Computational Fluid Dynamics for Incompressible Flows
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Lecture 1
Discretization vorticity-stream function equations using FDM

Hello everyone. So, in last few lectures we have learned the discretization of hyperbolic equations, parabolic equations and elliptic equations. So, we have learned some explicit and implicit schemes and you know that the finite difference approximation of a certain order to discretize any derivative.

So, now, today we will discretize the Stream Function Vorticity Equation, just to solve some fluid flow problem. So, we will consider steady incompressible flow and we will consider momentum equation first. Then we will convert it into stream function vorticity equations. Then we will discretize that equations using finite difference method. So, you know what is vorticity, right from fluid mechanics knowledge. So, vorticity is twice of the angular velocity or curl of velocity.

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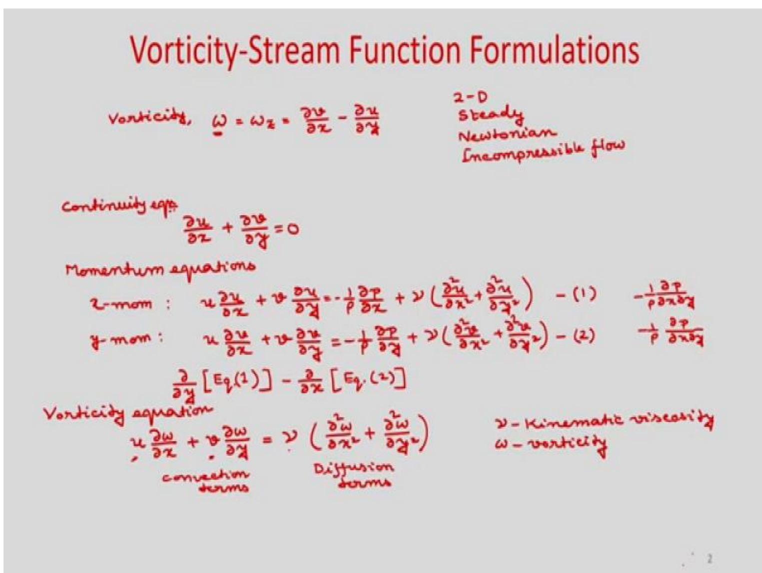

So, in two dimensional situation you can write, the vorticity, vorticity will denote at omega. So, we are considering two dimensional. So omega z, so this is your del v del x minus del u del y. So, this omegas, we are just representing as vorticity and we are considering 2D, steady, Newtonian and incompressible flow.

So, with this assumptions let us write the governing equations of free flow. So, that you know that one is continuity equation. Then we have two momentum equations. So, we have continuity equation del u del x plus del v del y is equal to 0. So, this is your continuity equation and you have momentum equations. So, momentum equations now x momentum, x momentum equations is steady we are writing. So, u del u by del x plus b del u by del y is equal to minus del p by del x plus.

So, this 1 by row you can write minus 1 by row and this is your mu by row, so kinematic viscosity nu del 2 u by del x square plus del 2 u by del y square. So x momentum equation, so we considered u velocity and y momentum equation you can write u del v by del x plus b del v by del y is equal to minus 1 by row del p by del y plus nu del 2 v by del x square plus del 2 v by del y square. So, in this equation there are 3 equations and 3 variables u, v and p, we have to solve. So, and fluid properties row and dynamic viscosity mu are known.

So, to eliminate the pressure from this term because we want to solve by simplified equations which conductive diffusive equation first. So, we will eliminate pressure from these momentum equations. To eliminate pressure, so what you do? You just take that derivative with respect to y of this equation 1 and you take the derivative with respect to x of this equation 2.

So, what you do? You take the derivative with respect to y, with respect to y of equation 1 equation 1 and subtract the second equation with respect to, so you take the derivative with respect to x of equation 2. Then you if you subtract then you can see that the pressure term will get eliminated, so you can see this will get del p by del x del y minus 1 by row and here you will get minus 1 by row del p by del x del y.

So, you can see that if you subtract then this will be eliminated and if you rearrange then you will get the equations vorticity equations. So, you will get u del omega by del x plus v del omega by del y is equal to nu del 2 omega by del x square plus del 2 omega by del y square.

