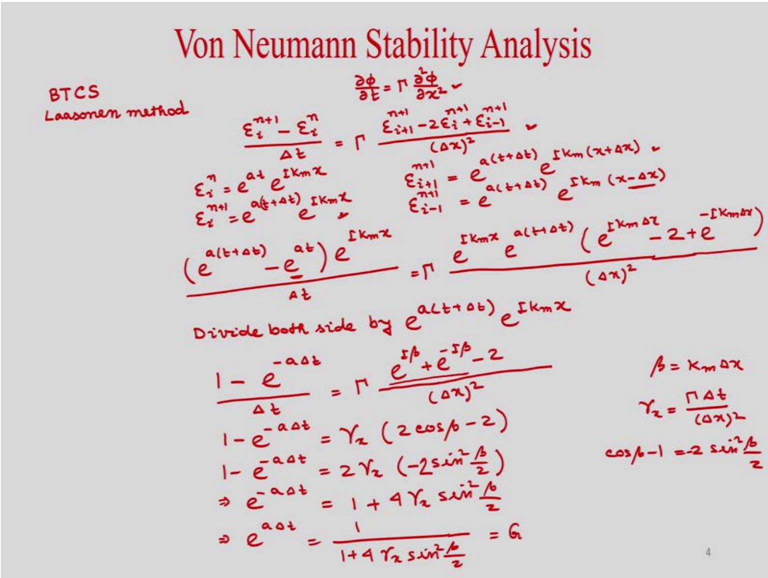
Computational Fluid Dynamics for Incompressible Flows
Professor Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati
Lecture 2
Von-Neumann stability Analysis of Different Schemes for Parabolic equations

Hello everyone, so today we will continue with the last lecture. So, in this lecture also we will study the von Neumann stability analysis of different schemes for parabolic equations. So, in last class we considered only the FTCS scheme Forward Time and Central Space, but today we will consider some other explicit and implicit schemes.

(Refer Slide Time: 00:54)



So, first let us consider BTCS backward time central space. So, we will considering Backward Time central space okay which is known as also laasonen method, laasonen method. Obviously we are considering one dimensional unsteady diffusion equation. So, the governing equation is del phi by del t is equal to gamma diffusion coefficient del 2 phi by del x square.

So, now we have already shown in the last lecture that the finite difference equation of this equation. The error also will satisfy this finite difference equation so, that means we can write finite difference equation so, it is epsilon i n plus 1 minus epsilon i n minus 1 because we are using laasonen method so it is epsilon i n divided by delta t.

So, this is your temporal term, so backward time with gamma epsilon i plus 1 minus 2 epsilon i plus epsilon i minus 1 divided by delta x square and what will be the time level it is n plus 1. So, now we know that epsilon i n is e to the power a t, e to the power IKmx. So, similarly you can write for epsilon i n plus 1 is equal to e to the power a t. So, plus delta t because n plus 1 and e to the power IKmx, now for epsilon i plus 1 n plus 1.

So, we can write e to the power a, so n plus 1, so t plus delta t and i plus 1 so, it will be e to the power IKm x plus delta x, because it is i plus 1. So, epsilon i minus 1 n plus 1 similarly, we can write a e to the power a t plus delta t into e to the power IKm x plus sorry x minus delta x plus i minus 1 so, it will be x minus delta x.

So, now you substitute this in the finite difference equation of this error. So, if you substitute then you can get e to the power a t plus delta t minus e to the power a t, e to the power IKmx divided by delta t, I have just taken common e to the power IKmx is equal to gamma. So, again if you take e to the power IKmx common then you can write and also you can write and take the common e to the power et plus delta t okay then you will get.

So, e to the power i, so e to the power IKmx already you have taken common So, it will be e to the power IKm delta x then minus, minus 2. So, e to the power I n plus 1 so, this we have taken common so, minus 2 plus e to the power minus IKm delta x okay because here minus delta x is there, so, it is e to the power minus IKm delta x divided by delta x square after substituting it, we got this expression.

Now you divide both sides with e to the power IKmx, e to the power a t plus delta t. So, divide both side by a to the power a t plus delta t, e to the power IKmx. So, if you divide both sides then you will get so, from first term you will get 1 minus e to the power minus a delta t because e to the power a t.

