Computational Fluid Dynamics for Incompressible Flows
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Module 10: Finite Volume Method - 1
Lecture 1
Introduction to Finite Volume Method

Hello everyone, so, till now we have learned finite difference method and applied this technique to discretize steady and unsteady diffusion equation, convective diffusion equation. We also solved what is a distinct function equations and Navier stokes equations using finite difference method. However, there are some limitation in using finite difference method, firstly it uses orthogonal Cartesian grid, and secondly that discretization may not necessarily be conservative, so to overcome this problem, we can use finite volume method to discretize the governing equations.

In finite volume method, the solution domain is subdivided into a finite number of small non overlapping control volumes by a grid. Finite volume method uses integral form of the governing equations and it inherently conjures the basic conjugation of governing variables, although finite volume method can be extended to complex geometry, where we can have arbitrary control volumes. In today's lecture however, we will use only Cartesian orthogonal grid. So, to start with first let us discuss about the terminology used in finite volume method. First, we will consider the grid in a two-dimensional domain, then in three-dimensional domain.

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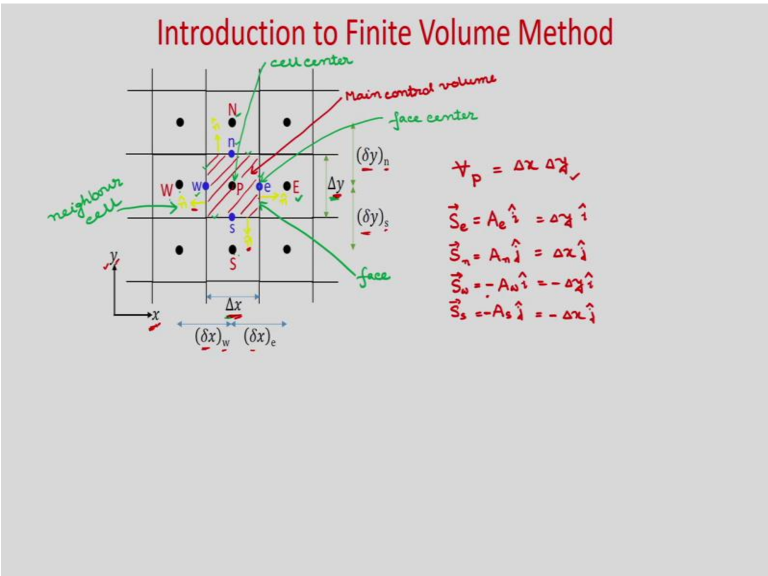

So, you can see, so this is just grid in two dimensions, this is the X direction and this is your Y direction, so you can see these are orthogonal grid in finite volume method. So, we considered the main control volume as this one and denote it by P and the centre of this main control volume is known as cell centre. So, this is your main control volume and it is denoted as P, the cell centre of this main control volume is denoted by P.

So, this is your cell centre, so this is your cell centre. So, you can see this is the volume, main control volume. So, for this particular case we have four faces. So, this is one face, this is one face, this is one face, this is one face, so this is your quadrilateral grid. So, in two dimension we have four faces and the centre of each face is known as face centre. So, you can see this is your one face centre and this is denoted as small e. So, this is your faces and this is your face centre.

So, we can see that in 2-D if we consider quadrilateral grid, then in the main control volume we have four faces and in each face we have the face centre, so four face centre we have and we have also four neighbours. So, the main control volume cell centre is P and its neighbour are denoted as capital E, which is actually in the right direction. Capital N in the north side, capital W, which is your west side, and capital S which is your south side.

So, east, west, north and south, these are the four neighbour cells, so these are known as neighbour cells. In 2-D and we have four faces, so we will have four neighbour cells. And we can have these face centre small e we have denoted as it is in east side as it is in the east side, small n in North Side, small w which is the face centre of the West face and small s is the face centre of South face. You can see that the main control volume, of size delta X into delta Y. So, you can see that we have the volume of this P cell is delta X into delta Y.