So, from this two momentum equations, we could derive the vorticity equation just eliminating the pressure. So, you can see it is u is the x direction velocity and v is the y direction velocity and omega is the vorticity and nu is the kinematic viscosity. So, here nu is kinematic viscosity and omega is in this case vorticity. So, we could derive this vorticity equation and which is conductive diffusive equation. Because in the left hand side convection term and this is your convection term and this is your diffusion term and this is your vorticity equation.

But how to find u and v now? Because vorticity equation you can solve for vorticity omega. But you need to know values of u and v. So, for that we will use stream function. So, you know what is stream function, stream function is a function which satisfies the continuity equation such that u is equal to del psi by del y and v is equal to minus del psi del x.

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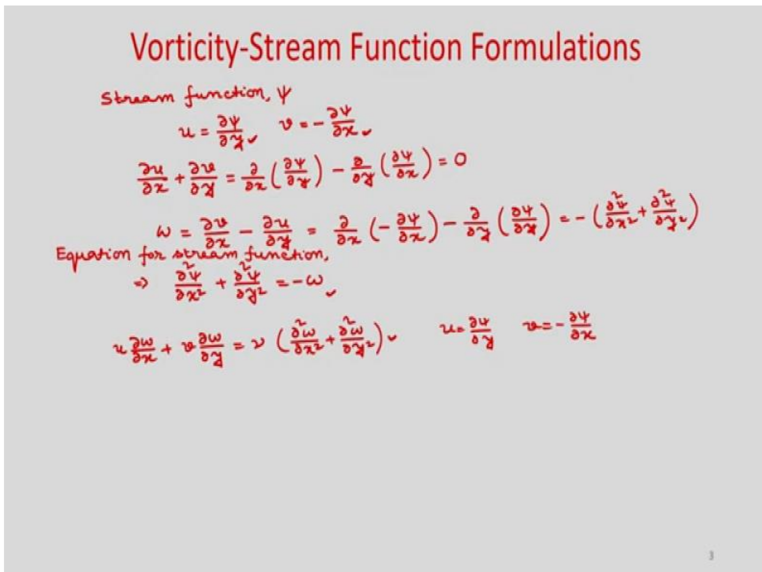

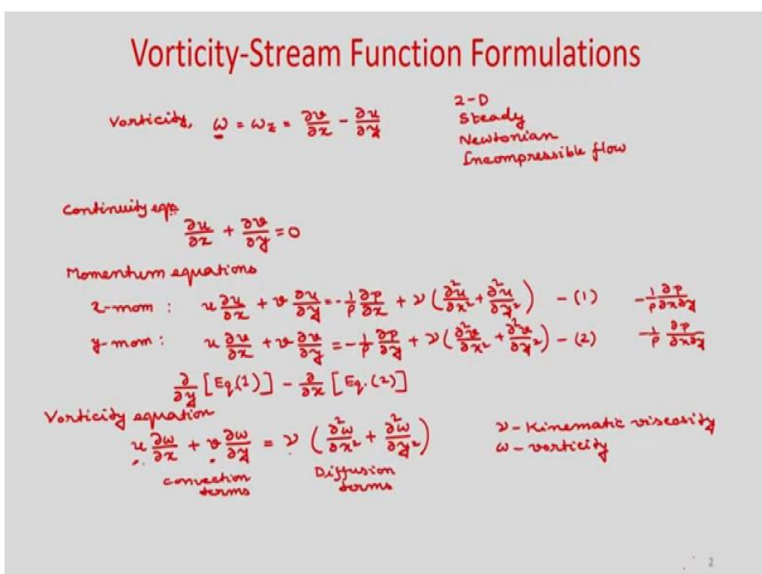

So, we define stream function, so stream function is psi. So, it satisfied the continuity equation such that u is equal to del psi by del y and v is equal to minus del psi by del x. So, you see whether it is satisfied continuity equation or not. So, you have del u by del x plus del v by del y is the plus del v del y, if it is 0 than it satisfies the continuity equation. So, if it is, so del of del x del psi by del y and minus del of del y del psi by del x. So, obviously this is equal to 0.

So, that means it satisfies the continuity equation. So, now we know the definition of vorticity. So, vorticity already we have defined omega is equal to del v by del x minus del u by del y. So, now you substitute v here. So, del of del x what is v? v is nothing but minus del psi by del x and minus del of del y what is u? u is del psi by del y.