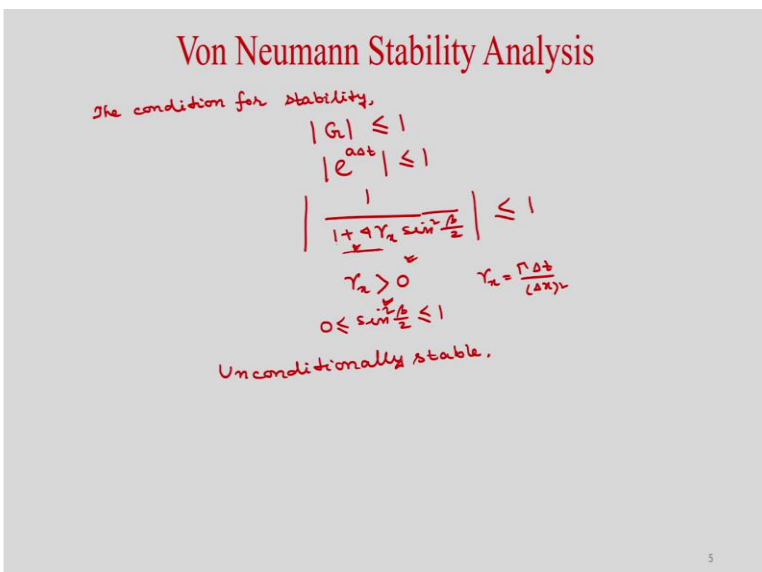
So, you have divided by e to power a t plus delta t, so, it will be e to the power minus a delta t divided by delta t is equal to gamma. So, this it will be e to the power i. So, you can write beta is equal to, beta is equal to Km delta x. So, you can write e to the power I beta plus e to the power minus I beta minus 2 divided by del x square. So, now you can write 1 minus e to the power minus a delta t is equal to gamma x where gamma x is gamma into delta t by del x square.

And now this term e to the power I beta plus e to the power minus I beta what you can write 2 cos beta so, you can write 2 cos beta minus 2. So, you can again write 1 minus e to the power minus a delta t is equal to, so if you take 2 common then 2 gamma x and cos beta minus 1 cos beta minus 1 it will be minus sin square beta by 2 cos beta minus 1 it is minus sin square beta by 2.

So, hence you can write e to the power minus a delta t is equal to 1 so this if you take this side, this minus will become plus so, it will be sorry it will be cos beta minus 1 it will be 2 sin square beta by 2, 2 sin square beta by 2 so, cos beta minus 1 will be minus 2 sin square beta by 2. So it will be 1 plus 4 gamma x sin square beta by 2.

So, this you can write as e to the power a delta t is equal to 1 by 1 plus 4 gamma x sine square beta by 2. So, the amplification factor now, we have found because this is your G right because e to the power a delta t is equal to G. So, this is 1 by 1 plus 4 gamma x sin square beta by 2.

(Refer Slide Time: 08:33)



So, condition for stability, the condition for stability your mod G should be less than equal to 1 or e to the power a delta t should be less than equal to 1, okay so that means 1 by 1 plus 4 gamma x sin square beta by 2 should be less than equal to 1. So, now you carefully look this expression gamma x, gamma x always positive because gamma x is your gamma delta t by delta x square.

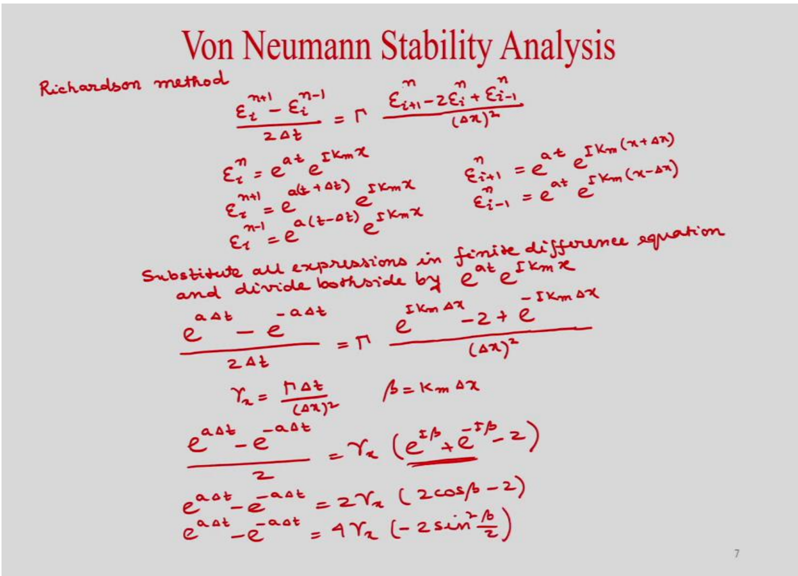
So, all are positive so gamma x always positive sin square beta by 2 what is the value sin square beta by 2 will be between 0 and 1. So, it will be between 0 and 1. So, you can see if it is 0 so, it will be 1 and if it is 1 and it is positive so, 1 plus something is greater than 1 in the denominator. So, 1 by that will be always less than or equal to 1 so, that means it is unconditionally stable.

