Computational Fluid Dynamics for Incompressible Flows

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Lecture 1

Solution of Navier-Stokes Equations using FDM

Hello everyone, so in last lecture we solved fluid flow problem using Vorticity-Stream Function Formulations. Obviously, you have seen that those equations are easy to solve, as no pressure gradient term appears in the governing equations. But, there are some limitations like if you want to extend it to three dimensional problems, then you cannot do. Because stream function is defined for two dimensional problems only.

At the same time, if you need to find skin friction coefficient or coefficient of drag or coefficient of lift, then you need to know the pressure distribution. If you solve vorticity stream function equation, then you will not get directly the pressure distributions. For, that you need to solve a separate Pressure Poisson equation. To overcome this problem, we need to solve the fluid flow problem using primitive variable approach. That means u, v, p, w system, we need to solve using the velocities u, v, w and p approach.

But, problem is that, that there is no obvious pressure equation. So, we need to derive the Pressure Poisson equation from the continuity equation. So, in today's lecture we will present a popular finite difference method that is known as, marker-and-cell method and we will solve two dimensional unsteady Navier-Stokes equations.

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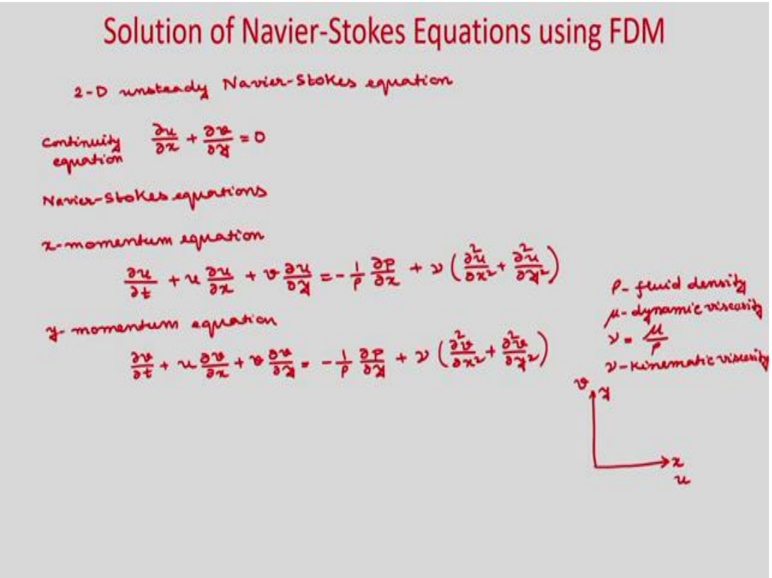

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\nu=\frac{\mu}{\rho}$$

So, let us write the two dimensional unsteady Navier-Stokes equations. So, we have the continuity equation, continuity equation del u by del x plus del v by del y is equal to 0. So, we are writing the governing equations in Cartesian coordinates, then we have Navier-Stokes equations. So, Navier-Stokes equations, first let us write x momentum equation, x momentum equation.

We are going to write this governing equation in non-conservative form. So, del u by del t plus u del u by del x plus v del u by del y and we have the pressure gradient term, minus 1 by rho del p by del x plus nu del 2 u by del x square plus del 2 u by del y square. Similarly, you can write y momentum equation, del v by del t plus u del v by del x plus v del v by del y is equal to minus 1 by rho del p by del y plus nu, nu is the kinematic viscosity, del 2 v by del x square plus del 2 v by del y square.

So, in this equation u is the velocity in x direction, v is the velocity in y direction, rho is the density of the fluid and nu which is kinematic viscosity. nu is equal to mu by rho, mu is dynamic viscosity. So, rho is the fluid density and mu dynamic viscosity of the fluid and now we define nu is equal to mu by rho, which is your kinematic viscosity and we have considered Cartesian coordinate.

