Computational Fluid Dynamics for Incompressible Flows
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Lecture 3
Von-Neumann stability Analysis of Different Schemes for Hyperbolic equations

Hello everyone, so we are actually studying the stability analysis using von Neumann stability method. In last two lectures we considered one dimensional unsteady parabolic equation and we have used different schemes and we have shown the stability criteria using von Neumann stability analysis. In today's lecture, we will consider hyperbolic equation. So, we will consider first one dimensional wave equation with a constant speed wave speed c.

So, this equation already you have discretize using different methods, but today we will use von Neumann stability analysis to find the stability criteria.

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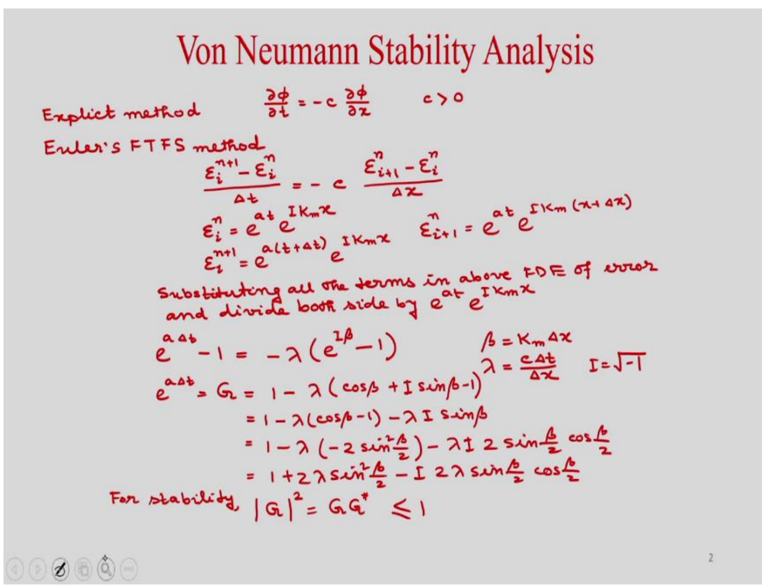

So, let us consider the governing equation 1 dimensional wave equation. So, del phi by del t is equal to minus c del phi by del x and here we consider c greater than 0 c is positive and c is the wave speed. And obviously c is known so, it is a linear equation and for that now, first let us take explicit method and Oilers forward time, forward space will take.

So, we are considering explicit methods first explicit method. So, we will consider first Oilers FTFS Forward Time and Forward Space. So, after discretizing, the finite difference equation, whatever will get the error will satisfy that equation, so, we will just write the finite difference equation for the error.

So, if you use Forward Time Forward Space and for the error if you write this finite difference equation, then it will be epsilon i n plus 1 minus epsilon i n divided by delta t. So, this is your forward time minus c and forwards space. So, epsilon i plus 1 minus epsilon i and as it is explicit so it is n divided by delta x.

So, this is the discretize equation and let us take that error epsilon i n e to the power at e to the power IKmx. Similarly, you can write epsilon i n plus 1 so, it will be e to the power a t plus delta t this is n plus 1 so, t plus delta t e to the power IKmx. And similarly you can write epsilon i plus 1 so, in spatial direction it is i plus 1.

So, it will be x plus delta x so, it will be e to the power at e to the power Ikm x plus delta x. So, you substitute all these and divided by epsilon at e to the power IKmx then substituting, substituting all the terms in above finite difference equation of error and divide both side by e to the power at e to the power IKmx, so after rearranging what we will get, we will get e to the power a delta t minus 1 is equal to minus lambda e to the power I beta minus 1.

So, where beta is nothing but Km into delta x and lambda is your c delta t by delta x. So, after rearrangement you will get this equation and you can write now e to the power a delta t which is nothing but your amplification factor. So, it is equal to G is 1 minus lambda and e to the power I beta in trigonometric function if you write what you will write cos beta plus I sine beta minus 1.

So, now, the imaginary part and real particular write separately. So, you can write is equal to 1 minus lambda cos beta minus 1 this is your real part and imaginary part is minus lambda I sin beta and I is nothing but root minus 1. So, minus lambda I sin beta, so what is cos beta minus 1 cos beta minus 1 is nothing but minus 2 sine square beta by 2. So, you can write 1 minus lambda cos beta minus 1 is 2 sine square beta by 2 minus lambda I 2 sine beta by 2 cos beta by 2.

So, then you can write 1 plus twice lambda sine square beta by 2 and minus I 2 lambda sine beta by 2, cos beta by 2 so you can see that first 2 terms is the real part and the last part is your imaginary part. So, what we will do now we will write the stability condition. So, stability condition is for stability your we can write that modulus of G square which is nothing but G and complex conjugate G star should be less than equal to 1. So, this should be less than equal to 1.