And the distance between the neighbour cell centre and the main control volume centre are denoted as small delta Y. So, capital N and capital P small delta Y North side and P and E is this distances is denoted as delta Y small s. Similarly, capital P and capital W this distance is denoted as delta X, small w and the distance between capital E and capital P these distance is denoted as delta X small e. So, these are actually the distance between two cell centres two neighbour cell centres.

And now, you have four faces and it will have the surface vector. In finite volume method we always use the normal which is outside the control volume that is considered as positive. So, in this particular case you can see the face normal always denoted as positive, which is outward. So, this is your positive, so this is face normal, this is your face normal. So, this is you can see now it is negative X direction. So, you can see this face normal face normal, this face normal is negative to X direction and this is normal is negative to Y direction, but always we consider the face normal as outward direction from the main control volume.

Now, let us calculate the surface area vector of each face. So, for this E you can see that S E in two dimension case obviously, it is the absolute surface area of east face A E into the unit vector in S direction that is your I. So, you can see that A E is the magnitude of this is East face and it is nothing but delta Y, because you can see this is your delta Y, so it will be delta Y I.

Similarly, now if you see North face, then you can write S north is equal to A north, so N is the magnitude of this surface vector and in the which direction in Y direction unit vector is J. So, it will be you can see the area of this face North face is delta X, so, this is your delta X J. Now, if we consider west and south face, then the face normal is negative to X and Y direction respectively.

So, we will write S W which is your west surface vector that is your minus A W I. So, this minus sign is coming as your face normal is negative to X direction, so it will be minus delta Y I. Similarly, S south you can write as S south, so J, so it will be negative because this surface normal is negative to Y direction. So, it will be, so this is your delta X, so it will be minus delta X into J and volume of the main cell is delta X into delta Y.

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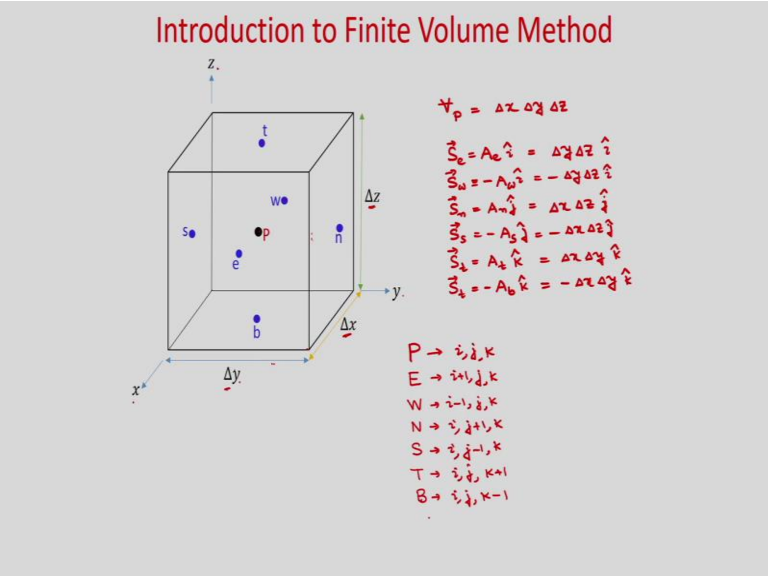

Now, if you consider the grid in three-dimension, so, you can consider the Z direction and in Z direction that you need vector is K and similarly you can write, so you can see this is the main control volume P, okay. So, this is your X direction, this is your Y direction, and this is your Z direction. So, you are in X direction, so, it is small e, East face, this is your small w, West face centre. In Y direction, this is your north, so this is your north face centre, this is your south face centre. And in Z directions, so this is your top, so top face centre and b for bottom face centre.

So, obviously in bottom you can see the face normal will be negative to Z direction. And the main cell is having the dimensions of delta X, delta Y and delta Z. So, you can write the volume of the main cell is delta X into delta Y into delta Z. And similarly, you can calculate the surface area vector of each face.

So, S E it will A E into I. And now east face if you see, so this is your East face, the front face is your east face, front faces is east face, so what is the magnitude of the surface area? That is your delta Y into delta Z, because this is your delta Y distance and this is your delta Z, so it will be delta Y into delta Z into I, so A is nothing but delta Y into delta Z.

