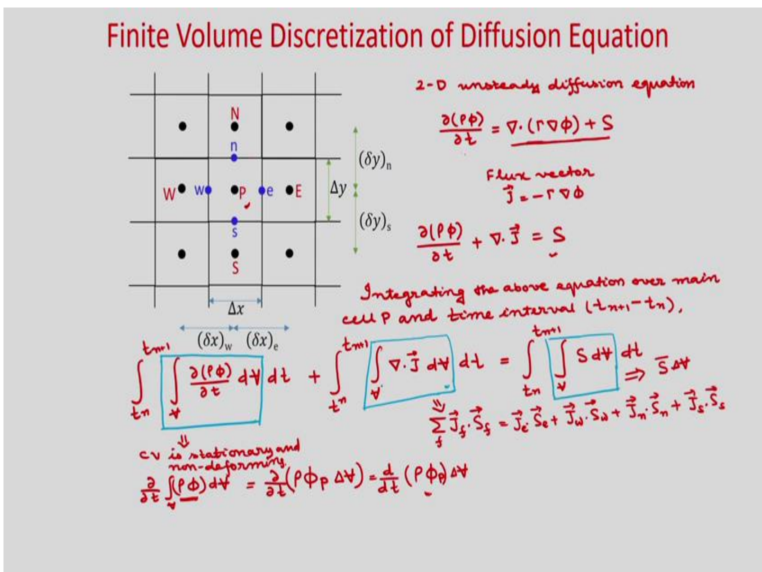

So, today let us consider, two dimensional unsteady diffusion equation. So, the equation is rho del phi by del t. So, this is your temporal term now, this term is added, and in last class already we have discretize these divergence of gamma grad phi plus S. So, in last class this first term, temporal term was 0. So, you know how to discretize these using finite volume method, only now we have to see that how we will integrate in the time level.

Obviously, new time levels will be extrapolated from the previous time level values, so you can see that now, we will define the flux vector J as minus gamma grad phi. So, the modified equation you can write, as divergence of J is equal to S. So, now this equation you integrate in this main cell P, this is your main control volume. So, in this main cell P we will integrate both over in volume and over time. So, integrating the above equation over main cell P and time interval tn plus 1 minus tn, what you will get now? So you will integrate over time, so time from tn to tn plus 1.

So again over the control volume, so this is your volume integral del rho phi by del t dv dt, so this is the temporal term, the second term now you integrate over the time tn tn plus 1 then volume integral then you have divergence of J dv, then dt and then you have source term so that also, you have to integrate from tn to tn plus 1 volume integral S dv dt.

So now, first let us integrate over control volume, each term. So, if you see the second term, this already we have learned how to discretize using finite volume method, what you do? You convert this volume integral to surface integral using Gauss divergence theorem and write the, in terms of face fluxes and these also we have integrated using the finite volume method and we have used average value at the cell center P. So, these two terms already we have learned in last class, how to discretize using finite volume method.

So, now let us see this term. So, now, let us integrate over the control volume. So, if you see this term, particularly, then this term you can write as say, in our case control volume is stationary, stationary and cv is stationary and non-deforming. So what we can do? We can write del of del t over integral over control volume rho phi dv. So now we can see that now, this value rho phi, you have to take the average value at the main cell P.

So that will take the mean value theorem and it will take the average value at cell P, so we can write del of del t, so we will take average value, so it will be rho phi P, so phi P is the average value at the cell center P and rho is constants only rho so into delta v, so that we can write and you can see now these value, we have to integrate over time so that you can write d of dt rho phi P delta v, so this is the first term, you can write this way.

Now, you consider the second term. So, now we are not going to write in detail. So, you use the Gauss divergence theorem and convert these volume integral to surface integral and write in terms of face fluxes. So, this, you can write, summation of f Jf dot Sf, summation of all the faces, so in this case we have four faces, East, West, sorry, we have East, North, West and South. So, this way, this will be just Je dot Se plus Jw dot Sw plus J north dot S north plus Js dot Ss.