Always it will be satisfied because gamma x is positive sin square beta by 2 is positive 0 to 1. So, and it lies in the denominator 1 plus 4 gamma x sin square beta by 2. So, it is always 1 plus something some value will come. So, obviously, it is in the denominator. So, 1 by this term always will be less than 1. So, it is unconditionally stable.

So, already that we have discussed while discretizing this equation using BDCS method we told that that it is unconditionally stable. So, your, this is your implicit scheme and this implicit scheme is unconditionally stable. So, this is your unconditionally stable. So, obviously you can see that there is no restriction to choose the delta t.

So, you can take as I, you can take a higher delta t but the problem will be there that there is a practical limit on this time step due to the truncation error, because if you increase the delta t your truncation error also will increase. So, there is a practical limit, but you can choose higher delta t in case of implicit scheme.

(Refer Slide Time: 11:19)



So, next let us choose Richardson method. Now, let us write the finite difference equation of the error. So, that will be epsilon i n plus 1 minus epsilon i n minus 1 divided by so, it is difference between n plus 1 and n minus 1 so, it will be 2 delta t is equal to gamma epsilon i plus 1 minus twice epsilon i plus epsilon i minus 1 n, n, n and delta x square, so, these already you have learned this Richardson method so, we have just written the finite difference equation for the error and you substitute the error now. So, we know epsilon i n is equal to e to the power at e to the power IKm x.

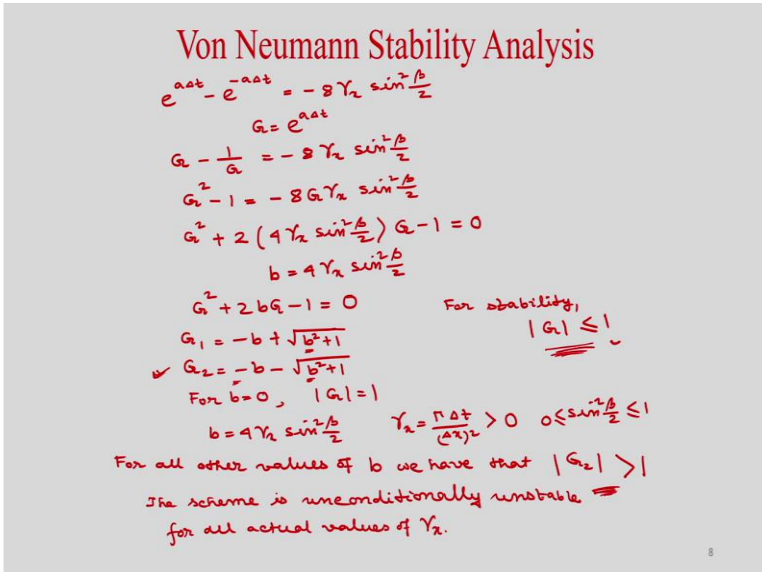
So, similarly, you can write epsilon i n plus 1 is equal to e to the power at plus delta t, e to the power IKmx and epsilon i n minus 1 now, so e to the power at plus minus delta t e to the power IKmx similarly, for epsilon i plus 1 n you can write e to the power at, e to the power Ikmx plus delta x and epsilon i minus 1 n is e to the power at e to the power IKmx minus delta x.

So, all these expression you substitute in the finite difference equation and divided by e to the power at, e to the power IKmx. So, substitute all expressions in finite difference equation, equation and divide both sides by e to the power at, e to the power IKmx. So, now I am going to write the final expression. So, you are going to get e to the power a delta t because it is n plus 1 minus e to the power minus a delta t because n minus 1 divided by 2 delta t is equal to gamma.