Similarly, you can write S W. So, now West face, so normal obviously in the negative X direction, so you can write minus A E I and it will be minus delta Y into delta Z I. So, we are considering orthogonal Cartesian mesh. So, obviously the magnitude of the surface area vector A E is equal to the A W, which is the magnitude of the surface vector of West face. Similarly, S N you can write A North and in J direction, so it is the unit normal in the J direction, so you can write.

So, now A N, so this face, right face, so, it will be delta X into delta Z. So, it is delta X into delta Z into J. Similarly, S S is South face it will be minus A N J and it will be minus delta X delta Z into J and similarly now for the top surface vector, so that this one on the top. So, you can see this is a surface area delta X into delta Y, so it will be A top. So, now unit need vector in that direction is k, so it will be delta X into delta Y into K and similarly bottom, so unit need vector in negative Z direction. So, A bottom, sorry this is you can write A W, S W and this is your A South, although in this particular case it is same.

And A bottom K is equal to minus delta X delta Y. And you can see that for this particular case, we are considering, in finite volume method, so generally we use the cell centre as P and the neighbours as east, west, north, south, top and bottom. So, the compass notation we can compare with the finite difference method as, so P, P is the main control volume, so this is your I, J and K.

So, similarly East, so it will be in I plus 4, J, K, west it will be I minus 1, J, K, so these are the indices. So, P, s, the east is just front side that is your neighbour cell W, neighbour cell in the west side. So, how many neighbours will be there for this three-dimensional case? It will there will be six neighbours, because six faces are there, so six neighbours will be there.

So, now not so it will be I, J plus 1, K. South, so it will be I, J minus 1, K. Then now top bottom, so, top will be just I, J, K plus 1 and bottom as I, J, K minus 1. So, in finite volume method generally we write the discretized equation in two-dimension up to south. So, P is the main control volume then east, west, north and south and if it is 3-D, then top, bottom will come. As you told that in finite volume method, we use the integral form of the governing equation and finite volume method indirectly conjures the basic conjurations of governing variables. So, let us see how it does.

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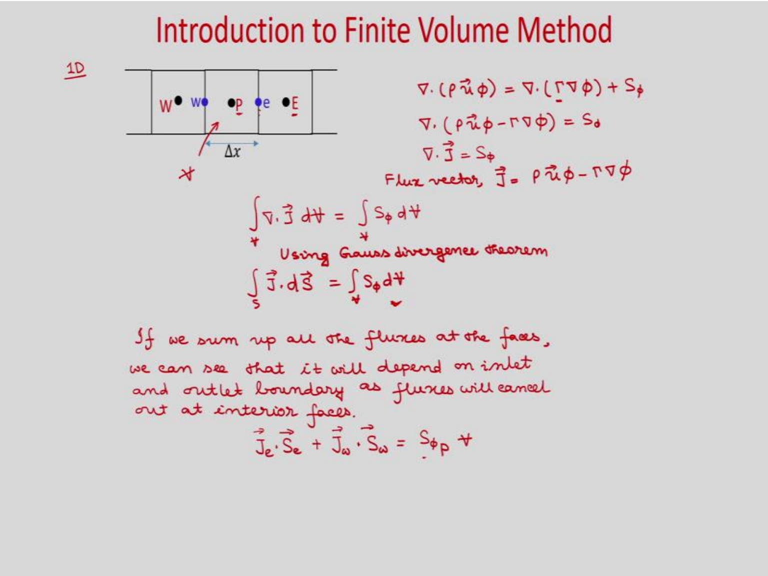

So, let us consider a 1-D case. So, in 1-D case you can see, capital P is the main control volume, capital E and capital W are the neighbours cells and delta X is the distances, small E and small W are the face centre. So, it is 1-D case, so now let us write the steady conductive diffusive equations. So, for general variable pi, so, what is that? So, it will be divergence of rho U phi is equal to divergence of gamma grad phi plus some system S phi.