So, this is the x direction, this is the y direction and we have velocity u in x direction and velocity v in y direction. So, in this equations how many unknowns are there? So, if you know the fluid properties, then rho and mu are known. Then there will be unknown u, v and p. So, we have three equations, one is continuity equation and two momentum equations. So, three equations and three unknowns. So, we will be able to solve these unknowns. But, there is no obvious pressure equation available. So, we need to derive this special equation from the continuity equation.

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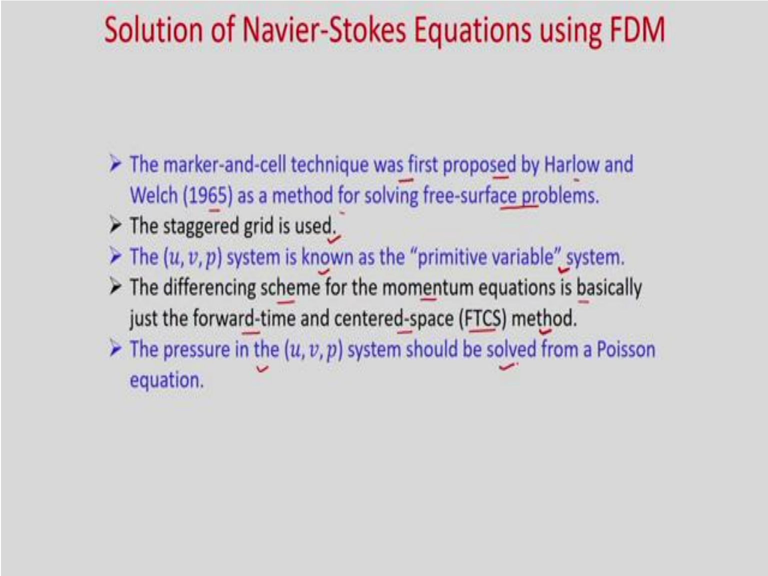

In this lecture will use marker and cell method. So, this marker and cell technique was first proposed by Harlow and Welch in 1965 as a method for solving free-surface problems. The staggered grid is used, we will discuss what is staggered grid. The u, v, p system is known as primitive variable approach and we solve this equation in primitive variable approach.

The differencing scheme for the momentum equations is basically just the forward-time and centered-space FTCS method. Both for convection terms and the diffusion terms and the pressure in the u, v, p system should be solved from a Poisson equation and that will derive from the continuity equation.

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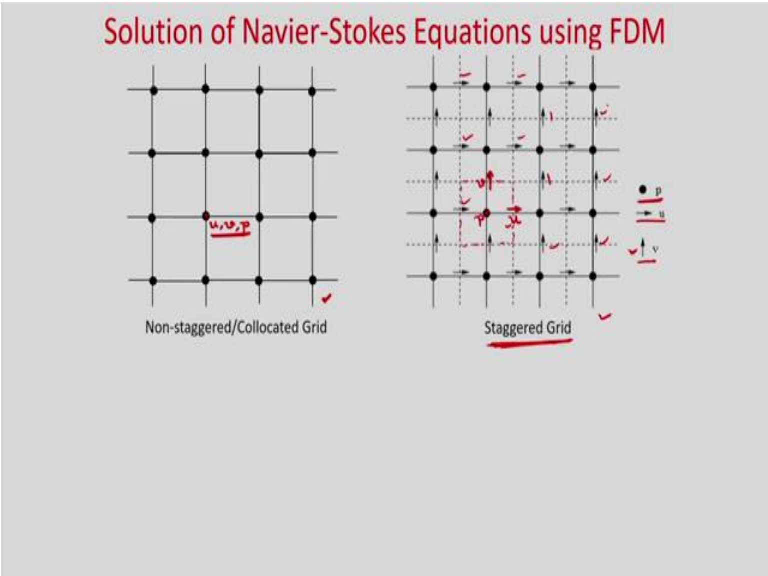

As earlier we discussed that, there are two types of grids based on the locations of these flow variables, collocated grid and staggered grid. What is collocated grid? In collocated grid all the solution variables u, v, p and any scalers like temperature or spaces. We solve at the same node point; whereas in staggered grid we solve pressure and scalers like temperature and spaces in a particular node.