So, this you are going to get e to the power Ikm delta x minus 2 plus e to the power minus IKm delta x divided by delta x square. So, now you write gamma x is equal to gamma delta t by delta x square and beta is equal to Km delta x. So, you can write e to the power a delta t minus e to the power minus a delta t divided by 2 is equal to gamma x. So, this you can right now e to the power i beta plus e to the power minus i beta minus 2.

So, now e to the power i beta plus e to the power minus i beta will be your 2 cos beta so, you can write e to the power a delta t minus e to the power minus a delta t is equal to gamma x then this you are writing 2 cos beta minus 2. So, now, cos beta minus 1 is equal to minus 2 sin square beta by 2. So, you can write e to the for a delta t minus e to the power minus a delta t. So, if you take 2 common it will be 4 gamma x and cos beta minus 1 will be minus 2 sin square beta by 2.

(Refer Slide Time: 15:53)



So, next we just write e to the power a delta t minus e to the power minus a delta t is equal to minus 8 gamma x sin square beta by 2. What is e to the power a delta t, e to the power a delta t is nothing but your amplification factor because G is equal to e to the power a delta t. So, you can write G and e to the power minus a delta t is 1 by G is equal to minus 8 gamma x sin square beta by 2.

So, you can write G square minus 1 is equal to minus 8 G gamma x sin square beta by 2 or you can write G square. You take it in the left hand side so it will be plus 2 into 4 gamma x sin square beta by 2 G minus 1 is equal to 0. So, let us write small b is equal to 4 gamma x sin square beta by 2, then you can write G square plus twice bG minus 1 is equal to 0.

So now you can find okay from this quadratic equation you can find what is the value of G So, G 1 will be your minus b plus root b square minus 1 so, you care minus 1 is there So, b square plus 1 okay and G 2 will be minus b minus root b square plus 1 so now, let us take so, for stability okay for stability mod G should be less than equal to 1 so that let us check.

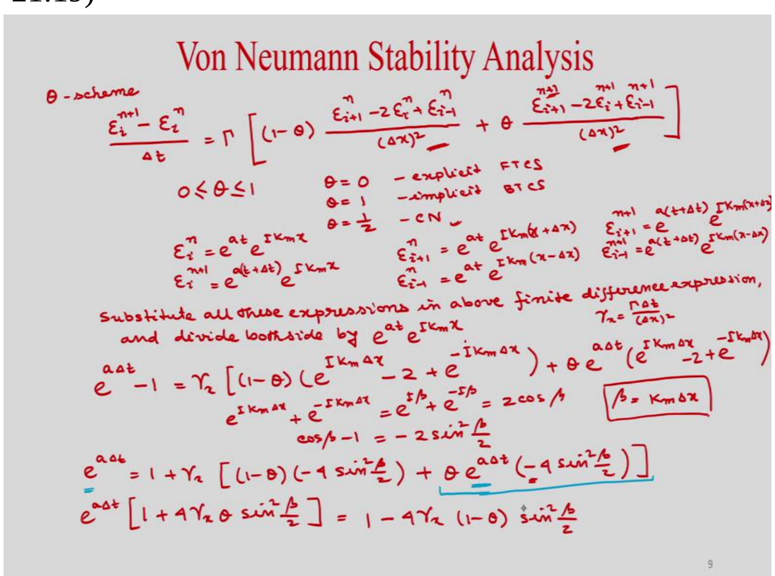
So now if 4 b is equal to 0 so if b is equal to 0 so you can see so for b is equal to 0, G will be just plus minus 1 because if you put 4 B is equal to 0, so mode G will be 1 so we know it is satisfied. Now, you see for all other values of b, so b is your 4 gamma x sin square beta by 2, gamma x is gamma delta t by delta x square so it is always greater than 0, so greater than 0 always it is greater than 0.

So, it is positive okay and sin square beta by beta square beta by 2 also it lies between 0 and 1. So, you can see b is always positive and it is 4 times gamma x sin square beta by 2. So, it is more than. So, when you are writing in the G so, it will be b square plus 1 so, it will be b square plus 1 and this is a positive term this so, b square term is positive term so positive 1 plus something it will be always positive term.

So, it is always positive term so, when you G 2 if you see, so G 2 will be negative B, B is positive, minus, again this is your positive so it will be negative so, for all other values of, for all other values of b we have that G 2 mode will be greater than 1 okay always it will be greater than 1 because you can see here.