So, now, you can rewrite it as divergence of rho U phi. If you take this diffusion term in the left-hand side, then it will be minus gamma grad phi is equal to S phi and this term will just denote as flux vector J is equal to S phi. So, flux vector J is your rho U phi minus gamma grad phi. So, this is the governing equation now, we need to integrate over this main control volume P. So, if you do that, so it will be volume integral divergence of J d V is equal to volume integral is d V.

Now, in the left-hand side this volume integral in divergence form we have written, so you can use Gauss divergence theorem and convert it into surface integral. So, using Gauss divergence theorem, we will convert this volume integral to surface integral J dot d S is equal to volume integral S phi d V. So, you can see this J dot d S now, it is surface integral. So, we have two surfaces in 1-D east and west, so we need to find or integrate these flux vector along these surfaces, east and west in one dimensional case.

So, now you can see that if you see that each of these small E, what is this face it is shared by P and E. So, the flux for P there will be one plus and for small E also at the east face there will be flux but you can see that the outward normal are opposite. So, for small P it is in the positive X direction and for capital E the face the face small E, the outward normal is negative x direction. The same flux you are calculating at the east face, but they are opposite in sign.

So, if you sum up the fluxes in all the faces, then what will happen the at the interior faces, all will get cancelled, because you are positive or negative and all fluxes will get cancelled, only will be you will be left with inlet and outlet fluxes. And hence, it conjures the conjuration variable. So, if we sum up all the fluxes at the faces, we can see that it will depend on inlet and outlet boundary as fluxes will cancel out at interior faces.

So now, you can see in this equation, so now up to these we have not done any assumptions just to have integrated over these, but you can see that when you are integrating these at the east face. So, we will consider the average value of these flux J at the face centre small E and we will write as J E dot S E. And here we will write plus J W dot S W is equal to, now these volume integral whatever we have will consider one average value and using mean value approximation, we will write S phi at the cell centre P into V, V is the volume of this cell, volume of this cell is V.

So now, you can see that we are approximated these flux vector these integral, we have approximated these flux at the face centre value and this is phi we have taken the average value at the cell centre S phi P using min value approximation and whatever this min value approximation we have taken now also that these are second order accurate these are second order accurate. And it is second order accurate part in the arbitrary control volume and non-uniform mesh.

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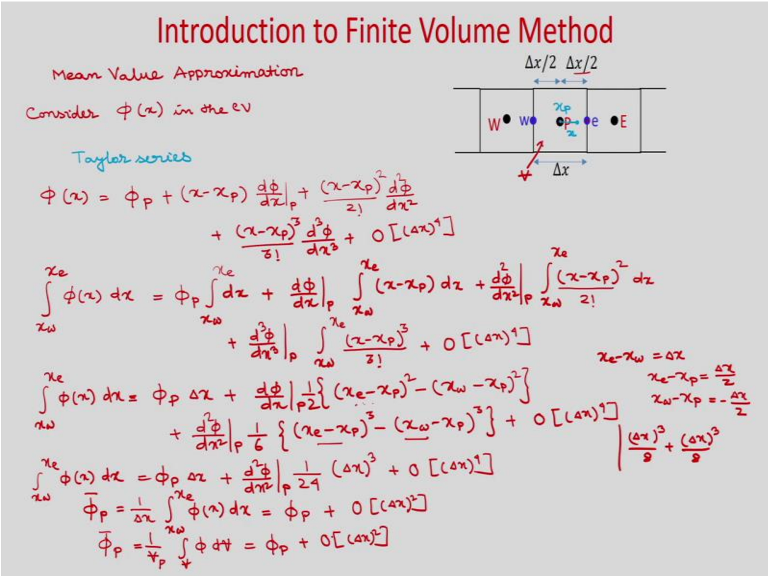

Now we will find the order of accuracy of these min value approximation, mean value approximation. Again, we will consider 1-D cell so, P is the means cell centre, capital E and capital W are the neighbours cell centre, small is the East face centre and small W is the West centre and delta X is the main control volume it is actually delta X into 1 into 1, this is the volume of this cell. And you can see that small E and small W will be located at a distance delta X by 2 from the main cell centre, because P is located at the centre of this main control volume.

