Computational Fluid Dynamics for Incompressible Flows
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Lecture 1
Von Neumann Stability Analysis of Different Schemes for Parabolic Equations

Hello everyone, so in last 2 modules we have learned different discretization schemes for parabolic equations and hyperbolic equations. So, those are marching problem and for marching problem, we have discussed what is the stability criteria. So, in today's lecture we will find the stability criteria using Von Neumann stability analysis. So, in today's lecture, we will study Von Neumann stability analysis of different schemes for parabolic equations.

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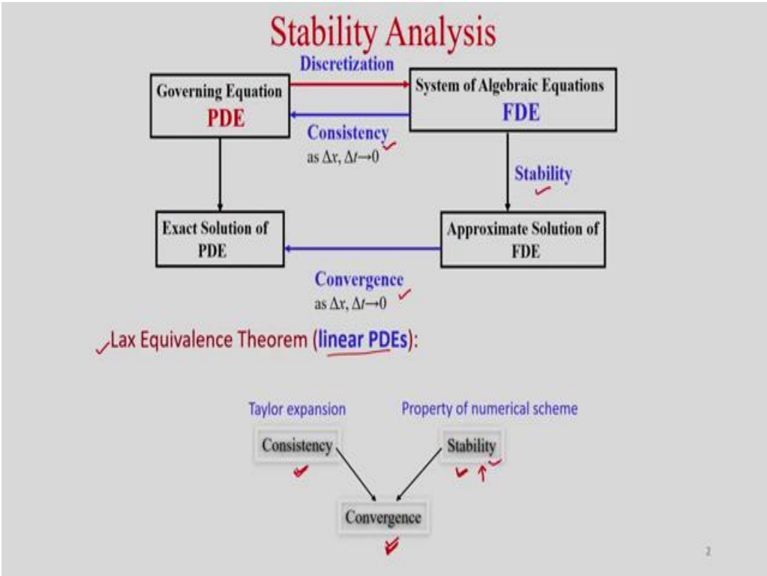

Already we have seen that if the schemes are consistent and if it is stable, then only you will get the convergence and it is given by Lax Equivalence theorem for linear PDEs, we have learned that consistency and stability will give you the convergence. So, if the scheme is consistent, but if it is not stable then you will not get the convergence. So, today we will learn a method to find the stability criteria.

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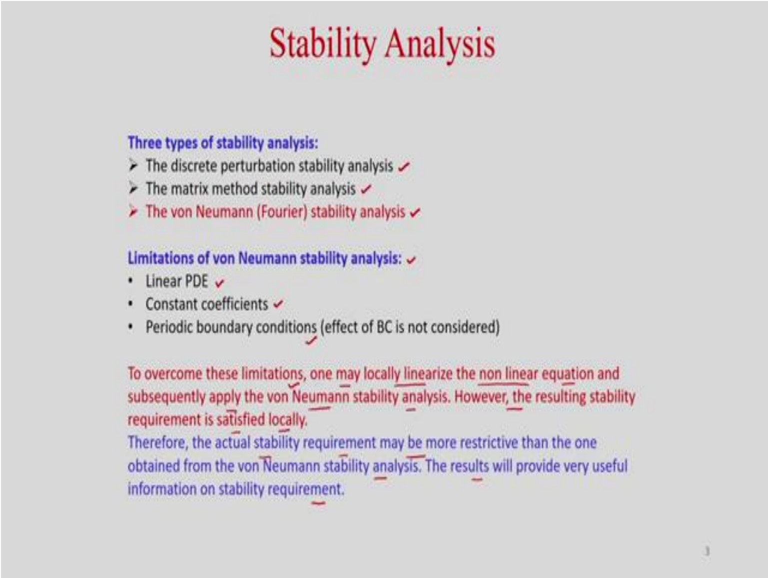

So, there are different types of stability analysis. So, these are the methods the discrete perturbation stability analysis, the matrix method stability analysis and the Von Neumann or Fourier stability analysis. So, in today's lecture, we will study only the Von Neumann stability analysis, which is considered to be more commonly used, but it is having some limitations. So, the limitation of Von Neumann stability analysis, these are applicable only to linear partial differential equations with constant coefficients and with periodic boundary conditions.

So, when we study the Von Neumann stability analysis, the effect of boundary conditions are not considered, to overcome these limitations one may locally linearize the nonlinear equation and subsequently apply the Von Neumann stability analysis. However, the resulting stability requirement is satisfied locally, therefore the actual stability requirement may be more restrictive than the one obtained from the Von Neumann demand stability analysis. The results will provide very useful information on stability requirement. So, using this Von Neumann stability analysis, you will get some criteria which you can actually use for the stability.