So, you can see that it is minus del 2 psi by del x square plus del 2 psi del y square. That means you can write as del 2 psi by del x square plus del 2 psi by del y square is equal to minus omega. So, this is the equation for stream function, equation for stream function. So, now this equation this is your obviously elliptic type equation. So, it is poison equation, so for a known value of vorticity omega you can solve this equation can find the stream function psi.

Once you know the psi, you can find the velocities. Because u v you can represent as a gradient of psi. So, from these expressions u is equal to del psi by del y and v is equal minus del psi by del x you can find the velocities once you solve this equation psi is known.

So, now if you go back and see the vorticity equation. Now, in vorticity equation u v you can get from the stream function. So, that means you need to solve two equations, one is your stream function equation, grad square psi is equal to minus omega and the vorticity equation which is you conductive diffusive equation.

So, that means now your this set of equations. So, you have u del omega by del x plus v del omega by del y is equal to nu del 2 omega by del x square plus del 2 omega by del y square. So, this is your vorticity equation and this is you stream function equation and you can find u v from del psi by del y and v is equal to minus del psi by x.

So obviously, we have written the governing equations in dimensional form and these equations if you solve. Then you will be able to get the velocity distribution in a domain. So, we will discuss two problems and we will discuss about the boundary conditions and we will show some results after solving those, these equations for a particular problem.

So, now we need to discretize this equation using finite difference method. So, you have two different equations. One is stream function equation and one is vorticity equations and now these two equations used as discretize, using finite difference method you know different scheme. So, depending on your choice you can use some scheme and discretize using finite difference method. So, first, let us discretize the stream function equation using finite difference method.

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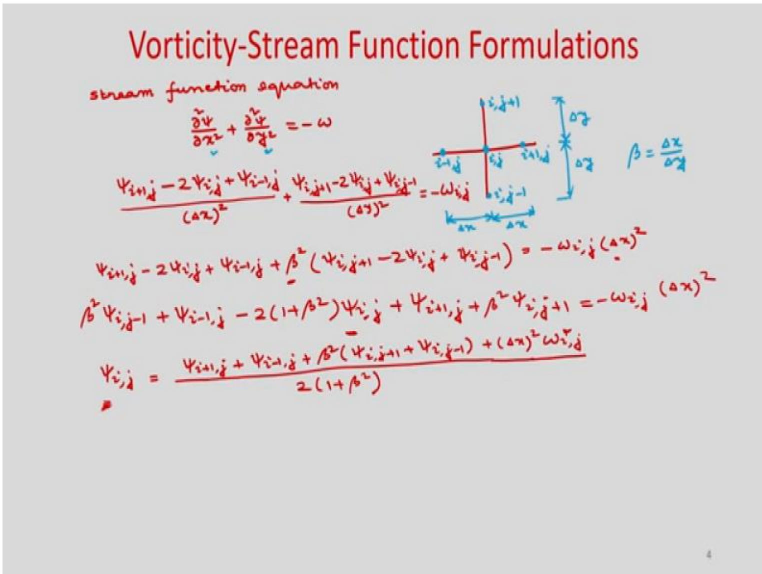

So, you have stream function equation, del 2 psi by del x square plus del 2 psi by del y square is equal to minus omega. So, we will take a grid 2 dimensional grid. So, you have ij i plus 1 j and this is your, i minus 1 j this is your i j plus 1 and this is your i j minus 1 and we will consider uniform grid.

So, that means here the delta x is constant, this is your delta x, this is your delta x and this is your delta y and this is also delta y. So, delta x may not be equal to delta y. So, we will define the beta is equal to delta x by delta y, the ratio of the step size in x and y direction delta x by delta y we are denoting as beta.

So, this is second derivative of psi. So, second derivative, so what we will use? We will use central difference approximation to discretize this second derivative. So, if you use second order method or central difference, so which is second order. So, you can write psi i plus 1 j minus 2 psi i j plus psi i minus 1 j divided by delta x square.

So, this is second order accurate this is you know, this approximation. Similarly in del 2 psi by del y square. So, psi i j plus 1 minus 2 psi i j plus psi i j minus 1 divided by del y square is equal to minus omega i j. So, this is the discretized equation, now you just write psi i plus 1 j minus twice psi i j plus psi i minus 1 j and we have defined beta is equal to delta x by delta y. So, we can write beta square psi i j plus 1 minus twice psi i j plus psi i j minus 1 is equal to minus omega i j delta x square.