So, the distance between small A small P will be delta X by 2 and the W and P the distance will be delta X by 2. So, now if we consider the control volume V as this one, and in this case it is delta X and we will consider any function phi X in this control volume. Consider the function phi X in the control volume, so now you expand this phi X. So, we will consider, so, here one point which is X distance and P is at X P distance, from the left-hand side the left boundary that distance of P centre is X P and this is your X.

So now, we can consider using Taylor series, we will write the phi X as phi P, about the phi P we are expanding. So, X minus X P, so this is your delta X, D phi by d X at P plus X minus X P whole square by factorial 2, D 2 phi by d X square and plus X minus X P whole cube by factorial 3 d cube phi by d X cube and order of approximation is delta X to the power 4. So, if you neglect the other higher order terms, then it will be order of delta X to the power 4.

So now, let us integrate this function in this control volume. So, now, as it is 1-D, we will write X W to X E, phi X d X so, what we can right now? So, phi P is the just the value of phi, so it is a discrete value so, you can take it outside the integral. So, it is be X W to X E d X plus now you can see d phi by d X. So, this is your point wise values, so it can be taken outside the integral, because d phi by d X at P you will calculate.

So, you can take it outside the integral it is as point wise values. So, you can write d phi by d X at P integral X W to X E, X minus X P d X. Similarly, d 2 phi by d X square you can take it outside d 2 phi by d X square at P, X W to X E, X minus X P whole square into d X and we can have the order of approximation or you can write another term. So d cube phi by d X cube at P, X W to X E, sorry here it will be factorial 2, so, divided by factorial 2 and this will be X minus X P whole cube by factorial 3 d cube phi sorry d cube phi we have written plus order of delta X to the power of 4.

So now, you if you integrate it, so what you will get? So, you can write X W to X E, phi X d X. So, you can see if you integrate this d X, so it will be X W minus X E. So, what is that? So, so, if you integrate it you will get X E minus X W, so X E minus X W. So, you can see X E minus X W is nothing but delta X. So, you can write phi P into delta X plus. Now if you integrate it, so it will be X minus X P the whole square by 2 and if you put the limit then you can write d phi by d X at P, so you can write as, so half X E minus X P the whole square minus X W minus X P whole square, then you can write this one.

So, 1 by 6, d 2 phi by d X square at P. 1 by 6 you can write 1 by 6. Now, you can write X E minus X P, so if integrating, so it will be cube minus X W minus X P cube, d 2 phi by d X square we have written this so you do not need to write. And you can write, so if you neglect the other terms it will be order of delta X to the power 4, delta X to the power 4. So, now you can see, the X E minus X P, what is the value? X E minus X P is that delta X by 2 and X W minus X P, it is minus delta X by 2.

So, if you see the distance between these X E and XP, what we can we are writing here. So, X E minus X P is delta X Y 2, this is the delta X by 2 and X W minus X P, X W minus X P, so it will be obviously minus delta X by 2, because from P it is going in this direction so, it will be minus delta X by 2. So now, if we put it here, so, you can see this is square, so delta X square by 4 and minus this is also delta X square by 4 so, it will become 0. So, second time will become 0, but the third time will not become 0.

So, you can write it as X W to X E, phi X d X is equal to phi P delta X, so, this time is getting 0. And here if you see, so it is delta X cube by h. So, it is factorial 2, so this is your factorial 2, so, it will be 2 into 3 so, it will be 6 and 6 into 8. So, it will be d 2 phi by d X square at P. So, now you can see this is your delta X cube by 8 and this minus minus plus, so this will be delta X cube by 8, because it will be 2. So, these two terms will be 2 delta X cube by 8 that means, delta X cube by 4, so and 4 into 6, so it will be 1 by 24 delta X cube plus order of delta X to the power 4.

Now you see, so now if we approximate this integral and if you take only the first term, then you can write phi P into delta X and you can neglect the other terms. So, what you can write, So, phi P which is your average value at the face centre of this function phi X, you can write 1 by delta X, X W to X E, phi X d X, so this if you write then you can write it as phi P, plus delta X you are dividing, so it will be order of delta X square.