But, we solve the velocities in staggered way. You can see this figure, so collocated grid sometime it is also referred as non-staggered grid. So, if you are solving, so these are the nodes you can see. So, here we have, we solve u, v and p for this particular case. But, you can also solve scalers like temperature and spaces at a particular node.

So, all these variables we are solving at c node, so in collocated grid. What is the problem in collocated grid? So, in collocated grid there will be pressure and velocity decoupling. So, to avoid this pressure and velocity decoupling, we use staggered grid. What is pressure and velocity recoupling that we will discuss in next slide. Before let us discuss about what is staggered grid. So, in staggered grid, you can see if it is the main node. So, if we solve the pressure, we solve the pressure or any scaler like temperature or spaces at this particular node.

But, we solved the velocities in staggered way. So, we solved u here and v here. So, if we see that if it is a cell. So with these dotted lines you can see, this is the dotted line. So, if it is a cell, then at the faces in the vertical face; we solve u velocity and in this horizontal face we solved the v velocity. So, you can see these velocities are solved in staggered way and pressure we solve at this centre of this cell. So, as velocities are solved in staggered way; that is why it is known as staggered grid.

So, using staggered grid you can avoid the pressure and velocity decoupling. So, you can see in this figure, all these horizontal arrows wherever it is there. We solved the u velocities, you can see all these places. But when we solve v velocity, it is in the vertical arrow. So, in vertical arrow the v velocity is noted. So, you can see this, this, this, so we solve all v velocities.

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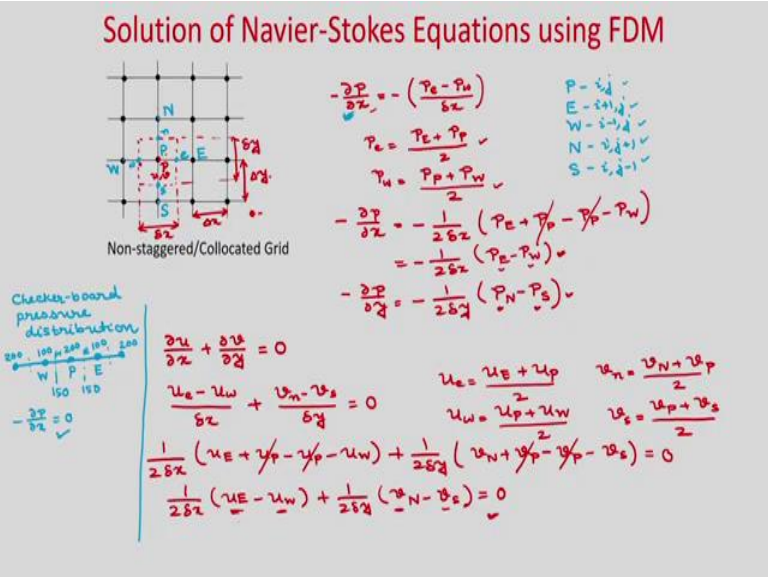

So, let us see what is the pressure gradient term in the collocated grid, where all the variables we solve at a particular node. We have, minus del p by del x or 1 by rho del p by del x. So, let us say the pressure gradient term you just consider, minus del p by del x. So, if we represent, so this is at cell.

So, we are considering in non-staggered or collocated grid. So, this is the cell and this is the cell centre which is your node. So, here pressure you are solving and u and v are also solving and we will denote this node as P, capital P. P means this is your i, j point and this is your E. So, this is your east, so it is i plus 1, j.

This point is W that means west. So, this i minus 1, j and this point is north. So, N this is your, i, j plus 1 and this point is south and S is i, j minus1. So, we will write this discretized equation in terms of P, E, W, N, S. Where those denotes with these in indices and at the face centre here. Let us say this is small e, this face centre is small w. This is your small n and this is the face centre it is small s. Now, if you are going to get this pressure gradient, how we will derive for particular cell?

