Computational Fluid Dynamics for Incompressible Flows
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Lecture 4
Modified equation, artificial viscosity, Numerical diffusion

Hello everyone. So in today's lecture we will discuss about the modified equation, artificial viscosity and numerical diffusion. So, first we will derive the modified equation so now we will determine the dominant error term present in the finite difference equation. First, we will substitute the Taylor series expansion in the finite difference equation, then after arithmetic manipulation will find the modified equation.

To illustrate these we will consider 2 examples first we will consider first order accurate scheme, which is first order upwind. And next we will consider second order accurate scheme which is your midpoint leapfrog method.

So, these examples will show the dominant error term and their relations with this numerical deficient error. So, first let us consider the Forward Time Backward Space scheme which is your explicit scheme first order upwind.

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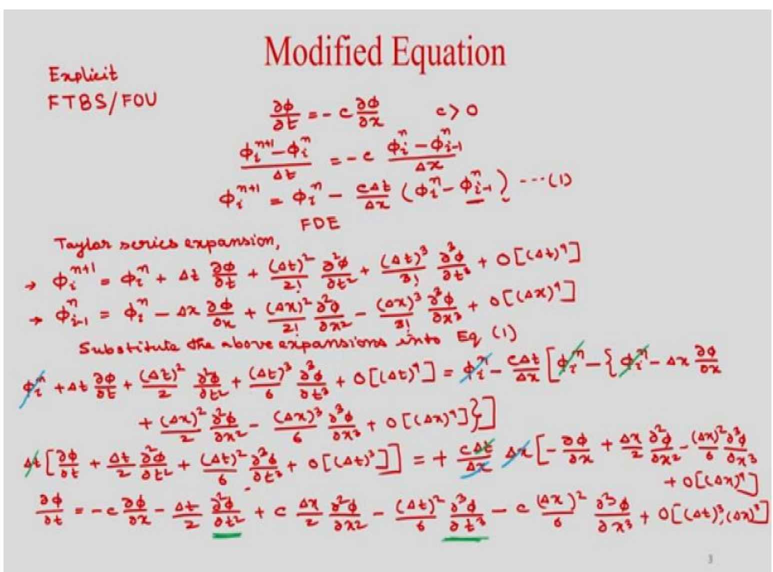

So, we will consider a FTBS this your explicit scheme and or first order upwind, so our governing equation is del phi by del t is equal to minus c by del phi by del x and c is the os speed and which is greater than 0. So, if you use first order upwind scheme and discretize this equation what you will get you will get phi in plus 1minus phi in divided by delta t is equal to minus c phi i. So, backward space so it is phi i minus 1n divided by delta x.

So, you will get phi in plus 1 is equal to phi in minus c delta t by b delta x and phi in minus phi i minus 1n. So, this is the finite difference equation finite difference equation using first order upwind scheme and let us say that equation number is 1. Now, you use Taylor series expansion to expand phi i n plus 1and phi i minus 1n.

So, use Taylor series expansion, Taylor series expansion to expand phi i n plus 1 and phi i minus 1n. So, phi i n plus 1 if you do then it will be phi in plus del t del phi by del n plus del t square by factorial 2 sorry it is t del 2 phi by del t square plus del t cube by factorial 3 del cube phi by del t cube plus high order term, del t to the power 4.

Now, similarly we expand using Taylor series expansion the term phi i minus 1n, so phi i minus 1n so it will be phi in minus delta t del delta x del phi by del x plus delta x square by factorial 2 del 2 phi by del x square minus del x cube by factorial 3 del cube phi by del x cube and other high other terms.

So, now substitute these expansion into the original fd, so substitute the above expansions into equation 1 equation 1, so this is your equation 1 you substitute back so then left hand side you substitute this one so you will get phi in plus del t del phi by del t plus del t square by 2 del 2 phi del t square plus del t cube by 6 del cube phi by del t cube plus high order term. And in the right hand side you have phi i n minus c delta t by delta x and you have phi i n minus phi i minus 1 n.

So, this you substitute this 1 phi I minus 1 n so it will get phi i n minus delta x del phi by del x plus delta x square by 2 del 2 phi by del x square minus del x cube by 6 del cube phi by del x square and other high order terms. So, now you do some arithmetic manipulation so you can see in the left hand side this phi n and this phi n will get cancelled.