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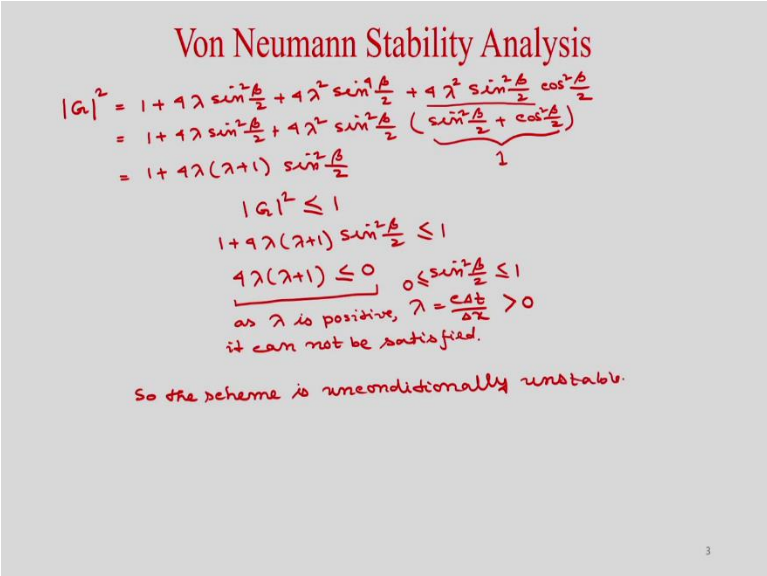

So, if you multiply G and G star, G star is your complex conjugate, then what you will you are going to get you are going to get mode G square is your 1 plus 4 lambda sine square beta by 2. So, it is real part square plus 4 lambda square sine to the power 4 beta by 2, so this is real part square plus imaginary part square so 4 lambda square, so I square will be minus 1.

So, it will be minus minus plus we can write 4 lambda square sine square beta by 2, cos square beta by 2 so, what we have done. So, modulus of G square we have written as real part square plus imaginary part square. So, you have written now, rearrange it so, what you will get, so 1 plus 4 lambda sine square beta by 2, here you just take common 4 lambda square sine square beta by 2 you take common so what you can write 4 lambda square sine square beta by 2 if you take common here first term we will get sine square beta by 2 and here we will get cos square beta by 2.

And this is nothing but 1 right it is nothing but 1. So, you can write. So, 1 plus now you take if it is 1 then you can take 4 lambda 4 lambda, lambda plus 1 sine square beta by 2. So, now condition for stability is mode G square should be less than equal to 1 that means 1 plus 4 lambda, lambda plus 1 sine square beta by 2 should be less than equal to 1 that means 4 lambda, lambda plus 1 should be less than equal to 0.

Because sine square beta by 2 maximum value is 1 right. So, it will be between 1 and 0 and lambda is c delta t by delta x. So, we have already considered c is greater than 0 that means c is positive transcript delta T is positive and that gets a delta x is positive then lambda is your positive so it is greater than 0.

So, you can see that this cannot be satisfied because as lambda is positive as lambda is positive it cannot be satisfied. So, that means this scheme is unconditionally unstable. So, the scheme is unconditionally unstable.

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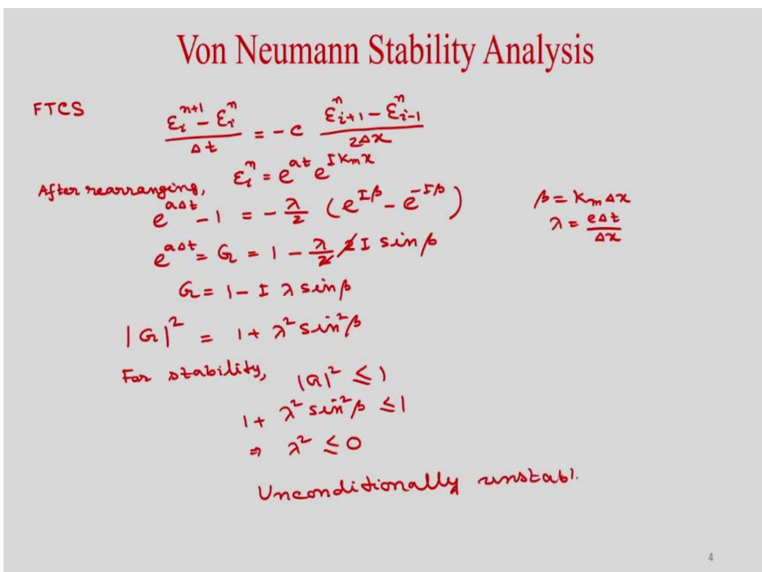

Let us consider another explicit method which is Forward Time and Central Space. Forward Time Central Space. So, it is equation will be epsilon i n plus 1 minus epsilon i n divided by delta t is equal to minus c epsilon i plus 1 minus epsilon i minus 1 and as it is explicit so, it is n divided by delta 2 delta x.