So it will be always greater than 1, so 1 plus b square so, it will be always greater than 1 and another b is there. So, mode G 2 will we always get then 1 that means you have this game is unconditionally unstable. So, this is your unconditionally unstable. So, the scheme is unconditionally unstable for all actual values of the gamma x. Because for stability mode G should be less than or equal to 1 but you are getting G 2 always greater than 1 so it is unconditionally unstable.

(Refer Slide Time: 21:19)



So, next let us consider theta scheme. So, it is combination of these explicit and implicit and putting a value of theta we can get implicit, explicit or Crank Nicolson. So, that scheme let us write so, it is your theta method or theta scheme maybe we have learned it is as a beta scheme but we are writing theta because beta is we are writing as k into delta x theta scheme.

So, these theta scheme, In earlier lecture you might have learned as beta scheme, but as here we have we are using beta is equal to k delta x so, we are writing theta scheme. So, for theta scheme for the error if you write the finite difference equation so, it will be epsilon i n plus 1 minus epsilon i n divided by delta t is equal to gamma 1 minus theta.

So, 1 term will be your explicit so, epsilon i plus 1 n minus twice epsilon i n plus epsilon i minus 1 n divided by delta x square plus theta epsilon i plus 1 minus twice epsilon i plus epsilon i minus 1 so, it will be implicit so n plus 1, n plus ,1 n plus 1 divided by delta x square okay so if theta lies between 0 and 1, for theta is equal to 0, theta is equal to 0 it is explicit already we have learned explicit scheme and theta is equal to 1 it is implicit scheme.

So, it is actually your explicit means forward time central space implicit means BDCS Backward Time Central Space and theta is equal to half means it is Crank Nicolson it is Crank Nicolson. So, now let us substitute the error. So, epsilon i n is equal to e to the power at, e to the power IKmx. Similarly epsilon i n plus 1 is equal to e to the power at plus delta t e to the power Ikmx x. And for other you can write epsilon i plus 1 in epsilon i minus 1 n. So, it will be just e to the power at, e to the power IKmx and e to the power at e to the power IKm sorry it is x plus delta x because i plus 1 and this is your i minus 1. So, x minus delta x. Similarly for epsilon i plus 1 and n plus 1. So, it will be e to the power at plus delta t okay because n plus 1 and i plus 1, so, we did the ikmx plus delta x similarly, epsilon i minus 1 n plus 1 you can write e to the power at plus delta t and e to the power ikmx minus delta x.

Again all these expression you put in a finite difference equation. So, you will get finally, if you put all these values and substitute all these expressions in above finite difference expression and divide both side by e to the power at e to the power ikmx, so final expression I am going to write. So, you will get e to the power a delta t minus 1 is equal to gamma x 1 minus theta e to the power ikm delta x minus 2 plus e to the power minus i k m delta x.

So, it is for explicit this term and now right for these term plus theta plus theta e to the power a delta t will be there a delta t because n plus 1 is there. So, 1 e to the power a delta t will be there and now e to the power i k m delta x minus 2 plus e to the power minus i k m delta x. So, we have written the final expression where gamma x is equal to gamma delta t by delta x square.

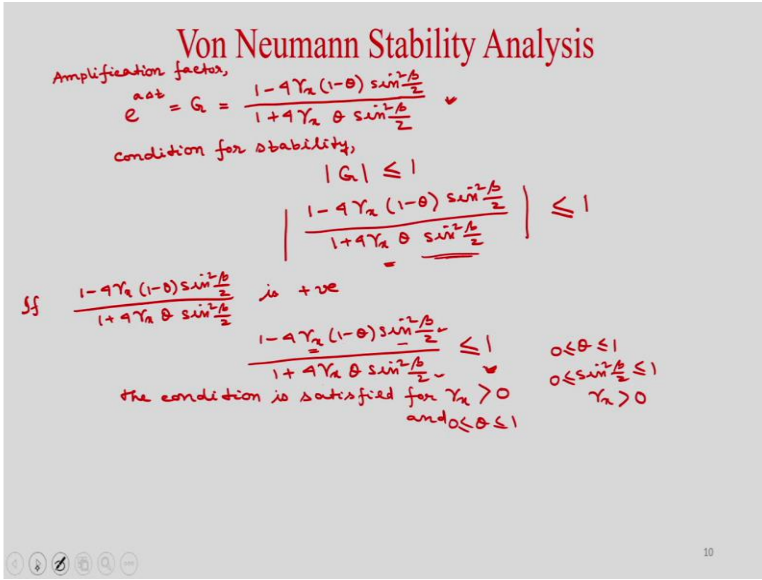
So, gamma x is 0 gamma delta t by delta x square now, you can write e to the power i k m delta x plus e to the power minus i k m delta x as e to the power i beta plus e to the power minus i beta and you can write as 2 cos square sorry 2 cos beta 2 cos beta where beta is your Km delta x beta is Km delta x and cos beta minus 1 again you write minus 2 sin square beta by 2.