Then you can see inside this phi i n and this is minus phi i n so this will get cancelled, so now you take delta t outside then you can write del phi by del t plus. So, it will be del t by 2 because we have taken outside delta t so del 2 phi by del t square plus delta t square by 6 del cube phi by del t cube plus order of del t cube. Because delta t you c have taken outside so you can write delta t cube is equal to so there will be minus c delta t delta x which is your coolant number lambda and you have minus here.

So, you can see this minus and this minus will become plus, so it will become plus and you can take delta x outside and you can write minus del phi by del x plus del x by 2 del x by 2 del 2 phi del x square minus delta x square by 6 del cube phi by del x cube and order of delta x cube. So, now both side you divide by delta t so you can see these will get cancelled.

So, this delta t this delta t and this delta x and this delta x will get cancelled. So, now you can write the equation as del phi by del t just after rearrangement you can write minus c del phi by del x and minus c del phi by del x then will be so this is your minus del t by 2. So, this I am taking del 2 phi by del t square plus c del x by 2 del 2 phi by del x square minus, so it will be this I am taking so it will be delta t square by 6 del cube phi by del t cube. And this will be minus c del x square by 6 del 2 del cube phi by del x cube and order of del t cube del x cube.

So, only in the left hand side we have kept del phi by del t and all other terms we have taken in the right hand side, now what you have to do so you can see here you have term del 2 phi by del square and del cube phi by del t square. So, these we have to substitute in terms of special derivative, so this time derivatives it is your second derivative and third derivative this you substitute with special derivative.

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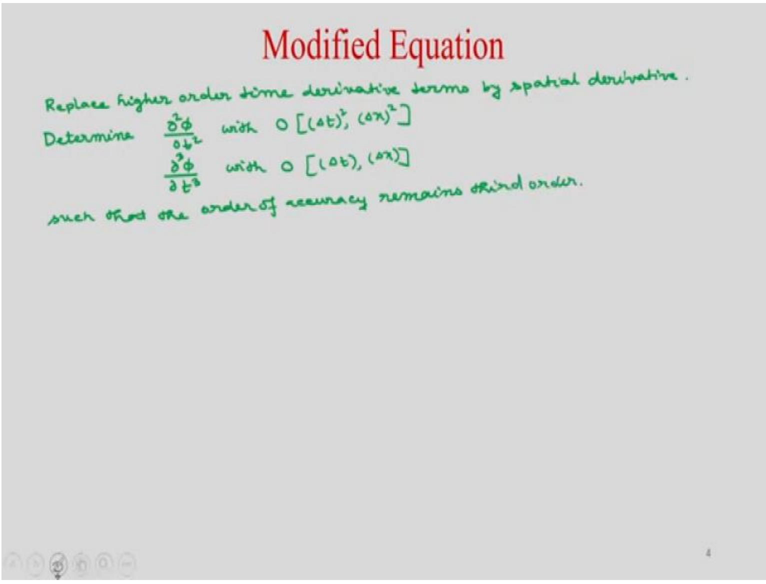

So that you can so replace high order time derivative terms by special derivative. So, determine del 2 phi by del t square with order of accuracy delta t square delta x square because what all accuracy we have to maintained del t cube and del x cube. So, this you obtained with this order of accuracy and del cube phi by del t cube with order of accuracy delta t and delta x such that the order of accuracy order of accuracy is remains third order, remains third order remains third order.

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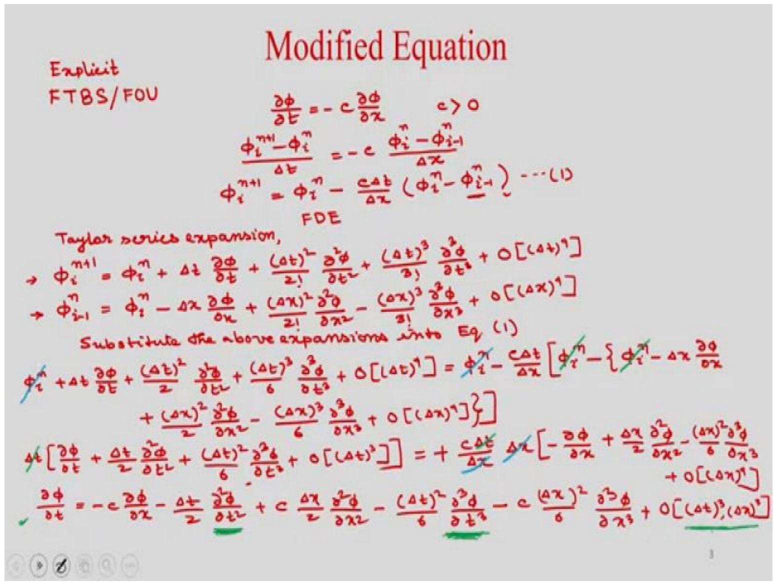

