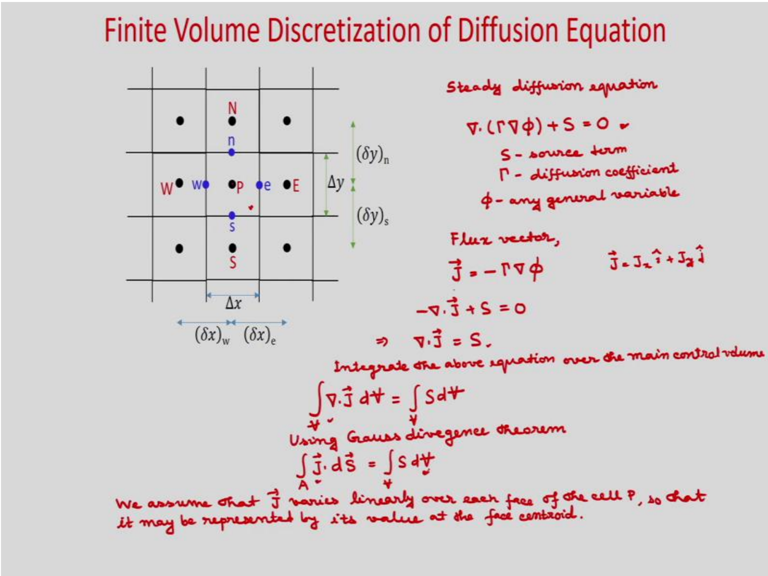
Computational Fluid Dynamics for Incompressible Flows
Professor Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati
Module 10: Finite Volume Method - 1
Lecture 30
Finite Volume Discretization of Steady Diffusion Equation

Hello everyone, in today's lecture, we will discretize diffusion equation using finite volume method. So, first we will consider steady diffusion equation for a general variable phi with a source term and we will use finite volume method to discretize this equation.

(Refer Slide Time: 0:54)



So, let us write the governing equation, so steady diffusion equation. So, you can write divergence of gamma grad phi plus S is equal to 0. So, S is the source term maybe heat generation in case of heat conduction equation and gamma is diffusion coefficient, so it is a thermal conductivity for heat conduction equation. So now, we have written this equation for any general variable phi, so phi is any general variable and this is the volume control volume we will use, so you can see this is the capital P.

And this is the cell centre of this main control volume and the neighbour control volumes are in east direction it is capital E, so that is the cell centre of neighbour control volume. Similarly, capital N north W, west and south and the face centre. So, as we have considered two-dimensional grid, so for this particular cell, we have four faces, so, one is small E that is the face centre, then small N face centre of the north face, small W is the West face and small S is the South face centre.

So, we will now integrate this equation over this control volume, so that is the main control volume, so in the main control volume we will integrate it. So now, let us define a flux vector J, which is represented as minus gamma grad phi. So, we will introduce a flux vector that is denoted as J as minus gamma grad phi. So, sometime it is known as diffusion flux vector because we are we have considered the diffusion equation, so J is minus gamma grad phi. So, obviously, J in two-dimensions you can write J X I plus J Y J.

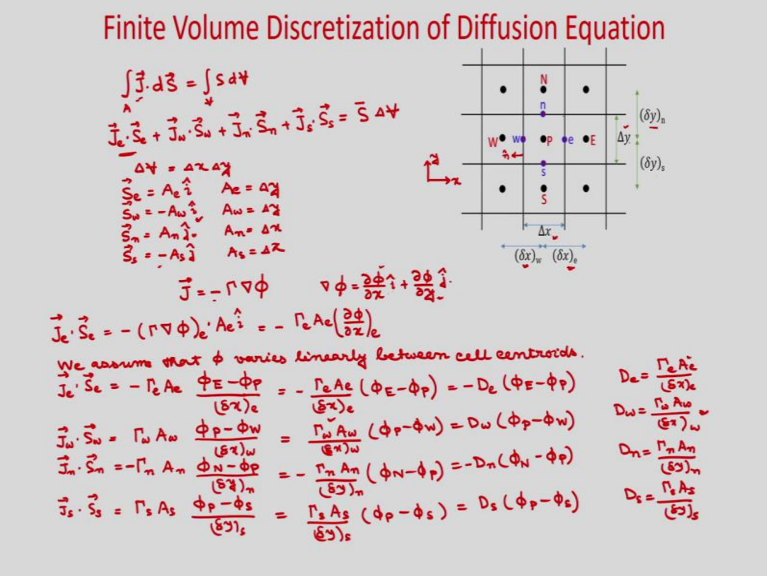
So, if you put it in this equation then what you will get? So, you will get minus divergence of J plus S equal to 0 or divergence of J is equal to S. In this form we have written, so that in general form we can integrate it and we have written in the divergence form. So, now let us integrate this equation over this control volume. So, integrate the above equation over the main control volume. So, we will integrate it. So, volume divergence of J D V is equal to volume integral of the source term into D V.