So, we have used central difference in spatial derivative and forward difference in the temporal derivative right and we have written the error finite difference we have written the finite difference equation for the error. So, after substituting epsilon i n is equal to e to the power a t e to the power Ikmx and corresponding epsilon i plus 1, epsilon i minus 1, epsilon i n plus 1 all these you write and substituted it and divide by e to the power at and e to the power IKmx then you will get e to the power a delta t after rearranging, after rearranging you will get e to the power at minus 1 is equal to minus lambda by 2 e to the power I beta plus sorry minus, minus e to the power minus I beta.

So, now you can write e to the power a delta t which is nothing but your amplification factor G is equal to 1 minus lambda by 2 and what is e to the power I beta minus e to the power minus I beta so, you know that it is 2 I sine beta so it will be 2 I sine beta so, beta we have written Km delta x and lambda is c delta t by delta x.

So, this is your G. So, now if you, you see this is your imaginary part, so it is 2, 2 will cancel out so G is 1 minus I lambda sine beta. So, mode G square you can write 1 plus lambda square sine square beta. So, for stability mode G square should be less than equal to 1 that means, lambda square 1 plus lambda square sine square beta should be less than equal to 1 that means, lambda square should be less than equal to 0.

So, you can see that it cannot be satisfied because lambda is positive because we have considered a positive wave speed c and delta t and delta x are positive so lambda is positive so it cannot be satisfied. So, it is unconditionally unstable, unconditionally unstable.

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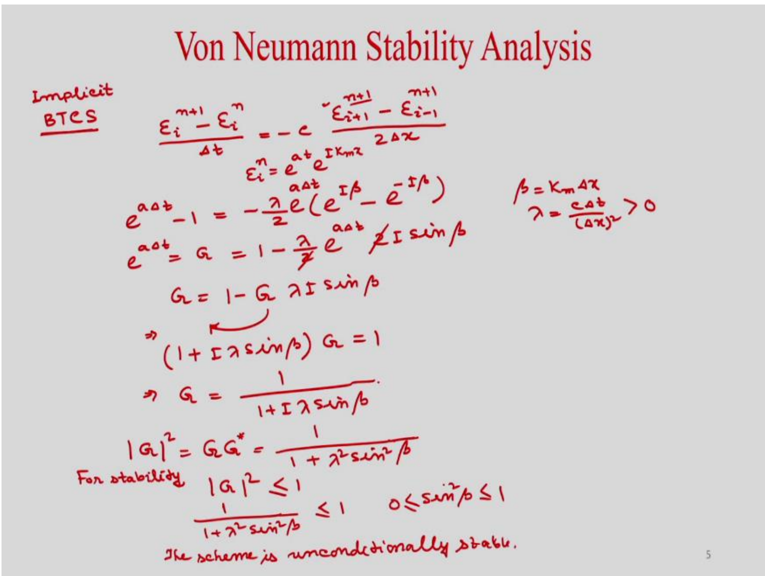

Now, let us consider the implicit BTFS. So, now consider implicit Backward Time Central Space, Backward Time Central Space. So, if you write epsilon i n plus 1 minus epsilon i n divided by delta t is equal to minus c epsilon i plus 1 n plus 1 minus epsilon i plus 1 sorry i minus 1 n plus 1 divided by 2 delta x. So, n plus 1 we have considered in the spatial derivative that means it is implicit.

So, now you substitute epsilon i n is equal to e to the power a t e to the power IKmx and corresponding epsilon i plus 1 i minus 1 at n plus 1 level and divide both sides by e to the power a t, e to the power IKmx then you will get e to the power a delta t minus 1 is equal to minus lambda by 2 e to the power I beta minus e to the power minus I beta, where n e to the power a delta t.

Because n plus 1 is there so, 1 e to the power a delta t will be there and beta is your Km delta x and lambda is your c delta t by delta x square which is greater than 0. So, now you can write e to the power a delta t which is your amplification factor is equal to sorry amplification factor is equal to 1 minus lambda by 2 e to the power a delta t and e to the power I beta minus e to the power minus I beta if you write in terms of trigonometric function then it will be minus sorry it will be 2 I sine beta.