So, you can see in the previous slide so this is your order of delta t cube and delta x cube that we have to keep to in order to keep this order of accuracy this we have to discretize with the order of delta t square and delta x square and this we have to write in terms of special derivative which is order of delta t and delta x.

So, this derivation I am not going to show so you can refer some books where you can find or you yourself you can do this mathematical formulation. So, what you do so you have to take the time derivative of this equation and also the special derivative and with some rearrangement you can find these time derivative in terms of special derivative. So, that derivation I am not going to show I am just going to write the final expression of these time derivative in terms of special derivative.

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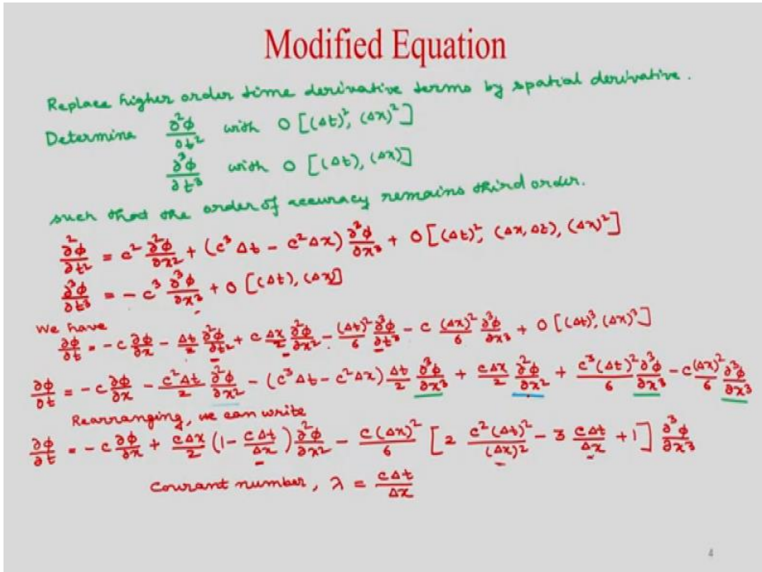

So, if you do that so you will get del 2 phi by del t square you will get c square del 2 phi by del x square plus c cube delta t minus c square delta x del cube phi by del x cube. Which will be the order of delta t square delta x delta t and delta x square. Similarly, you can so that del cube phi by del t cube will be minus c cube del cube phi by del x cube which is order of delta t delta x.

So, now let us write whatever we derived in last slide so del phi by del t is equal to minus c del phi by del x minus delta t by 2 del 2 phi by delta t square plus c delta x by 2 del 2 phi by del square minus delta t square by 6 del cube phi by del t cube minus c delta x square by 6 del cube phi del x cube. So, now and order of delta t cube and delta x cube.

So, now let us put these values del 2 phi del t square as this one and these you put this one. So, now you can write del phi by del t is equal to minus c del phi by del x will keep it as it is. Now, del 2 phi by del t square so this you use this right hand side terms. So, it will be minus c square delta t by 2 del 2 phi by del x square then you have minus again minus c cube delta t minus c square delta x and we have del t by 2 here so it will be del t by 2 del cube phi by del x cube.

So, here we are replacing the temporal derivative with the special derivative, then next term you keep as it is so it will be c delta x by 2, del 2 phi by del x square. Now, this you replace with this term so it will be minus and this minus and this so this is minus and this is your minus it will be become plus c cube delta t square divided by 6 del cube phi by del x cube and this last term minus c delta x square by 6 del cube phi by del x cube.

So, now you rearrange it so you can see you have del 2 phi by del x square del 2 by del x square and here you have del cube phi by del x cube del cube phi by del x cube del cube phi by del x cube. So, you rearrange it and rearranging we can write you can write del phi by del t so minus c del phi by del x plus c delta x by 2 if you take common so it will be so see this term will be 1 and minus now you can see it will be c delta t by delta x.