Because delta x square I have multiplied both side. So, delta x square by delta y square I have denoted as beta square and delta x square is coming here minus omega i j into delta x square. So, after simplification you can write it as psi i j minus 1 plus psi i minus 1 j minus 2 into 1 plus beta square psi i j plus psi i plus 1 j beta square psi, this is your beta square and psi i plus 1 j and plus beta square psi i j plus 1 is equal to minus omega i j delta x square.

So, this is the discretize equation. So, now you have to choose whether you will use implicit method or explicit method to solve it. So, you can see that psi i j you need to find. So, if you explicit type method if you use then you can write psi i j, which is your unknown. So, psi i j is equal to you can write psi i plus 1 j plus psi i minus 1 j plus beta square. So, these I am taking this side was beta square psi i j plus 1 plus psi i j minus 1 and this minus it will become plus. So, it will be delta x square into omega i j divided by 2 into 1 plus beta square.

So, if you add using some solver which is explicit type solver. Then you can solve for psi i j at time at iteration level k plus 1 and all you can take at best available value. So, here you can see that although you are solving for a psi. But you have to know the value of omega and this omega you have to get it from the vorticity equation and you can see that this way, this two equation are couple. Because when you are solving, the stream function equation you need to know the value of vorticity omega.

Similarly, when you are solving for Vorticity equation you need to know the velocity is u v which in turn actually you need to know the stream function. So, these are coupled equations, so you have to solve simultaneously. So, now we have discretized this equation, stream function equation and psi i j you can solve using this expression for the known value of omega i j. Now let us discretize the vorticity equation.

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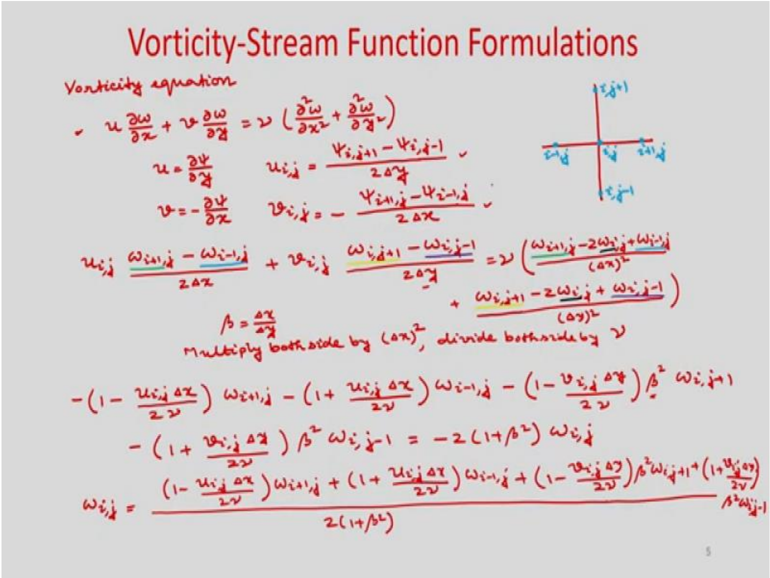

So, now we will discretize Vorticity equation, so what is the Vorticity equation? u del omega by del x plus v del omega by del y is equal to nu del 2 omega by del x square plus del 2 omega by del y square. So, u is del psi by del y. So, you can discretize this, in 2 dimensional grid if you use.

This is your, i plus 1 j, this is your i minus 1 j, this is i j plus 1 and this is i j minus 1. So, now if you want to find the value u at point i j, then you can discretize using u i j, u i j del psi by del y it is the first derivative, so you can use some second order approximation, so that you can use central difference, so you can write u i j.

So, it is a first derivative del psi by del y. So, you can use some second order approximation. So, we will use central difference. So, you can write psi i j plus 1 minus psi i j minus 1 and the distance between this 2 points is 2 delta y. So, this is your u i j. Similarly, v is equal to minus del psi by del x. So, v i j and if you use the central difference approximation then you can write psi i plus 1 j minus psi i minus 1 j divided by 2 delta x.