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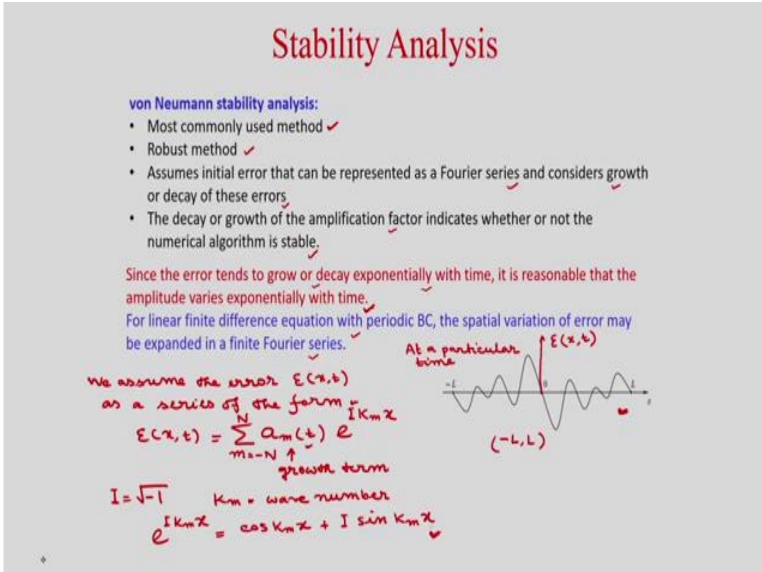

So, the Von Neumann stability analysis is most commonly used method, it is a robust method, assumes initial error that can be represented as a Fourier series, that is why it is name is Fourier stability analysis also and considers growth or decay of these errors and the decay or growth of the amplification factor indicates, whether or not the numerical algorithm is stable. Since the error tends to grow or decay exponentially with time, it is reasonable that amplitude varies exponentially with time.

For linear finite difference equation with periodic boundary condition the spatial variation of error may be expanded in a finite Fourier series. So, if you consider a periodic domain as shown in this figure, you can see it is minus L to L, one domain we have considered and if this is the error epsilon is the error x t, so at a particular time at a particular time this is the error distribution in the mesh.

So, we can write the error at a time in a mesh like so we assume the error epsilon x t, epsilon x t like as a series of the Forum, so it is a Fourier series we will assume, so a epsilon x t you can write summation of a m t which is your growth term, so it is exponential and e to the power i k m x and m varies to minus n to n. So, this a m t is the growth term.

And this i, i is your unit imaginary number, so i is equal to root minus 1, k m is your wave number and e to the power i k m x, you can write as cos k m x as trigonometry function and plus i sin k m x. So, the error we have expressed in terms of a Fourier series which contains on exponential term which is the growth term, exponential in time and e to the power i k m x, which is actually given by this expression.

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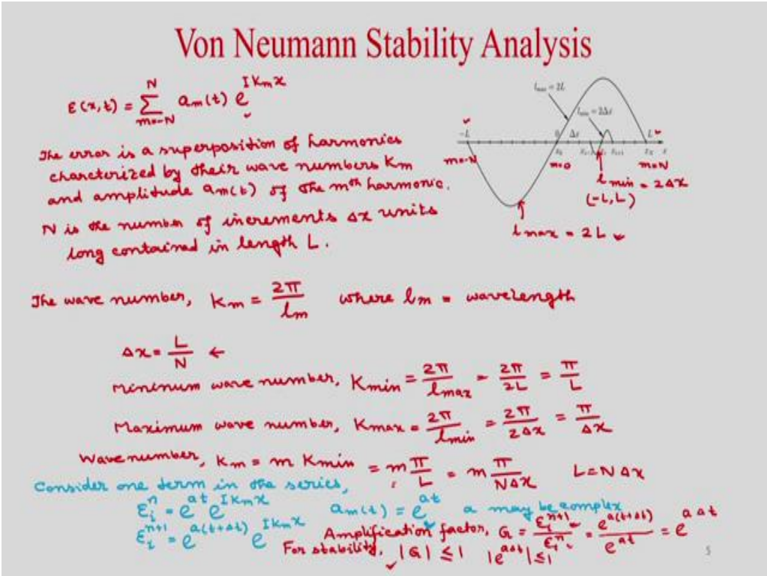

So, if you consider a periodic domain from L to minus L and if these are m is equal to 0 to m is equal to N and m is equal to minus N then whatever we have written epsilon x t is equal to summation of m is equal to minus N to N a m t e to the power i a m x, so the error is a superposition of harmonics characterized by their wave numbers km and amplitude a m t of the m th harmonic. And N is the number of increments delta x units long content in length L.