So, you can see it will be c square delta t by 2 del 2 phi by del x square minus if you take see delta x square by 6 common. Then you can write 2 into c square delta t square by delta x square minus 3 c delta t by delta x plus 1 into del cube phi by del x cube. So, now you can see that this term you can replace with the coolant number coolant number so you can write lambda is equal to c delta t by delta x, so this you just replace with lambda.

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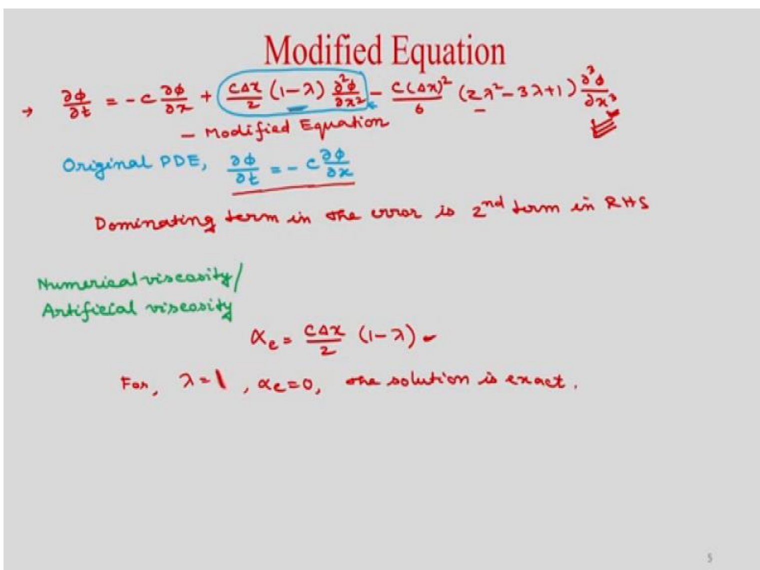

So, if you write it then you will get del phi by del t is equal to minus c del phi del x plus c delta x by 2 1minus lambda del 2 phi by del x square minus c delta x square by 6 2 lambda square minus 3 lambda plus 1del cube phi by del x cube. So, you can see this is known as modified equation this is known as modified equation, what is your original equation the original pd is we stared with del phi by del t is equal to minus c del phi by del x.

So, carefully you look it so after substituting the Taylor series expansion into the finite difference equation we got this equation which is known as modified equation. So, you can see that we started with the original pd del phi by del t is equal to minus c del phi by del x, so this is your original equation. Then we use some discretization scheme which is your first order upwind and we wrote the finite difference equation. Then when we substitute back the truncation error sorry Taylor series expansion into the finite difference equation we got this modified equation and this modified equation is for the discretization scheme first order upwind.

So, you can see that there are other terms are introduced so those are you can see this is this term there this is additional term so additional terms are introduced due to the finite approximation of the partial difference equation. So, this term so you can see that so dominating term of the error dominating term in the error is second term in the right hand side.

So, you can see that the dominating term is this one, so this the dominating term and which contains the second derivative of phi del 2 phi by delta x square. And this del 2 phi by del x square what is the coefficient c delta x by 2 1 minus lambda and this coefficient is known as numerical viscosity or artificial viscosity. So, this is your known as numerical viscosity or artificial viscosity artificial viscosity. So, the coefficient of 2 phi by del x square is known as artificial viscosity and this term is if you write with alpha e is equal to c delta x by 2 1minus lambda.

So, the coefficient in the second derivative this term is known as artificial viscosity or numerical viscosity and this is introduced due to the truncation error of the finite difference sorry it is introduce due to the truncation error of the specific scheme first order upwind. So, this alpha e which is known as artificial viscosity that you can see that if lambda is equal to 1 then this is 0.

So, for lambda is equal to 1you will get the exact solution, lambda is equal to 1alpha e is equal to 0 so the solution is exact, solution is exact and the effect of this artificial viscosity is to dissipate the solution. So, whatever solution you have the gradient will be dissipated, so this is the solution is exact and the gradient will be reduced and this is actually gives the dissipation error so that will discuss later.

So, first we considered first order upwind scheme and applied to this partial differential equation first order wave equation and in the process we have found that there are some additional terms which is coming due to the truncation error and the dominating error terms is the second order derivative del 2 phi by del x square and its coefficient is known as artificial viscosity. And the modified equation or this approximation is given in this equation.