So, it will be 2 I sine beta. So, these 2, 2 will get cancelled. So, you will get G is equal to and e to the power a delta t is also G so, it is G lambda I sine beta that means, if you take this side this if you take this side then you will get 1 plus I lambda sine beta into G is equal to 1 that means G is equal to 1 by 1 plus I lambda sine beta.

So, now if you write mode G square it is multiplication of G and the complex conjugate of it, so you will get this as 1 so it will be square so it will be plus lambda square sine square beta. So, 1 plus lambda square sine square beta so that means you will get, so mode G square for stability, mode G square should be less than equal to 1.

So, that means 1 by 1 plus lambda square sine square beta should be less than equal to 1 and sine square beta varies between 1 and 0. So, that means you can see that lambda is positive. So, in the denominator always it will be 1 plus something, so this will be always satisfied. So, this term 1 by 1 plus something will be less than equal to 1 so, it is always satisfied that means it is unconditionally stable. So, the scheme is unconditionally stable. So, Backward Time Central Space is unconditionally stable.

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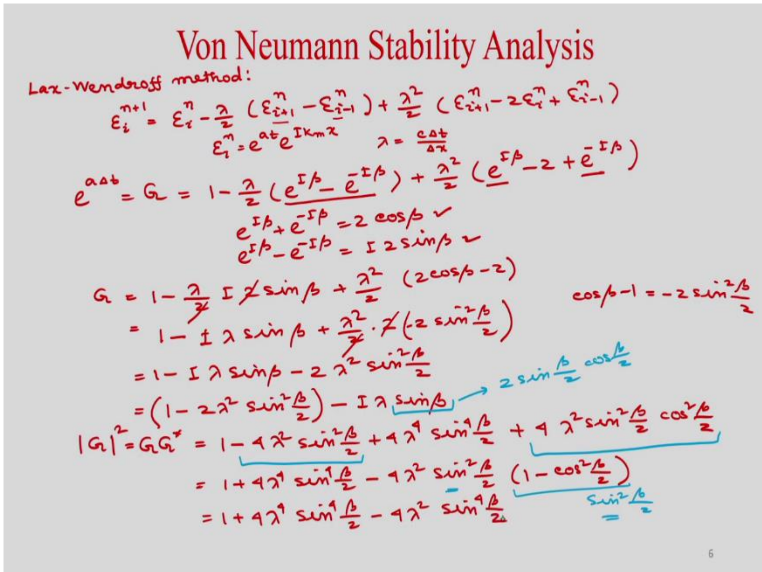

So, now we will consider lax-wendoff method and the scheme already we have discussed in earlier classes. So, I am going to write the finite difference equation for the error only. So, you can write lax-wendroff method. So, I am not going to write the derivation of the scheme just I am going to write the equation of finite difference finite difference equation for the error, so it will be epsilon i n plus 1 is equal to epsilon i n minus lambda by 2 epsilon i plus 1 n minus epsilon i minus 1 n plus lambda square by 2 epsilon i plus 1 n minus twice epsilon i n plus epsilon i minus 1 n.

So, you can refer the earlier lecture to find the finite difference equation of lax-wendroff method. So, I have written only the finite difference equation for the error now, you substitute epsilon i n is equal to e to the power a t e to the power Ikmx and corresponding epsilon i n plus 1 epsilon i plus 1, i minus 1. So, all these times you substitute and rearrange it you will get e to the power a delta t which is nothing but your amplification factor is equal to 1 minus lambda by 2, lambda is nothing but c delta t by delta x.

So, you are going to get e to the power I beta minus e to the power minus I beta plus lambda square by 2 e to the power I beta minus 2 plus e to the power minus I beta. So, now you just write e to the power I beta plus e to the power minus I beta what it is that it is twice cos beta and e to the power I beta minus e to the power minus I beta is I 2 sine beta. So, this term and this term if you can write twice cos beta and together this time you can write I 2 sine beta.

So, you can write G is equal to 1 minus lambda by 2, I 2 sine beta plus lambda square by 2 so this you can write twice cos beta minus 2 if you take common 2 outside so cos beta minus 1 so cos beta minus 1 you can write, minus 2 sine square beta by 2, so you can write so these 2, 2 will get cancelled.

So, you will write 1 minus I lambda sine beta and this is your lambda square by 2 into 2 and cos beta minus 1 will be there. So, that you can write as minus 2 sine square beta by 2, so 2 so it will be minus 2 sine square beta by 2. So these 2, 2 will get canceled. So, we can write 1 minus I lambda sine beta and it will be minus 2 lambda square sine square beta by 2.