So, if you substitute all these expression then you will get, so e to the power a delta t minus 1 you take in the right hand side it will be 1 plus gamma x it will be 1 minus theta, so, there will be 2 common and 2 and 2 minus 2 sin square theta by 2 so, it will be minus 4 sin square beta by 2 plus so, it will be e to the power theta, e to the power a delta t.

And again here minus 2 take common and cos beta minus 1 and that will be minus 2 sin square beta by 2 so it will be minus 4 sin square beta by 2 okay. So, now you can see that so, this term okay this term containing e a delta t you take in the left hand side okay then this e a delta t you can take common, so, you can write e to the power a delta t.

Then this will be 1 and this minus will become plus so, it will be plus 4 gamma x, gamma x is there into theta sin square beta by 2 and in right hand side you will get 1 is there 1 minus 4 gamma x 1 minus theta sin square beta by 2.

(Refer Slide Time: 29:07)



So, now you can write e to the power a delta t which is your amplification factor, amplification factor is equal to G is equal to now, you can write as 1 minus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus 4 gamma x theta sin square beta by 2, so we have found the amplification factor G.

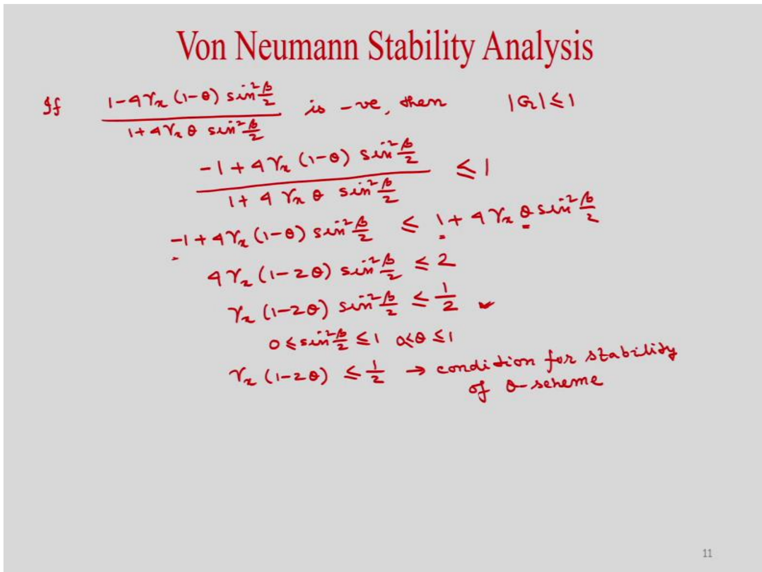
This is the expression, so for condition for stability is condition for stability is mode G should be less than equal to 1 that means 1 minus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus 4 gamma x theta sin square beta by 2 should be less than equal to 1 say this whole term. So, if it is positive then what will happen?

So, if it is 1 minus 4 gamma x, if 1 minus theta sin square beta by 2 divided by 1 plus 4 gamma x theta sin square beta by 2 is positive. If it is positive, then it should be less than equal to 1. So, 1 minus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus 4 gamma x theta sin square beta by 2 should be less than equal to 1.

So, you can see that theta lies between 0 and 1, sin square beta by 2 also between 0 and 1 and gamma x is positive. Gamma x is greater than 0, gamma x is greater than 0. So, if you see this expression, so 1 minus something because gamma x is positive sin square beta by 2 by positive and 1 minus theta is positive, either 0 or positive 1 minus theta. So, you can see that 1 minus something positive will be less than 1 and 1 plus 4 gamma x theta sin square beta by 2 always it will be the denominator term will be higher than the numerator term.