Now, let us see the last term, which is your source term, and also we will, use the average value at the cell center. So all these, you can see that we have used second order accurate scheme. So, this term only in this bracket we will write S bar delta v. So all this time we have discretize using volume integral, now we have to integrate over the time.

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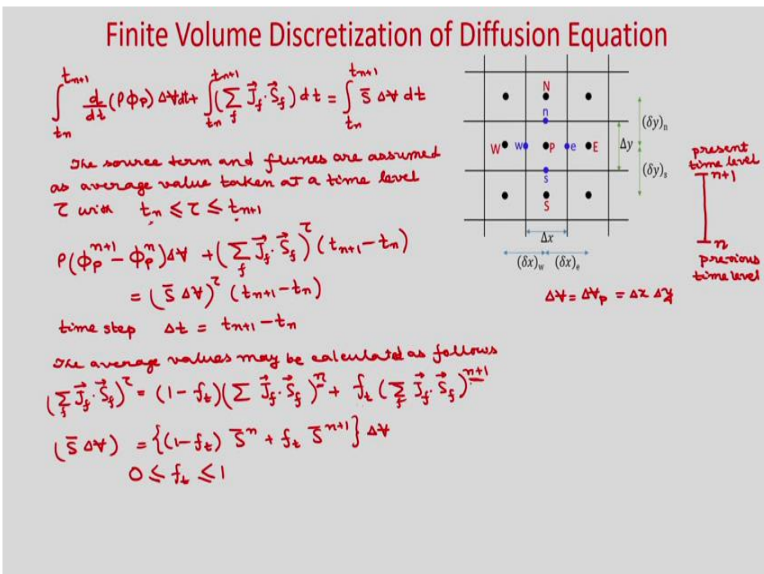

So, let us write, so, it will be now, tn to tn plus 1, d of dt rho phi P, delta v this is actually volume of the main cell P, delta v is nothing but delta v at p and that is nothing but for this particular case delta x into delta y.

The second term tn to tn plus 1, summation of f Jf dot Sf, dt, here dt is missing so, you have to write dt here is equal to tn to tn plus 1 S bar delta v dt. So, now see we have to integrate each term from the time level tn to tn plus 1. So, obviously if you want to integrate, so we have to use some average value at tau which tau maybe in between tn and tn plus 1.

So, the source term and fluxes are assumed as average value taken at a time level tau with tn less than or equal to tau less or equal to tn plus 1. So, you can see that we are going from tn to tn plus 1 and we have to take the average value of this source term and fluxes at a time level tau, so tau maybe in between tn and tn plus 1.

So we can write now the first term. So, if you integrate, so you can see, so these if you integrate over the time, so dt dt will cancel then it will be d of rho phi P and delta v does not depend on time. So obviously you can write rho phi P n plus 1 minus phi P n delta v. So, we are now marching in time from time level n to n plus 1, so n is the previous time level and this is the present time level.

Now, second term, you see, so we will take the average value, Sf at time tau and as we integral over delta t, so it will be tn plus 1, minus tn, is equal to S bar delta v. So, this also, we have to take at a particular time tau, tn plus 1, minus tn. So, now let us say that, delta t, the time step, delta t is equal to obviously tn plus 1, minus tn and so now, how will you calculate this value of this fluxes and source term at time level tau.

So, obviously you have to take either from tn or tn plus 1, because you are having the values at time level tn or at tn plus 1 which is your current time level. So, all these fluxes and the source term you have to take the average between these two points.

So, the average value, maybe calculated as follows. So, summation of Jf dot Sf at time level tau, will calculate as 1 minus ft, summation of Jf dot Sf at time level n plus ft summation f Jf dot Sf at time level n plus 1. So, you can see, we are considering these average value which is at time level tau, because we have the values available either at n time level or n plus 1 time level, because we are moving from n to n plus 1.

So, obviously, we are taking a factor 1 minus ft, of nth time level and ft times with n plus 1 time level. So, obviously, you can see that if ft is 0, then obviously, you will get only a time level n and if ft is 1, then you will take only from n plus 1 time level. So similarly, S delta v, you can take as, 1 minus ft, S at n time level plus ft at S n plus 1, into delta v.