So now, for the flux vector this integral will convert volume integral to surface integral using Gauss divergence theorem. So, using Gauss divergence theorem this integral you convert into surface integral. So, it will be J dot D S is equal to volume integral S D V. So, you can see that up to this point during the integration, we have not made any assumptions. But now you have to make some assumptions, because J we have to integrate over the faces of the control volume.

So, but at each point, the value of J is not known. So, we will use we will use some assumptions. The first assumptions is that J varies linearly along the faces. So, we will first assume that whatever J is the flux vector that varies linearly over the faces. And the second assumption is that the mean value of this J will take at the face centre small E for the east face. So, my assumptions are, we assume that J varies linearly over each face of the cell P, so that it may be represented as by its values at the face centre.

And for this source term also we will take the mean value and we will assume that the mean value of this source term is over the control volume is S bar. So, obviously you can see that if we are taking the mean value at the either cell centre for this volume integral and the face centre for this surface integral. So, both will give the second order accuracy.

(Refer Slide Time: 8:14)



So, let us write that this your J dot D S, so that is equal to integral S D V, this is your volume integral, it is area integral. So, these J now you need to integrate over these faces, so now we will take the mean value at this phase E, north, west and south all these four faces so, now you can write it as J E dot S E plus J W dot S W plus J N dot S N plus J S dot S S is equal to now, this S is the source term and average source term will take into delta V.

So, obviously you can see in this case, what is delta V? Delta V is your main control volume is P and for that we have delta X and delta Y, so it will be delta X into delta Y and the surface area S E will be your A E I, where A E, obviously for this east face you can see, so this is your delta Y. So, A E is delta Y. But as we discussed S W will be minus A W I, why? Because for this phase, the outward normal is in the negative X direction. So, this this is your X and this is your Y, then it is always we take the surface normal outside. So, obviously, it is negative X direction, so we will get minus A W I and A W is your delta Y.

Similarly, S N is equal to A N J, where A N is equal to you can see this is your delta X and S S is your minus A S, similarly, because your normal is in the negative Y direction, so, minus A S J and A S is your delta X. So, we know the volume as well as the surface area vector for each faces.

Now, let us make the dot product with J. So, what is J? So, J already we have defined as minus gamma into grad phi. So, now this we have to make the dot product with the surface vector. So, for the first term let us first write. So, J E dot S E. So, J this you can write minus gamma grad phi at east dot S E, S E is A E is the magnitude of the surface vector, so that is nothing but delta Y in this case and this is your unit vector in the X direction that is I.

So now, you know grad phi for this two-dimensional case it is, del phi by del X I plus del phi by delta Y J. So, now if you make dot product, so you will have only S E in only unit vector I. So, this I if you make the dot product, so I dot I will be 1 and J dot I will be 0. So obviously, I dot I is 1, so del phi by del X will remain. So, you can write minus gamma E, A E and we can write del phi by del X, these gradient you have to find at the face centre E.

So now, you can see gamma maybe constant, then gamma will be uniform, but now in this case, in general way we have written that as east space gamma. So, if your deficient coefficient is varying then you can have different values at different faces, but if it is constant it will be the same value. And these del phi by del X at east face now, you have to find. So, del phi by del X is the gradient you have to find at the face centre E.

So, now again we have to make assumptions. So, we will assume that the phi, the value of phi varies linearly between the cell centroids. So, if it is linear, then you can take the derivative and easily you can find between the two cell centres, one is neighbour E and the main cell centre P. So, now we will assume that phi varies linearly between cell centroids.