So, obviously always it will be satisfied because in the numerator we have less value than the denominator. So, it is always it will be satisfied less than equal to 1 it is always satisfied. So, the condition is satisfied 4 gamma x greater than 0 and theta less than 1 and greater than equal to 0. So, now you consider that this term is negative, then if it is negative what you can write.

(Refer Slide Time: 32:57)



So, if 1 minus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus 4 gamma x theta sin square beta by 2 is negative. Then obviously in the numerator you can write minus 1 plus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus because mode G is less than equal to 1 right so it is your G so, it will be if it is negative then you have to write them a negative values then minus 1 plus 4 gamma x 1 minus theta sin square beta by 2 divided by 1 plus gamma x theta sin square beta by 2 should be less than equal to 1.

So, you can write minus 1 plus 4 gamma x, 1 minus theta sin square beta by 2 should be less than equal to 1 plus 4 gamma x theta sin square beta by 2. So, if you take this side, so you will get 4 gamma x, 1 minus 2 theta because 1 theta is here and minus theta so, it will become minus 2 theta sin square beta by 2 and this side 1 is there it is minus 1 so, it will be less than equal to 2. So, that means, you can write 4 gamma x not 4 so, you can write gamma x 1 minus 2 theta sin square beta by 2 so, if you divide 4 this sides should be less than equal to half, so you can see that sin square beta by 2 is between 1 and 0 and your theta is also between 1 and 0.

So, this is the condition for stabilities. So, this is the condition for stability for any value of theta, so sin square beta by 2 maximum it will be 1. So, your (())(35:13) condition will be gamma x 1 minus 2 theta should be less than equal to half because sin square theta by 2 maximum value it will be 1. So, gamma x into 1 minus 2 theta should be less than equal to half.

So, this is the condition for stability, condition for stability of theta scheme gamma x 1 minus 2 theta should be less than equal to half. So, now we know that theta is equal to 0 means, you have explicit method theta is equal to 1 implicit methods and theta is equal to half, it is Crank Nicholson so, let us find the condition for stability for this schemes.

So, you can see so for theta is equal to 0 it will become FTCS we have already found. So, if it is theta is equal to 0 you can see gamma x would be less than equal to half gamma x is equal to less than equal to half and that already we have found okay for theta is equal to 1 so it is BDCS Backward Time Central Space.

So, you can see theta is equal to 1 so, this is 0 and it is always less than equal to half that means it is unconditionally stable, unconditionally stable and this is the stability criteria and for Crank Nicholson for theta is equal to half. Crank Nicholson, so you can see if it is half then it will also become 0. So, sorry for theta is equal 1 for theta sorry for BTCS scheme theta is equal to 1, so it is always negative so, it will be negative gamma x minus gamma x that means less than equal to half.

So, it will be always satisfied So, it is unconditionally stable and for Crank Nicholson theta is equal to half this 1 minus 2 theta 1 minus 2 theta will be 0. So, that will be always less than equal to half so, this is also unconditionally stable, unconditionally stable. So, that means, you can write that unconditional stable. So, for gamma x 1 minus 2 theta so, that means from half less than equal to theta, less than equal to 1.

So, it is if it is between 1 and half you can see 1 and half it is 1 and half. So, those theta varies between 0 and 1, 0 and 1. So, 1 and half it is unconditionally stable, unconditionally stable and if it is between 0 greater than equal to 0 and so if it is between 0 less than half less than half then it is conditionally stable, conditionally stable and what is the condition for stability it is gamma x 1 minus 2 theta should be less than equal to half.

So, for the theta scheme so finally, our conclusion is that if theta lies between 1 and half so, it is unconditionally stable. And if it is between 0 and less than half then it is conditionally stable and the condition for stability is gamma x into 1 minus 2 theta should be less than equal to half . So, in today's lecture, we have seen the stability criteria for three different methods.

First we use BTCS scheme which is your implicit method and we have shown that it is unconditionally stable then second we have considered Richardson scheme and we have shown that it is unconditionally unstable and finally, we consider theta scheme which actually gives for theta is equal to 0 explicit, theta is equal to 1 implicit, and theta is equal to half Crank Nicholson.

So, if theta in between 1 and half then it is unconditionally stable but if it is greater than equal to 0, but less than equal half then it is conditionally stable and condition for stability is gamma x 1 minus 2 theta should be less than equal to half. So, for other scheme similarly, you can do the von Neumann stability analysis whatever way we have carried out in these 2 lectures and you can find what is the stability criteria. Thank you