So, you can see ft lies between 0 and 1. So ft is a factor which you are taking the average between n and n plus 1 time level values.

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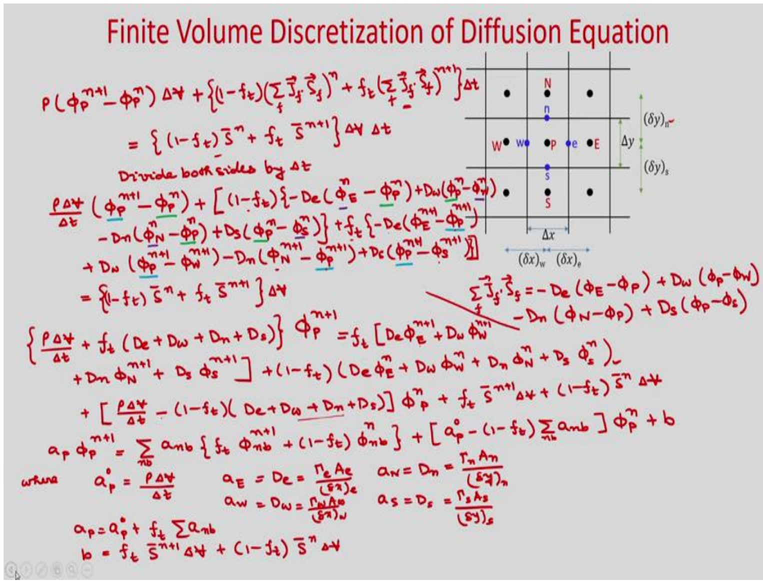

So, let us put all the values now, rho phi P, n plus 1 minus phi P n delta v plus, so now you write 1 minus ft, summation of Jf dot Sf, so which time level we will take? It is at nth time level, plus ft summation of Jf dot Sf at time level n plus 1, and we have delta t, is equal to now source term, we have 1 minus ft, Sn plus, ft S bar n plus 1. We have delta v and delta t. So now you divide both sides by delta t. So what you will get, divide both sides by delta t, so what you are going to get, rho delta v by delta t phi P n plus 1 minus phi P n.

Now, delta t we have divided, so only we will have 1 minus ft 1 minus ft, and what is Jf dot Sf? So we know already Jf dot Sf all the faces, so Je dot Se plus Jw dot Sw, Jn dot Sn plus Js dot Ss. So, that if you write, so you will get minus D. So, this we have derived in last class, phi P n plus Dw phi P minus, sorry this time level, do not write.

So, phi P minus phi w plus Dn this is minus Dn and phi N minus phi P plus Ds phi P minus phi S, so, this is the fluxes at the faces, Jf dot Sf, so, that already we have discretize in last class so that we have just written here. Now you put these values at n and n plus 1 time level wherever applicable.

So, now you can see this is a time level n so you can write minus De phi E, so we are taking a time level n minus phi P n plus Dw phi P n minus phi wn, then we have minus Dn phi n minus phi P plus Ds. So, these at time level n phi P n minus phi S n. So, now we have this term so you can write plus ft, so we can write minus De, so at which time level at n plus 1 time level so phi E n plus 1 minus phi P n plus 1 plus Dw phi P n plus 1 minus phi W n plus 1 minus Dn phi n at n plus 1 minus phi P at n plus 1 plus Ds phi P n plus 1 minus phi S at n plus 1.

So, these are the flux terms and we have now this source term, so is equal to 1 minus ft S bar n plus ft S bar n plus ft S bar 1 delta v. So, now what we will do, we will take the values of phi P at time level n plus 1 in the left hand side and all other terms will take in the right hand side so that we can write this equation in ap phi P form.