Because we have use central difference approximation for this first derivative and the distance between that 2 points is 2 delta x. So, psi i plus 1 j minus psi i minus j by 2 delta x and psi you already you can find it, so psi you can find it from the stream function equation. Already we have discretized that equation. So, from the known values of psi you can find the values of u and v. Now you discretize this equation.

So, here also let us use the central difference for the first derivative and central difference for the second derivative. So, that overall order of accuracy will be delta x square and delta y square. Because the stream function equation already we have discretize using second order approximation. So, the order of accuracy is delta x square and delta y square and when we are going to discretize this vorticity equation. Let us take central difference approximation, so the order of accuracy will be delta x square delta y square.

So, we will take the central difference approximation for both first derivative and second derivative. So, you can write, u i j del omega by del x, so omega i plus 1 j minus omega i minus 1 j. So, central difference divided by 2 delta x plus v i j then omega now del omega by del y. So, this i j plus 1 minus omega i j minus 1 divided by 2 delta y. Now in the right hand side we have the second derivative. So, you will use central difference, so you will use omega i plus 1 j minus twice omega i j plus omega i minus 1 j divided by delta x square.

So, this is the first term and the second term you can write, omega i j plus 1 minus twice omega i j plus omega i j minus 1 divided by delta y square. So, you can see overall the approximation is second order delta x square delta y square. So now, you rearrange it and you define beta is equal to delta x by delta y. So, if you rearrange it you are going to get, so you can see what are the terms are there. So, omega i plus 1 j here, omega i plus 1 j here. This we will take common, then omega i minus 1 j here, omega i minus 1 j here.

Then we have omega i j plus 1 here omega i j plus 1 and now we have omega i j minus 1 and we have omega i j minus 1 and we have omega i j here and omega i j here. So now, you rearrange it and write. So, multiply both side by delta x square. So, that we can write in terms of beta, so delta x square. So you can see this omega i plus 1 j. So omega i plus 1 j, so if you write omega i plus 1 j. So, it will be multiply both side by delta x square and divide both side by nu. So, that here nu will come in the denominator.

So, now you can write, so what is the coefficient of omega i plus 1 j? So omega i plus 1 j so it will be minus 1 minus u i j. So, we have multiplied by delta x square. So, it will be delta x divided by 2 into nu it will be omega i plus 1 j. Similarly, omega i minus 1 j. So, i minus 1 j, so it will be negative, so you can, it will be omega 1 plus u i j delta x by twice nu omega i minus 1 j.

Similarly now you write omega i j plus 1. So, omega i j plus 1, if you write then it will be minus. So, it will be beta square, so it will be beta square, so it will be 1 minus, so v v i j v i j. So we have multiply with delta x square. So, it will be delta x square. So delta x by delta y will be beta, so it will be delta x square. So, beta square and this is delta y if you get it. So, it will be v i j, so delta y divided by 2 into nu and beta square will be there. So, it will be beta square omega i j plus 1.

Because here delta y is there. So, if you multiply both side delta y. So, denominator and numerator, so it will be delta y by delta y square and delta x square you have a. So, delta x square by delta y square will be beta square and that we are taking common. Similarly you write for omega i j minus 1. So, it will be minus 1 plus v i j delta y by twice nu. Then beta square omega i j minus 1 and now you have in right hand side. So, you have omega i j, so nu we have already divided. So, it will be omega i j. So, it will be minus 2 into 1 plus beta square into omega i j.

So, now you can write for omega i j. So, omega i j is equal to so minus in both side it is there. So, 2 into 1 plus beta square and so it will be, so it will be 1 minus u i j delta x divided by twice nu omega i plus 1 j plus 1 plus u i j delta x by twice nu omega i minus 1 j plus 1 plus v i j delta y by twice nu beta square omega i j plus 1 and you have plus 1 plus v i j delta y by twice nu and multiplied by beta square omega i j minus 1. So, now you can substitute the values of u i j and v i j. Because u i j, v i j you have written in terms of psi. So, if you substitute it then you are going to get