So, it will be minus del p by del x, so obviously for this particular cell it will be pe minus w and the distance if we are assuming that uniform cell. Then uniform grid we are assuming uniform grid. So, it is also delta x and it is also delta x and this is your delta y and this is also delta y. So, in that case you can see, pe minus pw divided by. Let us say that the grid distance is delta x and this is your delta y and the distance between these faces east and west. This is your delta x and this is your delta y.

So, del p by del x, you can write as minus pe minus pw divided by delta x. So, this is we have used central difference method for the first derivative. So, obviously pe minus pw divided by delta x. But, the pressure at the faces are not known. Because at point e small e and small w, pressure is not known. So, we have to interpolate. So, let us use average at this point small e, pe as pe plus pP divided by 2 and w as p. So, at this point you can see. So, this is the average if you take.

Then pP plus pw divided by 2. So, pP plus pw divided by 2. So, now because pressures are known only at this point, capital E, capital P and capital W, capital N, capital S. So, where the circular nodes are there only pressures are available at this point. But, in the small e, small w, small n, small s, here you do not know the value of pressure. So, we are just taking the average. So, if you substitute it here, then what you are going to get? Minus del p by del x is equal to minus. So, let us take outside 2 and delta x.

Then you will get pE plus pP minus pP minus pW and this pP will get cancelled. Then you will get minus 1 by 2 delta x pE minus pW. Similarly, you can derive minus del p by del y will be minus 2 by delta y and pN minus pS. Now, you see we are solving the, full Navier-Stokes equations that means those momentum equations at the cell centre p. Where velocities we are solving and when we are calculating the pressure gradient for this momentum equation, pressures are taken away from this point p.

You can see when we are solving u momentum equation at point p pressures are involved east and west. There is no pressure at p and similarly when we are solving y momentum equation del p by del y, pressures are involved at north and south not at P. So, you can see that although we are solving the velocities at a particular cell. But the pressure at that point we are not incorporating in the pressure gradient. In that way, pressure and velocity may decouple and you can see the points discrete points, what we are considering, this is actually 2 into delta x.

So, if you are solving the problems for a particular grid, but when you are calculating the gradient you are using some coarse grid. Because p and pw, pn minus ps, so it is 2 delta x and 2 delta y apart. So, obviously you can think of that you are using a coarse mesh which actually not true. The other way you can see when we are solving the, when we are invoking the continuity equation at this cell. So, what is continuity equation? del u by del x plus del v by del y is 0. So, when you are invoking this continuity equation at this particular cell.

You can write this equation as, ue minus uw by delta x plus del v by del y. So, it will be vn minus vs by delta y is equal to 0. So, velocities we are solving at main nodal point p, but after discretizing this equation the velocities are involved at the faces, small e, small w and v at small n and small s. But, velocities are not known at those points. So, again you need to do the interpolation. So, you can see if you do the interpolation, then will use ue as, uE plus so you can see at E. So uE plus uP divided by 2.

Similarly, uW you will use uP plus uW by 2 and vN as, vN plus vP divided by 2 and vS as, vP and vS divided by 2. So, now these if you substitute in the continuity equation, you will see that the velocities is at point p does not appear. So, you substitute here, so this will be 1 by 2 delta x and it will be uE plus uP minus uP minus uW plus 1 by 2 delta y. So, vN plus vP minus vP minus vS is equal to 0. So, you can see this will get cancelled, then you are going to get uE minus uW plus 1 by 2 delta y vN minus vS is equal to 0.

So, if you see this equation carefully, u you are taking at east face, west face and v you are taking at north face and south face. But, essentially you are solving or you are trying to impose this continuity equation at this control volume. Where you have the main node p. So, obviously you can see that here it is not correct that, there is no velocities which actually appears from the main control volume.

So, this way there will be some problem in the solution; there may be some oscillatory problem in the solution and also you can see that velocities you are taking from far away points. So, that is also one drawback. In some situations of the flow fluid, if you have some oscillatory distribution or oscillatory velocity distribution, then what will happen you see.

So, this is known as Checker-board pressure distribution or Checker-board velocity distribution. So, let us write here Checker-board pressure distribution. So, let us say one dimensional case and you have these are the points, main points, main node and these are the faces.