So, now you can see it your imaginary part and the real part, real part is 1 minus twice lambda square sine square beta by 2. And imaginary part is I lambda sine beta. So, now we will write mode G square. So, modulus of G square, which is nothing but G into G star so you can write 1 minus 4 lambda square sine square beta by 2 so it is real part square plus 4 lambda to the power 4 sine to the power 4 beta by 2 plus, so here 2 is missing so 2 we have to write here.

So it will be 4 lambda square so I square will be minus 1 so it will be plus 4 lambda square. And sine beta we can write as sine, so it will be sorry so you can write plus, so you can see it is sine beta, so sine beta can write as 2 sine beta by 2 cos beta by 2. So, you can write this as 4 lambda square sine square beta by 2, cos square beta by 2, so now we will rearrange it.

So, you can see here 1 minus 4 lambda square sine square beta by 2, here you take 4. So, this term and this term you take 4 lambda square sine square beta by 2 outside, so what you can write is equal to 1 plus 4 lambda to the power 4 sine to the power 4 beta by 2, here if you take minus 4 lambda square sine square beta by 2, then it will be 1 minus cos square beta by 2, 1 minus cos square beta by 2.

And 1 minus cos square beta by 2 what you can write, you can write it is as sine square beta by 2 so, sine square this beta by 2 and sine square beta by 2 you can write sine to the power 4 beta by

2. So, you can write 1 plus 4 lambda to the power 4 sine to the power 4 beta by 2 and this will be minus 4 lambda square, sine to the power 4 by beta by 2.

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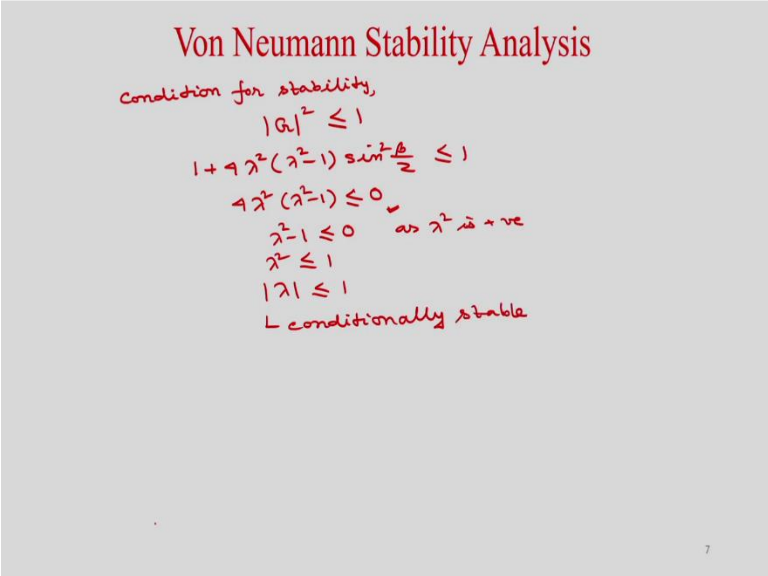

So now, what is the condition for stability, condition for stability is modulus of G square should be less than equal to 1. So, you can write 1 plus 4 lambda square if you take common then the lambda square minus 1 sine square beta by 2 should be less than equal to 1. So, that means 4 lambda square lambda square minus 1 should be less than equal to 0. So, that is the condition for stability.

So, lambda is positive. So, lambda square cannot be less than equal to 0. So, that means lambda square minus 1 should be less than equal to 0 as lambda square is positive, lambda is positive. So, lambda square should be less than equal to 1 that means mod lambda should be less than equal to 1. So, this is the condition, conditionally stable and lambda mod lambda should be less than equal to 1.

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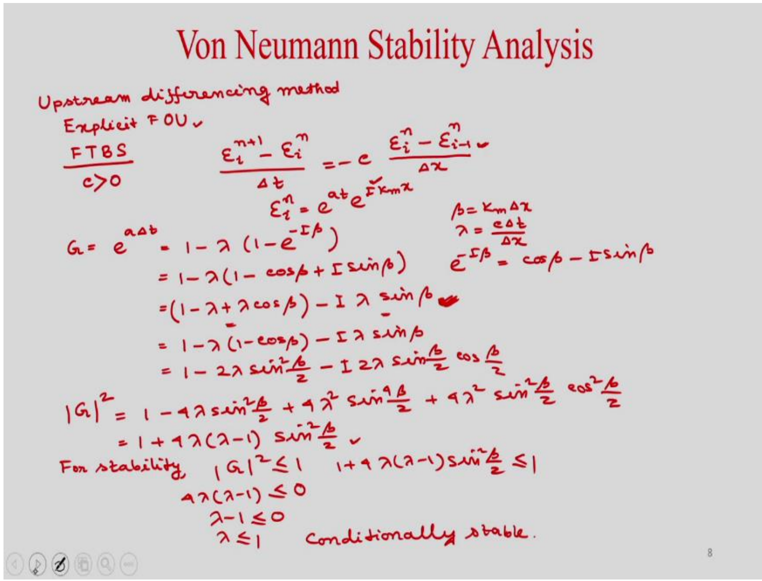