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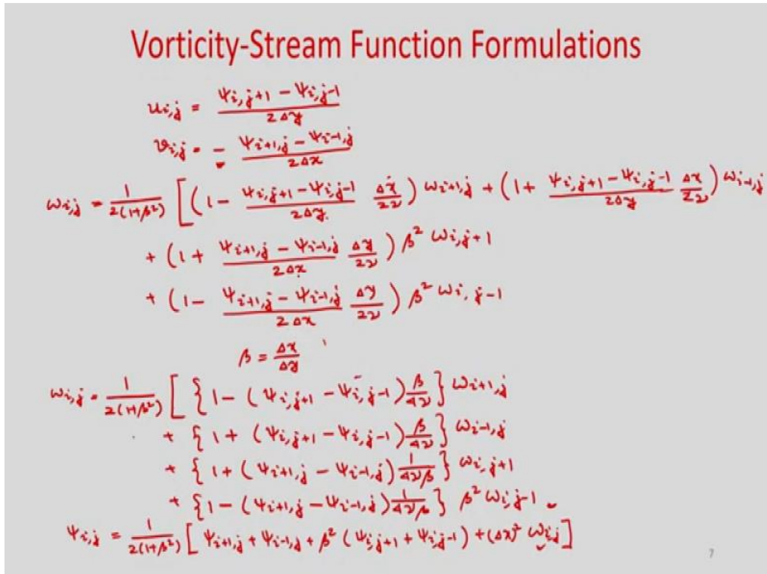

Omega i j we have written, but now we can substitute u i j and v i j. So, u i j is psi i j plus 1 minus psi i j minus 1 divided by twice delta y and v i j as minus psi i plus 1 j minus psi i minus 1 j divided by 2 delta x. So, now we will substitute this u v in the expression of omega i j. So, omega i j is 1 by 2 into 1 plus beta square 1 minus.

Now you put the value of u. So, it will be psi i j plus 1 minus psi i j minus 1 divided by 2 delta y delta x by twice nu. So, this is omega i plus 1 j, then plus now 1. So, minus so it will be plus. So, psi i j plus 1 minus psi i j minus 1 divided by 2 delta y delta x by 2 nu omega i minus 1 j. Then next term now 1, so it was minus but velocity v i j is having negative sign so this minus minus will become plus.

So, psi i plus 1 j minus psi i minus 1 j divided by 2 delta x delta y by twice nu. Then beta square omega i j plus 1 and 1 minus. So, here plus was there but now v i j you have negative. So, this negative here it will appear negative. So, psi i plus 1 j minus psi i minus 1 j divided by 2 delta x delta y by twice nu beta square omega i j minus 1.

So, now you substitute beta is equal to delta x by delta Y, here delta x delta y and here delta y delta x, so you can put beta. So, if you write omega ij, so you will get 1 by 2 into 1 plus beta square 1 minus psi i j plus 1 minus psi i j minus 1 beta by 4 nu, delta x by delta y we are putting beta omega i plus 1 j plus 1 plus psi i j plus 1 minus psi i j minus 1 beta by 4 nu omega i minus 1 j plus 1 plus psi i plus 1 j minus psi i minus 1 j by 1by 4 nu beta.

So, beta delta y by delta x is equal to 1 by beta omega i j plus 1 plus 1 minus psi i 1 j minus psi 1 i minus 1 j 1 by 4 nu beta beta square omega i j minus 1. So, this the discretize equation for omega i j in dimensional form and whatever psi we already derived, psi ij is 1 by 2 into 1 plus beta square psi i plus 1 j plus psi i minus 1 j plus beta square psi i j plus 1 plus psi i j minus 1 plus delta x square omega i j. So, this is the discretize equation for psi. So, you can see in the psi discretize equation omega i j is coming and in omega psi values are coming. So, it is coupled and here you need to solve this psi i j and omega ij for all interior points.

So, whatever these equations we have derived for the stream function and the vorticity. So, these are applicable for the interior points. But we need to solve some free flow problem. So, in the next class we define, some free flow problem and for those, problem we will define the boundary conditions and we will discuss the boundary conditions for omega and psi.

Because when you are solving these equations these vorticity equation and stream function equation, you need the boundary as well in terms of psi when you are solving for stream function and you need the values of vorticity at the boundary when you are solving the vorticity equation.

But, you may know the boundary condition in terms of velocities for a particular fluid flow problem. So, from these velocity boundary condition you need to derive the stream function boundary condition as well as the vorticity boundary condition. Then only these two equations whatever we have derived today for the interior points you will be able to solve for a particular problem provided the boundary conditions are given in terms of stream function and vorticity. So, in next class we will discuss two problems and we will describe the, we will discuss the boundary conditions for those two problems. Thank you.