In marker and cell, we are using cell. So, this is your P, this is your E, this is your W and these are small e and small w and consider some pressure distribution, which is appears like it is 200 at this point it is 100, this is 200, this is your 100 and this is your 200. So, some oscillatory pressure field is there and you have this type of pressure distribution. So, if you see that pressure, if you calculate at East point, what you are going to get? So, it is just average 200 plus 300 divided by 2.

So, that means it is 150 and at this west face it is 150 and this del p by del x if you calculate, you are going to get 150 minus 150 means 0. So, del p by del x you are going to get 0. So, you can see that in the momentum equation. When you are solving the pressure gradient term is becoming 0. But, fluid flows due to the pressure gradient, so that means here pressure and velocities are decoupled.

They are not talking each other. So, this is the problem in the collocated grid and similar will happen in the when you have a oscillatory velocity field and when you will invoke the continuity equation, due to this oscillatory pressure field it may satisfy but which is not true. So, due to this problem of pressure and velocity decoupling in the collocated grid, we use staggered grid. In this lecture, when you are using marker and cell method, we will use staggered grid.

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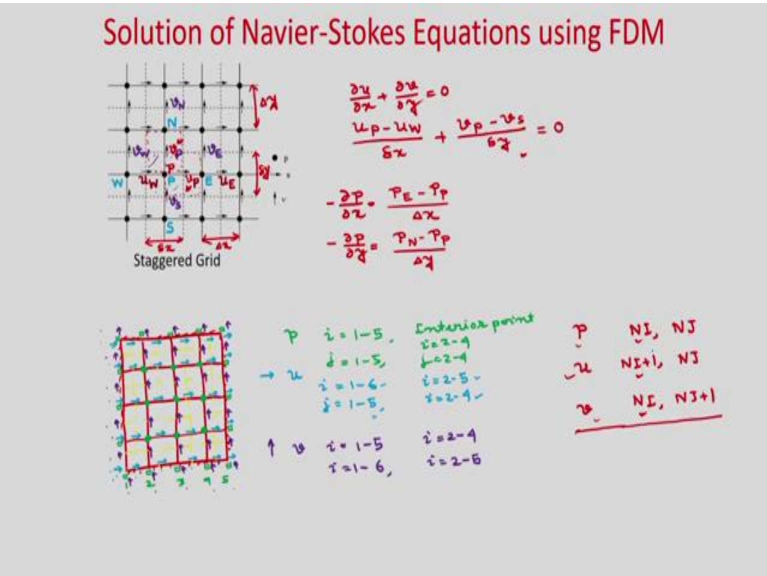

So, in this staggered grid you can see that, if you have this main control volume p is here and you are solving for pressure here. This is your east, this is your west, this is general notation whatever we discussed in earlier lecture. This is your north and these are south. These are the points for pressure and so you are solving pressure here and you are solving the velocities u here and v here.

So, here you can see that u when you are solving at this point, this is the control volume or cell. It is not control volume; it is cell for this finite difference method. So, this is the cell, this is the main cell for this velocities u and this is the main cell or velocity v and they are in staggered way. So, you can see now, so if it is, so for u this is the main control volume. So, uP and this is your uW and this is your u east and for v velocity.

So, this is your vP this is the main control volume, this is your v north and this is your v south and for this v east and this is your v west. So, the index whatever we are using we have different cells for pressure u velocity and v velocity and in their main cell we are denoting as capital P that means i, j and this is i, j. Now, if you are solving for this Navier-Stokes equation for this main control volume. So, now if you are invoking, so now if you are invoking the continuity equation at this point.

So, what you are going to get, del u by del x plus del v by del y is equal to 0. So, in the main control volume now you see for the pressure. So, it is uP minus uW divided by delta x. So, this is the delta x and this is your delta y and this is your delta x and this is your delta y. So, now uP minus uW divided by delta x plus del v by del y. So, that means vP minus vS divided by delta y is equal to 0. Now, you can see that, the velocities are already available at those points.