So, now you can write J E dot S E is equal to minus gamma E A E. Now, you see del phi by del X at this face centre E you need to find. So, the gradient at this face T. So, del phi by del X if you take then it will be just second order if you use phi E minus phi P divided by the distance delta X E. So, the distance between these two cell centre is delta X E. Similarly, the between the P and W it is delta X W and the cells centre distance between North and P, so it will be delta Y N and it will be P and S it is delta Y S.

So, we have written this terminology in for a non-uniform grid. So, if it is in the uniform grid then obviously delta X will be delta X W is equal to delta X E. So, now del phi by del X now we will write phi E minus phi P divided by delta X E. And you know that the order of accuracy of this discretisation is second order accurate. So now, you can write as minus gamma E, A E by delta X E, phi E minus phi P.

So now, you can denote this term as some coefficient D. So, you can write D at face E, so it is phi E minus phi P. So, that means your D is gamma E, A E by delta X E. So, you see A E is nothing but delta Y for this particular case because A E is the surface area of this east face, so that is your delta Y. Now, similarly you discretize for the other faces. So, you can write J W dot S W. So, now it will be minus gamma W, E W, so at this face this is your west space.

So, if you are taking del phi by del X at this face centre W, then it will be phi P minus phi W divided by delta X W. So, this is also second order accurate. So, phi P minus phi W divided by delta X W and this you can write as minus D W, phi P minus phi W. So, D W is gamma W, A W divided by delta X W. So, J W dot S W, so what is S W? S W is minus A W I, minus A W I.

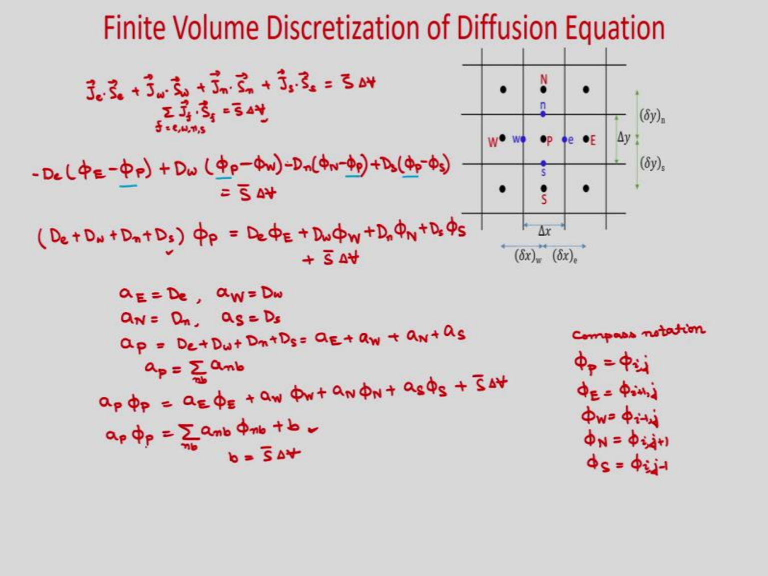
And J W is your this is the gradient, so obviously, if you take the dot product, so, you will only del phi by Del X will remain but with a minus sign. So, this minus sign and this minus sign will become plus, so it will be gamma W, A W because surface vector is negative and the plus vector is one negative is there so, it will become plus and del phi by del X. So, del phi by del X, so del phi by del X at east space how we will determine.

So, del phi by del X it will be phi P minus phi W divided by delta X W. So, it will be phi P minus phi W divided by delta X W. So, it will be gamma W, A W by delta X W, phi P minus phi W and it will be D W, phi W, phi P minus phi W and D W will be gamma W, A W by delta X W. Similarly, for the north face you do. So, north face, so north face area is A N J and now J dot J will be 1 and I dot J will be 0, so these only del phi by del Y will remain, so del Phi by del Y will remain.

So, you will get gamma N, A N with a negative sign because this J is minus and now del phi by del Y. So, del phi by del Y at this face centre N, so it will be phi N minus phi P divided by the distance between these cell centres that is delta Y N. So, it will be phi N minus phi P divided by delta Y N. So, you can write minus gamma N, A N by delta Y N. So, it will be phi N minus phi P and this with this diffusion coefficient you can write D N, phi N minus phi P where D N is gamma N, A N by delta Y N in this case delta Y N is delta X.