Now, let us consider these Oilers sorry explicit first error points scheme So, that means, upstream differencing method, so this is your known as explicit FOU, so that means it is Forward Time and Backward Space you are using Forward Time Backward Space we are using for the wave speed c greater than 0, c greater than 0.

So that is known as explicit first error point. So, you can write the error equation epsilon i n plus 1 minus epsilon i n divided by delta t is equal to minus c, now it is backward space so, it will be epsilon i n explicit So, epsilon i minus 1 n divided by delta x so, we have written for constant wave speed c which is greater than 0.

So, constant positive wave speed, so this we have written so now you substitute the error so if you write it so, you are going to get e to the power a delta t after rearranging you will get 1 minus lambda 1 minus e to the power minus I beta, beta is Km into delta x and this is coming from I minus 1, so e to the power minus I beta and lambda is your c delta t by del x and what is e to the power minus I beta, e to the power minus I beta is nothing but your cos beta minus I sin beta.

In terms of trigonometric function, so, this you can write 1 minus lambda 1 minus cos beta plus I sine beta. So, now real part and imaginary part you separate it out So, you will get 1 minus lambda 1 minus lambda plus lambda cos beta right minus minus plus this is your real part and imaginary part is minus I lambda cos beta minus I lambda sorry sine beta minus I lambda sine beta.

So, this is your real part and this is your imaginary part. So, you can write now, 1 minus lambda 1 minus cos beta 1 minus cos beta minus I lambda sine beta. So, 1 minus cos beta. 1 minus cos beta, so you can write 2 sine square beta by 2. So, 1 minus cos beta you can write 2 sine square beta by 2. So, it will be 1 minus 2 lambda sine square beta by 2 and this you can write minus 2 lambda sine beta by 2 cos beta by 2.

So, similarly you can write so this is nothing but your amplification factor so G modulus G square you can write now 1 minus 4 lambda sine square beta by 2 plus 4 lambda square sine to the power 4 beta by 2 plus 4 lambda square sine square beta by 2, cos square beta by 2. Similarly, you rearrange it and you will get sine square beta by 2 plus cos square beta by 2 is equal to 1 after taking common.

So, these if you rearrange it, finally we will get 1 plus 4 lambda, lambda minus 1 sine square beta by 2, I did not write the in between steps, so you can do it and after rearranging you will get 1 plus 4 lambda, lambda minus 1 sine square beta by 2. So, for stability this should be less than equal to 1 mod G square should be less than equal to 1. So, it will be 1 plus 4 lambda, lambda minus 1 sine square beta by 2 should be less than equal to 1.

So, that means, you 4 lambda, lambda minus 1 should be less than equal to 0. So, you can see that lambda is positive so, lambda cannot be less than 0, so lambda minus 1 will be less than equal to 0 that means lambda should be less than equal to 1. So, this is conditionally stable and condition for stability is lambda minus lambda should be less than equal to 1 so it is conditionally stable.

So, now we will use graphical method where we will use the real part and imaginary part and this will represent this G in a, in a complex plane. And we will try to find graphically what is the condition for stability. So, already you can see that we have written real part and imaginary part. So, let us write these and start from here.

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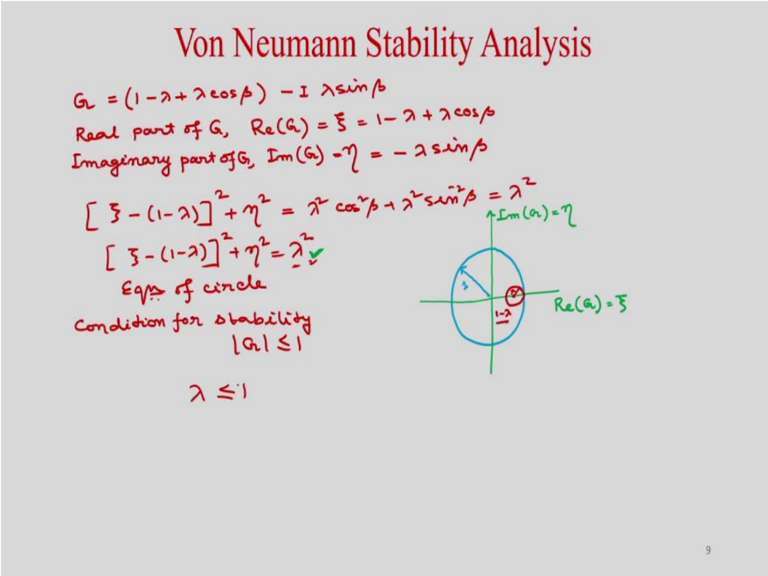