At the face center of the main cell where we are solving the pressure. So, obviously we do not need any interpolation. We do not need any interpolation, so obviously there will be no pressure and velocity decoupling and essentially you are using the nearest grid point velocities and it is second order accurate as you are using central difference. Now, you consider the pressure gradient, now you consider the pressure gradient in the u momentum equation. So, if you consider minus del p by del x. Now where you are solving these velocities? So now this is a different cell.

It is different cell, so minus del p by del x when you are writing for this uP, you are going to write pE minus pP divided by delta x and minus del p by del y, you are going to write for the v momentum equation, del p by del y will appear for v momentum equation. So, pressure will write, pN minus pP divided by delta y. Now, you see already the pressures are available at those points, you do not need any interpolation. So, there will be no pressure and velocity decoupling. So, this is the advantage of using staggered grid.

Although you can use collocated grid and you can overcome this pressure velocity decoupling using (())(29:49) interpolation. So, that we will not discuss here. Now, will discuss about only the MAC algorithm in this lecture using staggered grid. Here let us discuss about what is the advantages and disadvantages of this collocated and staggered grid. Obviously, collocated grid is easy to code, you can see that we have already we are solving all the variables u, v, p, temperature or spaces at the same node.

So, that data structure to maintain the data structure it is easy. Because you are having the same index for the u, v, w. But, when you are solving using staggered grid, so you can see that velocities are in staggered way and keeping of this data structure is very difficult because you have separate cell for u, v and p and accordingly you need to keep the indices. So, data structure somewhat complicated in case of staggered grid. But, advantages you have seen that in staggered grid pressure and velocity decoupling will not be there.

But, in collocated grid that is the problem. Now, in the boundary conditions; so if you have staggered grid, then you can see that if you have the boundaries. So, now let us consider a cavity, let us consider a square cavity. So, these are the boundaries. So, obviously at the boundaries you will apply no-slip boundary condition. You have this is the grid, let us say it is uniform grid.

Now, where you are solving the pressure, pressure you are solving at this points and these are the boundaries, these are the boundaries for pressure. So, you need to apply pressure boundary conditions at this points. So, if you see in terms of i, j, you have how many i varies for pressure, i varies here 1, 2, 3, 4, 5. So, 1 to 5 and interior points you have interior points you have, i is equal to 2 to 4. Because in these points you will solve. So, these are 2 to 4. Now, you see it is for pressure, now for u velocities. So, u velocity is you are solving; first let us make a grid like this. So, that you can identify the cells easily.

Now, you see u velocity is you are solving at the face centres; the vertical face you have in this case edge, at this point you are solving, these are the points. Now here, so all these are interior points and boundary points you can see, boundary points in the y direction any way you need to you have velocities at the boundaries. So, horizontal faces you have the u velocities. So, this is your u velocity. So, and here also you need to solve, so you can see, if you apply at the boundary, then it is not actually lying on the boundary.

But, u velocity boundary condition you need to satisfy at the wall. Now, you see p j is varying, in j is varying 1, 2, 3, 4, 5. So, 1 to 5 and interior points j varies 2 to 4. But, for u you see 1, 2, 3, 4, 5, 6. So, it varies i varies 1 to 6 and interior points varies 2 to 5 and j varies now you see j 1, 2, 3, 4, 5. But, it varies 1 to 5 and interior points varies 2 to 4. So, you can see that index are not same for u velocities. Because in i direction it grid points 1 to 6, but j 1 to 5, accordingly interior points also varies.

Similarly, if you now see velocities v. So, velocities v to c. So, velocities v we are solving at the horizontal surface top in a cell. So if this, so this is the horizontal face. So, here you are solving, so these are interior points. Here you are solving, so these are interior points and exterior point in the vertical surface in a way, actually it is falling on the boundary itself, you can see here. But, the index varies differently and at this boundary horizontal boundary, now you have velocities here you need to satisfy, v velocities no-slip conditions.