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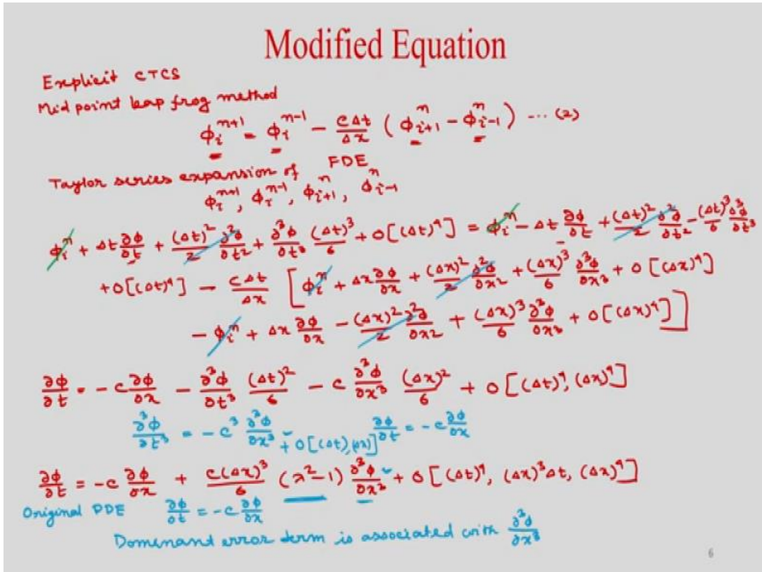

Now, let us consider central difference central space which is known as midpoint leapfrog method and let us find what is the modified equation and what is the leading error term or dominant error term. So, again we will consider explicit scheme CT Central Time Central Space which is known as midpoint leap frog method mid-point leap frog method. So, the equation will be phi i n plus 1 is equal to phi i n minus 1 minus c delta t by delta x delta x will be phi i plus 1 n minus phi i minus 1.

So, we will apply the similar procedure, so we will expand this term this term this term and this term using Taylor series expansion and we will substitute back into this finite difference equation this is your finite difference equation. Let us say this equation is 2 then if you use the Taylor series expansion, phi i n plus 1 phi i n minus 1 phi i plus 1 n and phi i minus 1 n. So, you substitute it here so you will get so first term if you write, so it will be phi i n plus del t del phi by del t plus del t square by factorial 2 del 2 phi by del t square plus del cube phi by del t cube the you will get del t cube by factorial 6 sorry factorial 3 it will be 6 plus order of del t 4.

So, this is the first term then in the right hand side first term phi i n minus 1 so similar way phi i n minus del t del phi by del t plus del t square by 2 del 2 phi by del t square minus del t cube by 6 del cube phi by del t cube and order of delta t to the power 4 minus C delta t by delta x. Then now special term.so it will be phi i n plus del x del phi by del x plus delta x square by 2 del 2 phi by del x square then plus del x cube by factorial 3 which is 6 del cube phi by del x cube. Order of accuracy is del t del x to the power 4.

And similarly minus phi i n so now you are substituting this, so minus phi i n so it will be minus minus plus so it will be delta x del phi by del x then it will be minus delta x square by 2 del 2 phi by del x square and again it will be plus delta x cube by factorial 3, so del cube phi by del x cube and plus order of del x 4. So, now you just rearrange it so you can see that from here this pi n this pi n you can cancel then this phi i n phi i n this also will get cancelled.

So, now the other terms you can see here this term also will get cancelled. So, now if you rearrange it you will get del phi by del t is equal to minus C del phi by del x minus del cube phi by del t cube del t square by 6 minus C del cube phi by del x cube del x square by factorial 3 means 6 and order of delta t 4 delta x 4. So, after rearranging you are going to get this equation. So, you can see here del phi by del t is there here del phi by del t is there so after rearrangement you will get this equation.

So, now this time derivative higher order time derivatives del 2 del cube phi by del t cube you replace it with the special derivative. And already we have written what is the expression so that if you find del cube phi by del t cube in terms of special derivative you will get del cube phi by del x cube. It is very easy to find this because you know del phi by del t is equal to minus C del phi by del x.