So, your G is real part is your 1 minus lambda plus lambda cos beta, so, this is your real part and imaginary part is minus I lambda, I lambda sine beta, so this is your real part so now we will represent real part as real part of G. We will repeat that real part of G is equal to zeta is equal to 1 minus lambda plus lambda cos beta and imaginary part.

Part of G is imaginary G is equal to eta and it is your minus lambda sine beta. So, now we will plot this G in the complex plane and we will find the stability criteria graphically. Now, from here you can see that zeta minus 1 minus lambda whole square plus eta square what is that? So, it will be lambda square so it is 1 minus lambda we have subtracted so it will be lambda square cos square beta plus lambda square sine square beta is equal to lambda square.

So, that means, you are going to get 1 equation of circle whose center is 1 minus lambda 0 and radius is lambda square, so you can see this is the equation of circle equation of circle x minus a whole square plus y minus b whole square is equal to r square so, that means your the coordinate or center of this circle is 1 minus lambda and 0 and the radius is lambda so if you draw and for conditional for stability is mode G should be less than equal to 1.

So, if you draw a unit circle. So, if this is your 1 so, in the complex plane. So, this is your real G x axis or it is your eta zeta and y axis is your imaginary G is eta. So, now for condition for stability that your G should be in this circle. So, if you draw this circle, so what we will get, you see, its center will be 1 minus lambda, 1 minus lambda distance and its radius will be lambda.

So, its radius is lambda its radius is lambda. So, this is the circle. So, you can see. So, what will be the value of lambda from this a relation say for stability your this G should be less than equal to 1. So, that means in this unit circle, it should lie. So, now when we have drawn this circle, this is the circle and it is a distance of this center is 1 minus lambda and its radius is lambda.

So, to satisfy this condition it should not go beyond 1 your lambda should be less than equal to 1 then only it will be satisfied so from graphically you can see that lambda should be less than equal to 1, lambda should be less than equal to 1.

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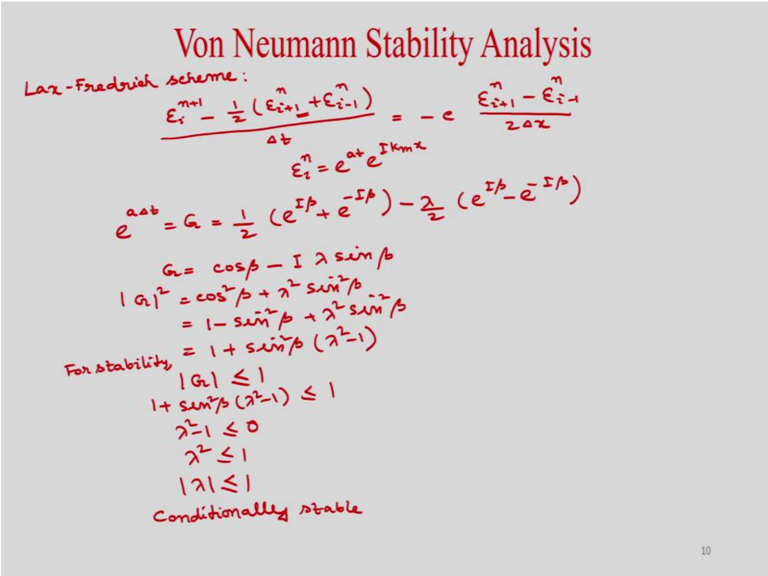

So, now we will consider another scheme Lax- Fredrich scheme. So, we are going to write first the finite difference equation for the error. So, this will be epsilon i n plus 1 minus so it will be average of epsilon i plus 1 n minus epsilon plus epsilon i minus 1 n divided by delta t is equal to minus c epsilon i plus 1 n minus epsilon i minus 1 n divided by 2 delta x.

So, similarly substitute the error epsilon i n is equal to e to the power a t, e to the power Ikmx and rearrange it. So, finally you will you are going to get e to the power a delta t which is nothing but your amplification factor is equal to half into e to the power I beta plus e to the power minus I beta so, this term is coming from here and here we will get minus lambda by 2 e to the power I beta minus e to the power minus I beta.