So, the wave number k m we can represent as, the wave number we can represent as K m is equal to twice pi by lm, where lm is your wavelength. So, if you see this figure we have considered a domain minus L to L up to L distance, so it is a periodic domain and for m is equal to 0 at this origin and N is equal to N here, so for this wave, what is your wavelength? So, it is the length L max, What is the wave length? You can see this is L and this is also L, so it is 2 L.

And the smallest wave you can have of the grid size delta x, so if it is delta x grid size, uniform grid size delta x, then for these wave, you can have k minimum, you can have the wavelength of this L minimum, you can have twice delta x. So, because you have the mesh and uniform mesh size delta x, so you can have the L minimum the wavelength minimum you can have twice delta x and maximum you can have wavelength twice L, because you have considered the domain of 2L size.

Now, so it is L by N, so obviously your minimum wave number, when you will get? When you have maximum wavelength, so minimum wave number, minimum wave number k min you will get when you have maximum wavelength. So, when you have maximum wavelength it is 2L, so you will get twice pi by 2 is L, so it will be pi by L. And maximum wave number you will get when you have minimum wavelength, maximum wave number you will get k max as when you have the minimum wavelength. So, you will get twice pi by twice delta x that means pi by delta x.

So, you can see from here that you can write the wave number, you can write wave number km is equal to m times k min. So, you can see that m times K min means it is pi by L. So, that means when m is equal to 1 you will get k m as Pi by L and when you have m is equal to n, so you can again write m into so you can see delta x is equal to L by n, so L is equal to n into delta x, so you can write m into pi by N into delta x. So, when m is equal to N you can see that you will get pi by delta x, pi by delta x, m is equal to N, so you will get pi by delta x.

So, this is a wave number minimum and maximum number you will get and it is a series, but as we have considered linear because this Von Neumann stability analysis is applicable to linear partial differential equation, so we can actually study the growth of only one term that will give you the characteristic of the series as a whole.

So, we will consider epsilon x t so we will consider only one term considered one term in the series, because it is linear, so all the terms will have the same characteristics, so if you consider only one term and see its behaviour, so the behaviour of the series as a whole will be the same.

So, we will consider epsilon i n as so we will consider this a m t, it is exponential term so we will consider as e to the power a into t, where a may be complex, a may be complex, then it will give the exponential growth or decay of the error, so we will consider epsilon i n as e to the power a t so one term we are considering and e to the power i k m x, so only one term here considering. So, similarly epsilon i n plus 1, what you can write? So, epsilon i n plus 1 that means t plus delta t, so you can write e to the power a t plus delta t, e to the power i k m x.

So, if it is so then what you can write the amplification factor you can find as amplification factor G, you can write as epsilon i n plus 1 divided by epsilon i n, so we are trying to see whether with time your error is growing or decaying, so you can see that we are seeing the growth sorry error at n plus 1 time level divided by the error at n th time level. So, if you see from here, so you will get e to the power a t plus delta t divided e to the power a t, because e to the power i k m x will cancel out, so this will give it to the power a delta t.

So, your growth factor sorry amplification factor is e to the power a delta t, delta is t is the time step. So, for stability, for stability your error should be bounded, it is not grow, so that means the mod of G should be less than equal to 1, because the error will either remain same or it will decay, because epsilon i by n plus 1 sorry epsilon i n plus 1 divided by epsilon i n should be less than equal to 1.

So, that means the for stability the criteria is mod G should be less than equal to 1 or mod e to the power a delta t should be less than equal to 1. So, this is your stability criteria. So, for stability the amplification factor, so mod G should be less than equal to 1, so error should be bounded, it should either remain same or it should decay with time, so that we are showing from epsilon i n plus 1 divided by epsilon i n.