So, you take this derivative del cube phi by del t cube then you will get this one which is order of accuracy is delta t and delta x. So, now this time derivative in terms of special derivative you have written and this you substitute in the equation. So, you are going to get del phi by del t is equal to minus C del phi by del x plus C delta x cube by 6 lambda square minus 1 del cube phi by del x cube and order of accuracy is delta t 4 delta x cube delta t and delta x 4.

So, this is the modified equation using this midpoint leap frog method. So, we have started with the equation del phi by del t is equal to minus C by C del phi by del x, which is your original PDE. So, this is your original PDE but while doing the finite difference approximation, so there are some errors introduced and you can see the leading order term in the error is third derivative del cube phi by del x cube and its coefficient is this one.

So, obviously you can see that this will having some effect. So it is the third derivative which is your odd term and the dominant term dominant error term is associated with del cube phi by del x cube, which is your odd derivative. So, now two examples we have considered one is a first order upwind which is your first order accurate scheme and this scheme is second order accurate scheme delta t square and delta x square, so this mid-point leap frog method.

And in both the method we have shown the modified equation, so there are some additional terms are introduced due to the truncation error and the leading order term in the first order upwind scheme we have got the second derivative. And the coefficient of the second derivative is your artificial viscosity.

But when we considered this leap frog midpoint leap frog method then the leading order term is associated with the third derivative which is your odd term. So, now let us discuss about the numerical diffusion error. So, this numerical diffusion is a combined effect of dissipation error and dispersion error.

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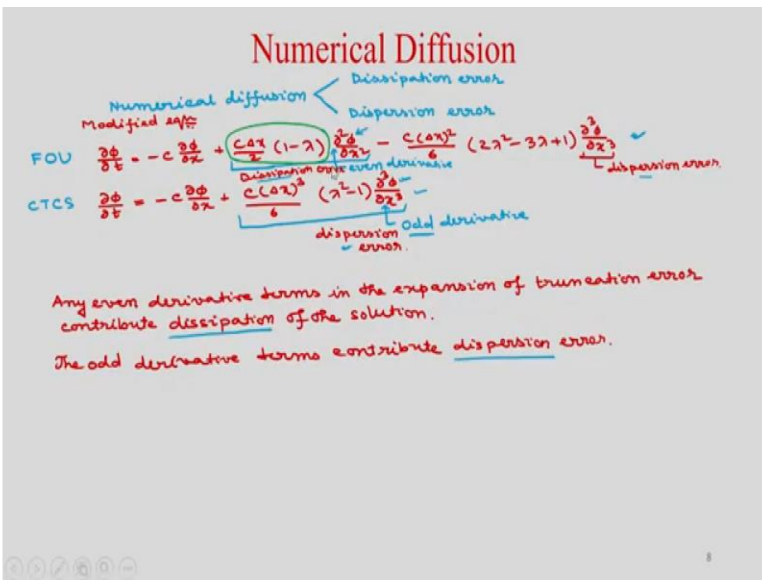

So, whatever numerical diffusion we are considering numerical diffusion, this is a combined effect of combined effect of dissipation error dissipation error and dispersion error. So, before discussing about this dissipation and dispersion error first let us write the modified equation what we derived using two different methods.

So, one is first order upwind another is your CTCS which is your midpoint leap frog method. So, if you write the equation modified equation modified equation so it is del phi by del t is equal to minus e del phi by del x plus C delta x by 2 1 minus lambda del 2 phi del x square minus C delta x square by 6 2 lambda square minus 3 lambda plus 1 del cube phi del x cube. And using midpoint leap frog method we have derived this modified equation del phi by del t is equal to minus C del phi by del x plus C delta x cube by 6 lambda square minus 1 del cube phi by del x cube.

So, now you can see that although the original equation is del phi by del t is equal to minus C del phi by del x but during approximation some errors are introduced and the leading order term using first order upwind scheme is associated with the even derivative del 2 phi by del x square. And leading order term in the error of leap midpoint leap frog method is associated with third derivative, which is your odd derivative.

So, you can see that this is your leading order term in the error and this is your leading order term in the error. And it is associated with you can see that it is even derivative, second order second derivative and this is your third derivative which is your odd derivative. So, this even derivative actually contributes in the dissipation error this term the leading order term will contribute in the dissipation error dissipation error. So, any even derivative will contribute in the dissipation error and any even odd derivative will contribute in the dispersion error.