So, this now what you can write G is equal to so half e to the power I beta plus e to the power minus I beta it is nothing but cos beta and minus it will be lambda by 2. So, it will be e to the power I beta minus e to the power minus I beta so, it will be I sine beta so, it will be I lambda sine beta. So, now you can write mod G square is equal to cos square beta plus lambda square sine square beta.

So, cos square beta you can write as 1 minus sine square beta plus lambda square sine square beta and this you can write 1 plus sine square beta lambda square minus 1. So, for stability mode G should be less than equal to 1 so that means 1 plus sine square so, for stability sine square beta lambda square minus 1 should be less than equal to 1.

And you can see that sine square beta by sine square beta will be maximum value 1 between 0 and 1 so maximum value 1. So, you can write lambda square minus 1 should be less than equal to 0 that means lambda square should be less than equal to 1 that means mode lambda should be less than equal to 1. So, the lax- Fredrich scheme is your conditionally stable scheme and condition for stability is modulus of lambda should be less than equal to 1 so it is conditionally stable.

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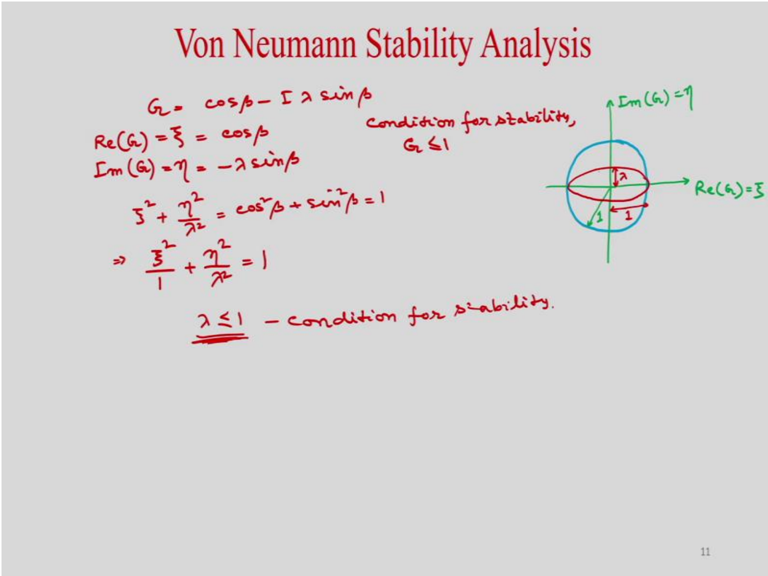

Similarly, if you use the complex plans for G and try to find the stability criteria, then you can write your G is we have already got imaginary part and real part. So, real part is your cos square beta and imaginary part is G is equal to cos beta minus I lambda sine beta. So, if you see your real part real part is zeta is equal to cos beta and imaginary part is eta is equal to minus lambda sine beta.

So, now condition for stability is G should be less than equal to 1 so, that means if you draw a unit circle so in the complex plane. So, it is unit circle of G and this your x axis is your real part. And y axis is your imaginary part it is your eta, this is your zeta and this is the unit circle. So, this is your 1. So, if you see so it will if you write eta square zeta square plus eta square by lambda square, what you can write.

So, the eta square is cos square beta and eta square by lambda square it will be sine square beta. So, that means it is 1 so, that means, it will be eta square by 1 plus eta square by lambda square is equal to 1. So, what is this equation, so this equation of ellipse equation of ellipse. So, now you can draw the ellipse. So, how will draw the ellipse so, it will be.

So, this is your 1 and this is your lambda. So, you can see from graphically that lambda obviously should be less than equal to 1 then only your, this will be satisfied. So, lambda should be mod lambda should be or lambda should be less than equal to 1, lambda should be less than equal to and those lambda is anyway positive so lambda should be less than equal to 1.

Because this lambda cannot be more than 1 because then it will go outside this circle because it is the unit circle. So, for stability your lambda values should lie inside this unit circle, so lambda value cannot be greater than 1. So, from the graphical you can see that lambda should be less than equal to 1, so this is condition for stability.

So, today we have used 1d wave equation which is your hyperbolic in nature and we use different schemes explicit and implicit and we have shown the stability criteria for those schemes. Although these we discussed in earlier lectures where we discretize that equation, but we did not show the stability criteria, we did not discuss how it has come.

So, now we have used von- Neumann stability analysis and you apply two different discretization methods and we have shown the stability criteria. So, for other schemes also similarly you can use this von- Neumann stability analysis and you can find the stability criteria. Thank you.