So, you can see where we have phi P at n plus 1. So, this 1, this, this, this and this, so this will keep in the left hand side and rest all other terms will keep in the right hand side. So, we will write rho delta v by delta t so, this is the first coefficient, then if you consider this one, so you will have minus minus plus so it will be plus De, then, so De but we will have ft here, so you can write it as ft De, then we have Dw, this is your Dn plus this is your Ds.

So, these all are at time level n plus 1 phi P n plus 1. And now rest all other terms you will take in the right hand side. So, what we will write equal to, so equal to, now, first you see the n plus 1 time level terms. So, we have, so ft and you see minus De phi so, if you take in the right hand side it will become plus.

So, you can write ft, so it will be De phi E n plus 1. Then plus this one phi W, so it is minus so this side it will be plus Dw phi W n plus 1. Then we have plus this one. So, Dn phi N n plus 1 plus this one, Ds phi S at n plus 1. So, all the terms at n plus 1, so, these you can see all our neighbor times phi E, phi W, phi N and phi S.

Now, you take the neighbor terms up at n time level. So, you can consider this term, this term, this term and this term. So, if you write it, so, you will get 1 minus ft is there, so, one minus ft and we can write De phi E at n plus Dw phi W at n plus Dn phi N at n and Dn phi N at sorry, phi South at n.

So, this is all neighbor terms now, we have some terms phi P at time level n, so, this one, this one, this one, and this one so, this we take in the right hand side. So, if we take in the right hand side you can see we have another term also phi P n here. So, all these you take in the right hand side so, you can see, so, this is your minus which will become plus so, if you take right hand side it will be rho delta v by delta t.

Then you see this is your, 1 minus ft, so, it will become minus 1 minus ft and again you can see this is your minus minus plus, so obviously right hand side, it is minus and we have put minus here. So you can write De plus Dw plus Dn plus Do. So, this is your phi P n and now the source terms. So you can write ft S n plus 1 delta v plus 1 minus ft S bar n delta v, so all other terms we have taken in the right hand side.

So, this is the final algebraic equation. So, now let us write in terms of the coefficients, so obviously, the coefficient associated with the phi P n plus 1 that is your diagonal coefficient and that will represent that ap, so we will represent as ap, so, this is your ap phi P n plus 1.

So, this is your ap then you have right hand side you see you can write it as summation of all the face neighbor you have all anb. Then we have ft phi nb n plus 1, and also you have 1 minus ft phi nb at time level n. So, up to this we have written. So now, you write plus so, this is your, we are denoting as ap 0, rho delta v by delta t, then we have minus 1 minus ft, and this is your summation of anb, all the neighbors.So, we have now phi P at n plus the source term b. So, this is the final algebraic equation which you need to solve and you can see we have the terms associated with n and n plus 1 time level.

So, now what are the coefficients? So, we told that we have ap 0 is rho del v by del t, what is aE? aE is your De which is, we have already calculated gamma E, aE by delta x e and aW is Dw gamma w aW by delta xw. Similarly, aN is Dn is equal to gamma N An divided by delta y n. So, this is the difference between phi and minus phi P.

So, and a South is Ds is equal to gamma s As by delta y s, and we have ap so ap you can write as ap 0 plus ft summation of anb, and the source term b you have ft S bar n plus 1 delta v plus 1 minus ft S bar n delta v. So, now let us see some special cases where ft is 0, ft is 1, and ft is half.

So, as you can see that if you put ft is equal to 0, obviously, you are going to get only the terms associated with n in the right hand side. So, obviously, it will be explicit and if it is ft equal to 1, then you have both n and n plus 1 so, obviously, it will be implicit and if you put ft is equal to half, then you are taking the average of at time level n and time level n plus 1 values, so obviously it will become Crank Nicolson and you can see that obviously explicit and implicit are these are first order Euler time integration we have done so, it will be order of accuracy is delta t, but Crank Nicolson is delta t square.