And here it is odd derivative so it will also contribute in the dispersion error. So, we can write any even derivative any even derivative terms in the expansion of truncation error expansion of truncation error contribute dissipation error, dissipation of the solution and the odd derivative odd derivative terms contribute dispersion error.

So, this is dissipation and this is your dispersion. So, from the modified equation you can see that this leading order term is even derivative so you will get dissipation error but the other term is associated with the third derivative so it will be dispersion error and when we use the modified equation for the midpoint leap frog method the leading term is associated with the even term so you will get sorry associated with the odd term so you will get dispersion error.

And combinely this dissipation error and dispersion error are known as numerical diffusion. So, you can see in this equation the numerical viscosity or artificial viscosity is only associated with the second derivative term. The coefficient associated with the second derivative is known as artificial viscosity or numerical viscosity but if you have the other higher order even derivative but that coefficient is not known as artificial viscosity only the coefficient of second derivative is known as artificial viscosity.

But the other higher order terms which are even that will contribute in the dissipation error, those will actually contribute in the dissipation error but the coefficient will not be artificial viscosity only the coefficient associated with the second derivative del 2 phi by delta x square is known as artificial viscosity and it will contribute to the dissipation error.

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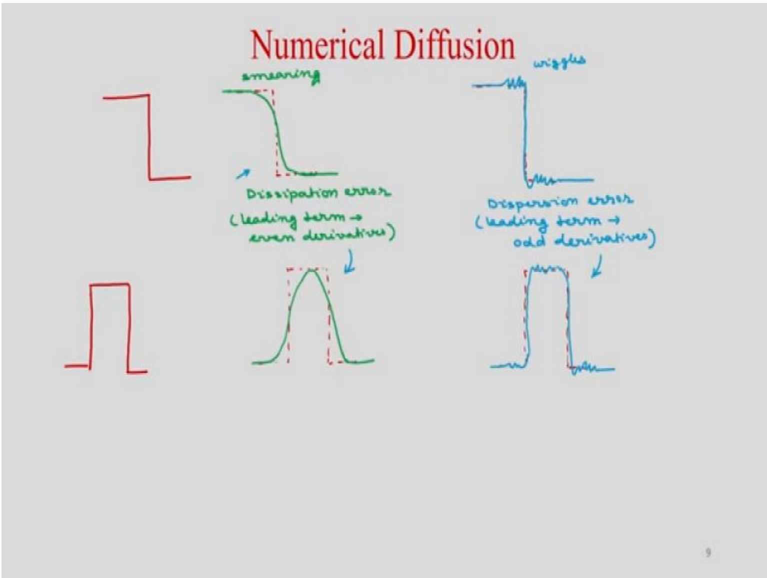

So, let us say you have any vibe and exact solution is exact solution looks like this. So, this is your one vibe so the exact solution looks like this, so if you use some discretization scheme where the leading term is associated with the even term so it will dissipative in nature. So, you will get the solution, so if this is your exact solution but you will get the solution like. So, you can see that the this solution is dissipated.

So, this is known as dissipation error dissipation error and leading term is even derivatives. So, this is known as smearing. Now, if your leading order term is odd then you will get dispersion error and the solution will look like is exact solution is this then you will get the solution some oscillation. These are known as wiggles. So, you will get solution like this and this is your dispersion error dispersion error and these are known as wiggles and leading term is associated with the odd derivatives.

So, if we use leap frog midpoint leap frog method then you will get solution like this and if you use first order upwind scheme then you will get the solution like this in the and you will get the dissipation error. And similarly if you take another wave like this which is your exact solution. But if you get dissipation error your solution may look like, so this is your exact solution, and this is your exact solution, but if you get dissipation error your solution will be like this.

So, this is your dissipation error. And dispersion error you will get, so you will get solution like this. So, this is your dispersion error. And this is your dispersion dissipation error. So, today will we have derived the modified equation for two different methods, one is first order upwind and leap frog midpoint leap frog method. And we have shown that the leading order term in the error of this first order upwind scheme is associated with the second derivative and its coefficient is known as artificial viscosity.

And these even derivative always contribute in the dissipation error. Then we considered in the modified equation of this leap midpoint leap frog method the leading order term is associated with odd derivative and this odd derivative contributes in the dispersion error. And this combined effect of dissipation error and dispersion error is known as numerical diffusion. Thank you.