So, whatever function you are integrating in the main control volume or whatever function you are integrating over the main control volume and if you take the average value at the cell centre, then you can see that it is second order accurate, so that we have proved and it is true for non-uniform grid as well and any arbitrary control volume. We have shown it for one dimensional grid, but you can show it for any arbitrary control value that the mean value at the cell centre if you take then that will be second order accurate.

And we can write in general phi P is equal to 1 by V it is a main control volume P, so volume integral phi d V as phi P, which is your second order accurate. And in the next lectures we will see that we will discretize this convection diffusion equation and we will use this mean value approximation and you can see that it is second order accurate. So, whatever we have shown, so that is true for any source term as well, because the source term we are integrating over the main control volume and the average value we are taking at the cell centre, so that will be your second order accurate.

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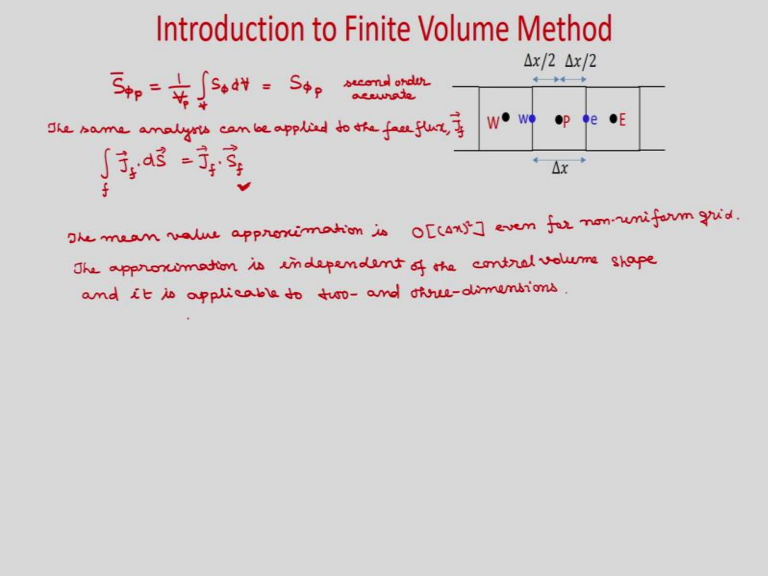

So, the S phi at P, which is if you are integrating over this control volume then you can write 1 by V, S phi d V over the volume and it will be S phi P and it is also second order accurate, this is also second order accurate. It is also true for the face integral. So, any quantity which you are integrating over the faces and if you take the average value at the face centre, so using this min-value approximation you can show that this is also secondary accurate.

So, whatever these fluxes we are integrating over the face, J E or J F, so that if you take the average value at the face centre, so that will be second order accurate. So, J F, so the same analysis can be applied to the face flux. So, J F you can write as face flux J F, so face flux J F. So, you can write integral faces J F dot d S, whatever you are doing so that we can write as J F dot S F, which is your second error accurate using min-value approximation.

And also the mean value approximation is order of delta X square second order accurate even for non-uniform grid and the approximation is independent of, approximation is independent of the control volume shape and it is applicable to two and three dimensions. So, this was found to be good compromise between accuracy and flexibility while keeping the method simple and relatively low computational cost.

So, now if you see the flux vector, we have conduction flux as well as diffusion flux. So, if we consider the diffusion flux only then there will be a gradient. So, at the face centre, so to calculate the gradient at the face centre, so we will see what is the accuracy of this discretization.