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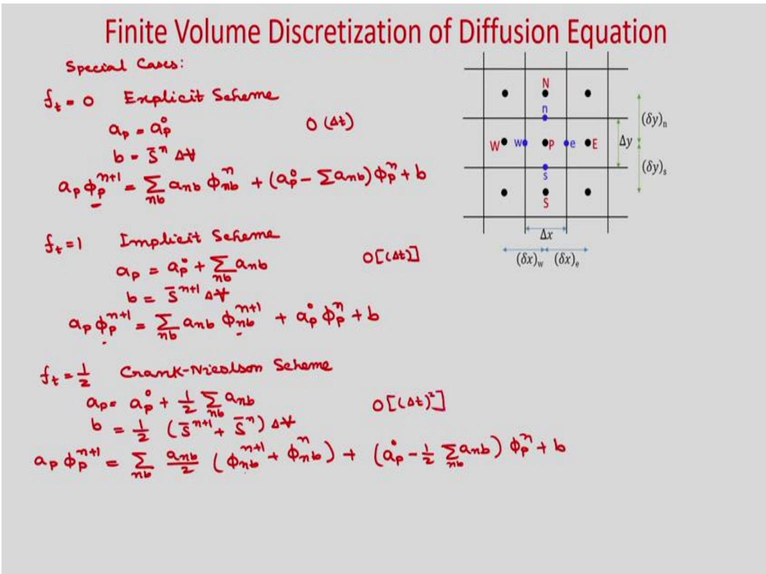

So now, explicit scheme, special cases. So, ft is equal to 0 which is your explicit schemes so if you put ft equal to 0. So, obviously, you can see ap will be your only ap naught, the source term will be S bar n delta v and the algebraic equation will become ap phi P n plus 1 is equal to summation of all the neighbors anb phi nb at time level n plus ap naught minus summation of anb phi P n plus b.

So, you can see only phi P n plus 1 is unknown and all other terms are available at time level n. So, obviously this is explicit schemes and it is very easy to code, easy and convenient to code and but order of accuracy is delta t, so order of accuracy is delta t.

Now, you see ft is equal to 1, so it will become implicit methods on implicit schemes. So, in this particular case your diagonal coefficient ap will become ap naught plus summation of all the neighbor coefficients. And the source term b will become S bar n plus 1 delta v and the algebraic equation will become ap phi P n plus 1 is equal to summation of all the neighbors anb phi nb at n plus 1 and we have ap naught phi P n plus b.

So, you can see here n plus 1 all the neighbor coefficient diagonal coefficients, sorry, all the neighbor values at n plus 1, the diagonal value phi P at n plus 1. So, obviously, it is implicit schemes and order of accuracy is delta t. Now, ft is equal to half, if you consider then it will become Crank Nicolson method, Crank Nicolson schemes, so it is better to write scheme.

Now, in this particular case now ap will become half of summation of anb neighbor ap is equal to ap naught plus half of summation of all neighbors and the b will become S bar n plus 1 by S plus S bar n delta v. And the algebraic equation will become ap phi P n plus 1 is equal to summation of all the neighbor anb by 2 phi nb n plus 1 plus phi nb at n plus ap naught minus half summation of anb nb phi P n plus b. So, you can see here now half of phi nb n plus 1 by nb n. So, this will become order of delta t square.

So, in today's class, we considered unsteady diffusion equation in two dimensional situation and we discretize the using finite volume method. So, we integrated the governing equation over control volume as well as over time and we wrote the fluxes, when we integrate over the time as the average value at time tau, but the values at tau you can take the average from either tn time level or tn plus 1 time level.

So, obviously, because at these two points, we know the values of the variables. So, we took a factor ft where we have taken 1 minus ft times the fluxes at n time level and ft times the fluxes at n plus 1 time level and similarly for the source terms. So, then we have discretize that equation and wrote the final algebraic equation and also we have written the coefficients.

As a special case, we have put ft is equal to 0 and we got the explicit scheme where only one unknown is there phi P n plus 1 and when ft is equal to 1 it will become implicit scheme and there are more than one unknowns at a time n plus 1 level and when we put ft is equal to half so it become Crank Nicolson method, we take half from n time level and n plus 1 time level of this fluxes and the source terms and the order of accuracy becomes second order. Thank you.