And similarly, J S the dot S S. So, now you can see S S is negative and J is negative, so negative-negative positive gamma S, A S and only del phi by del Y will determine, because J dot J will be 1 and del phi by del Y at these face centre if you determine, so, it will be phi P minus phi S divided by delta Y S. So, it will be phi P minus phi S divided by delta Y S. So, it will be gamma S, A S divided by delta Y S, phi P minus phi S. So, it will be D S, phi P minus phi S, where D S is gamma S, A S by delta Y S.

(Refer Slide Time: 21:10)



So, now all the flux vectors with the dot product with the surface vector we have determined now you put together, so what you will get? So, it is your J E dot S E plus J W dot S W plus J N dot S N plus J S dot S S equal to S bar is the average value of the source term into delta V. At sometime these in compact form also you can write summation of J F dot S F, so J f dot S F is the face. So, face is east, west, north, south, is equal to S bar delta V, sometime in compact form we can write this way.

Now, put all these values. So, what is J E dot S E? This is minus D E, phi E minus phi P plus D W phi P minus phi W minus D N, phi N minus phi P plus D S, phi P minus phi S is equal to S bar delta V. So, what we will do now, all the coefficient of phi P we will put together and we will keep in the left-hand side, rest all other times we will take in the right-hand side. So, if you take only the phi P, so you can see this is your phi P, this is your phi P, phi P, phi P all this you take in the left-hand side.

So, what will the coefficient? So, with that you see minus-minus, plus, so it will be D E here plus D W, this is your plus D N and this is your plus D S into phi P. So, this is your left-hand side and right-hand side now, this East. So, if you take in the right-hand side it will be D E phi E, then this is your minus D W, phi W, so it will be plus D W phi W, this is your plus D N, phi N plus D S phi S plus you have the source term S bar delta V.

So, now you can see that the coefficient of the neighbours you can see the coefficient of phi E is your D E and the phi W, D W, phi N D N and phi S, D S. So, these will represent at the coefficient A. So, A east will be D E then A W is D W and A north, D N and A south as D S.

So, you can see if we represent these coefficient as A E, A W, A N and A S, left-hand side if we represent as A P, the coefficient of phi P is A P is D E plus D W plus D N plus D S, which is nothing but A E plus A W plus A N plus A S.

So, for this particular discretization you can see the diagonal coefficient is A P and that is equal to summation of all the coefficients, summation of all the neighbour coefficients that means, A E plus A W plus A N plus A S, so, phi P is the summation of all neighbour coefficients. So, that you can sometime write A P is equal to summation of A N B where neighbours, so neighbour is east, west, north, south. And now the equation you can write as A P, phi P is equal to A E, phi E plus A W, phi W plus A N, phi N plus A S, phi S plus S bar delta V.

So, this equation in compact form is written as A P, phi P is equal to summation of all the neighbours A N B, phi N B plus B, where B is your source term, in this case it is S bar delta V. So now, you can see this is the final discretized equation using finite volume method. So, obviously P represents I J, so P represent I J, capital E is I plus 1 J. So, if you write the compass notation, so phi P is phi I J, phi E is phi I plus 1 J, phi W is phi I minus 1 J, phi N is phi I J plus 1 and phi S is phi I J minus 1.

So now, you need to solve this discretized equation using suitable solver. And in finite volume method generally, we will write in this A P, phi P form, where phi P denotes phi I J. So, in today's lecture, we considered steady diffusion equation and we introduced one flux vector which is J is equal to minus gamma grad phi and integrated the governing equation over the main control volume. And then we converted the volume integral to surface integral using Gauss divergence theorem, then we have written the fluxes at the faces.

So now, fluxes is at the faces, we have determined using two assumptions. First, assumption is that, J varies linearly over the faces and second assumptions is that we take the mean value at the face centre of the corresponding faces. So obviously, this gives the second order accuracy. Then we have retain the dot product of the flux vector and the surface method, and there one gradient comes at the east faces. And those gradients again, to calculate those gradient we assumed that phi varies linearly between cell centroids and we could find the derivative of phi with respect to X and Y which is second order accurate.

And finally, we have written the discretized equation in a A P, phi P form, where A P, phi P is equal to summation of A N B, phi N B plus B, where B is the source term S delta B. So, you can see that whatever assumptions we have taken, so that leads to second order accuracy. So already we have shown that the mean value theorem gives you the second order accuracy. Thank you.