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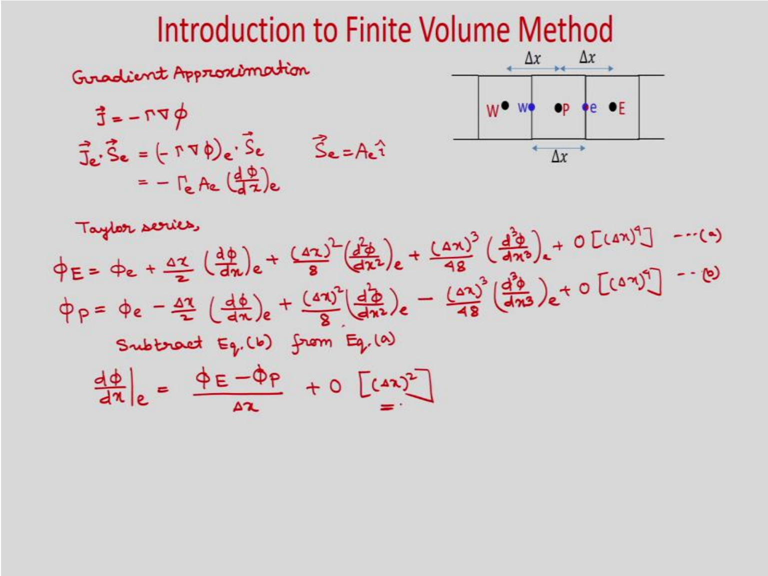

So, this is your gradient approximation. So, let us consider only diffusion flux. So, whatever the integration we have done the flux vector contains conductive flux and the diffusive flux, but now, let us consider the diffusion flux only. So, J for diffusion flux if you considers it will be gamma minus gamma grad phi. So, if you write the J E dot S E. So, the flux vector diffusion flux vector dot the S E, so at this east face we are calculating this J E dot S E.

So, what you can write? Minus gamma grad phi at east face dot S E and S E is nothing but A E into I and grad phi if you see it will be del phi by del X I plus del phi by del Y J. So, you can write it as minus gamma E, A E del phi by del X at E. So, you can see, we need to calculate this del phi by del X at the east face centre. So, to calculate that we will use Taylor series again.

So you can write phi E if we expand it about small phi E, so it will be phi E plus delta X by 2, d phi by d X at E plus delta X square by 8, d 2 phi by, so here you can write d phi by d X, because you are considering only one dimension. So, d 2 phi by d X square at east plus delta X cube by 48, d cube phi by d X cube at east and other higher order term. Similarly, phi P you can expand about small E, so, it will be phi E minus delta X by 2. So, delta X by 2 is the distance from small E to small P.

So, you can write d phi by d X east plus delta X square by 8, d 2 phi by d X square at east minus delta X cube by 48, d cube phi by d X cube at east plus order of delta X to the power 4. So now, if you tell that this is your A and this is your B, so, now subtract the second equation, subtract equation B from equation A. So, what you will get? So you subtract you can see that this term will get cancelled, the other terms will get cancelled, so you will get d phi by d X at East will be phi E minus phi P divided by delta X plus order of delta X square.

So, you can see the assumptions that the phi that is linearly between grid points then it will lead to truncation truncation error of delta X square, but it is true for only for uniform grid. So, if you use uniform grid, then the gradient at the face centre if you calculate, then you will get secondary accurate, but non-uniform grid you will get the order of accuracy as delta X.

So, today we have started with the terminology of these finite volume cells, then we introduced the volume and area for 2-D and 3-D finite volume cells, then we have used steady convective diffusive equation and introducing the flux vector, we integrated the governing equation in the main control volume P and we have written in the surface integral form of this flux vector and when we integrate the source term, we have used the average value of the cell centre.

Now, these approximation depends on these mean value approximation. So, using this mean value approximation, we have shown that if you use the average value at the cell centre, then it gives second error accuracy, even for arbitrary control volume or non-uniform rate. Same thing is also applicable when you are using the surface integral and taking the average value at the face centre and that is also we have discussed the flux vector when you are integrating over the faces, then the average value at the face centre whatever we have taken that is second order accurate.

While discretizing the diffusion flux vector the gradient of phi at the face centre it will come and if we use uniform grid, then if we take the difference between the neighbour values divided by the distance between those cells, then that gradient will be second order accurate, but if we use non uniform grid, then it will be no longer secondary accurate it will be first order accurate.

So, in the next few classes, we will consider steady diffusion equation, then unsteady diffusion equation and convective diffusive equation, and we will discretize using this finite volume method and we will use this mean value approximation. Thank you