So, now you see this is your v velocities, i varies i 1, 2, 3, 4, 5. So, 1 to 5, interior points varies 2 to 4 and j varies 1, 2, 3, 4, 5, 6. So, i varies 1 to 6 and interior point varies 2 to 5. So, you can see the bookkeeping of this data structure is very difficult. Because indices are different for pressure, velocity and in velocities also in i and j direction indices are different. So, you can see if you have pressure, if you have total number of points for pressure.

If it is NI and NJ, then for u you will get NI plus 1, NJ and for v you will get NI, NJ plus 1 and the locations are also different. These are in staggered way that you have to remember. So, very carefully you need to write the code in staggered grid and accordingly you need to apply the boundary condition. Because boundary will be at point i is equal to 1, for velocities i is equal to 1 and i is equal to NI plus 1. So, that is the boundary but for v you can see, i is equal to 1 and NI and for pressure boundary it is, i is equal to 1 and NI.

So, this you have to keep the bookkeeping. So, that you do not do any mistake while writing the program. So, now we discussed about the staggered grid and non-staggered grid. What is pressure velocity decoupling and we have also discussed, what are the advantages and disadvantages of these different types of grids.

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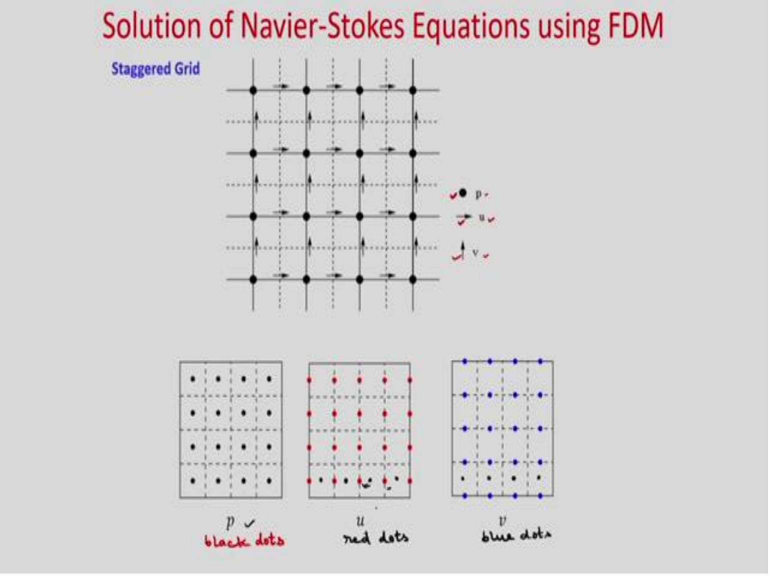

So, you can see we will be dealing with, so now in MAC algorithm we will use staggered grid. Where pressure will be at the main cell and velocities will be at the faces of this cell. So, this is the staggered grid. So, where this circular dot is pressure, this arrow horizontal arrow is u and vertical arrow is v and if you see for the pressure, for the pressure these are the cells and at black circular dot you are going to solve pressure at the interior points and here for velocity u, you can see wherever you are solving at the red dot.

So, here black dots, so those are the interior points. For velocity u, now you are solving the velocities at the faces. So, you can see, so this was your, so this is your pressure. So, at the face you are solving u velocity, so these are the red dots and when you are solving v velocity. So, v velocity you are solving at the horizontal face.

So, if these are pressure, then v you are solving at the blue dots and you have different cells for p, u and v. So, today in this lecture we have discussed about why we need to solve the primitive variable approach and then we have discussed about two different types of grid arrangement. One is collocated grid and staggered grid.

In collocated grid the problem is that velocity and pressure may decouple. So, to avoid that problem we used staggered grid, where there will be strong coupling between pressure and velocities. However, in staggered grid, book keeping of data structure is very difficult.

Because you have different grid points for pressure and velocities u and v. Pressure you solve in the main cell and velocities you solve in the staggered way. Separate cells are there for p, u, v. MAC algorithm is developed using staggered grid and in the next lecture we will discretize the Navier-Stokes equation using MAC algorithm, thank you.