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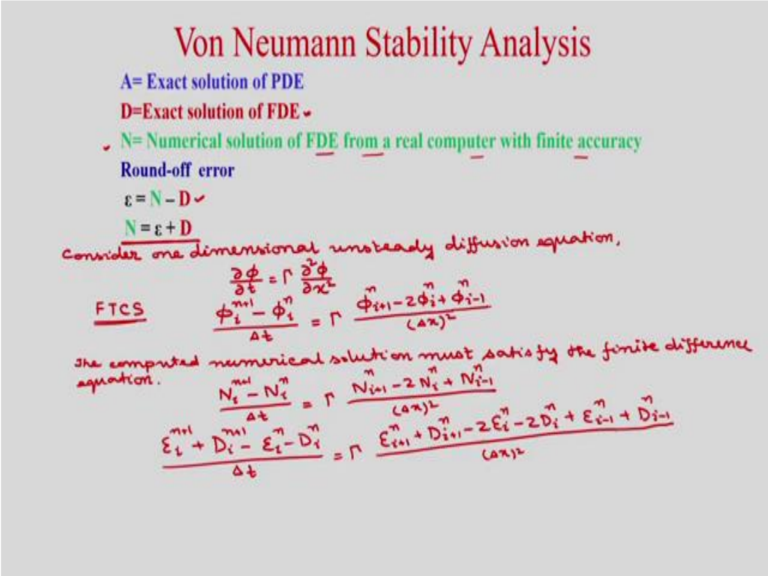

So, now we have seen the stability criteria, so the amplification factor should remain bounded, it should not grow with time, we have also seen that the roundup error epsilon as N minus D, where N is the numerical solution of finite difference equation from a real computer with finite accuracy and D is the exact solution of the finite difference equation. So, when we will solve the finite difference equation in a real computer you will get the solution N, because N is the numerical solution of the finite difference equation from the computer.

So, if you consider some equation governing equation or governing partial differential equation, then this numerical solution should satisfy that equation, so first we will consider one dimensional unsteady diffusion equation, so consider one dimensional unsteady diffusion equation. So, for a model equation, we have already written that it is del phi by del t is equal to gamma, where gamma is the diffusion coefficient del 2 phi by del x square, gamma is the diffusion coefficient.

So, if you discretize this equation, then what we will get? Using let us say forward time central space we are using forward time central space, so phi i n plus 1 minus phi i n divided by delta t is equal to gamma phi i plus 1 minus 2 phi i plus phi i minus 1 at time level n divided by delta x square. So, this we have seen, it is the first derivative we have used first order accurate scheme and for the spatial derivative we have used central difference approximation.

So, whatever numerical solution you are getting N, so it should satisfy this governing equation, so we can write the computed numerical solution numerical solution must satisfy the finite difference equation, so you will get you will get Ni n plus 1 minus Ni n divided by delta t is equal to gamma delta x square Ni plus 1 n minus twice Ni n plus Ni minus 1 n, as n is the numerical solution of finite difference equation, so it is satisfying the governing equation and we know N is equal to epsilon plus D.

So, if you substitute N is equal to epsilon by epsilon plus D, then you can write epsilon i n plus 1 plus D i n plus 1, where D is the exact solution of the finite difference equation minus epsilon i n minus d i n divided by delta t is equal to gamma epsilon i plus 1 n plus D i plus 1 n minus 2 epsilon i n minus 2 D i n plus epsilon i minus 1 n plus D i minus 1 n. So, your numerical solution now had satisfied this governing equation. Now, you have the exact solution of the finite difference equation that is D, so it should also satisfy the governing equation.

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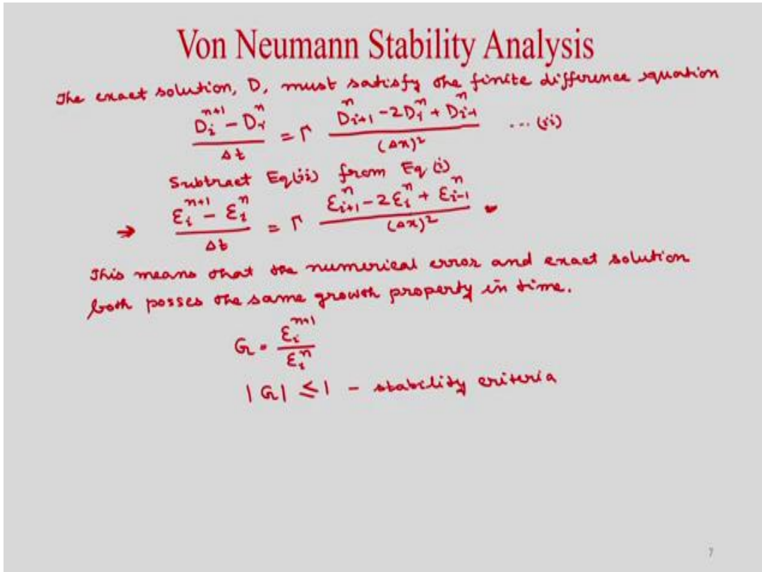

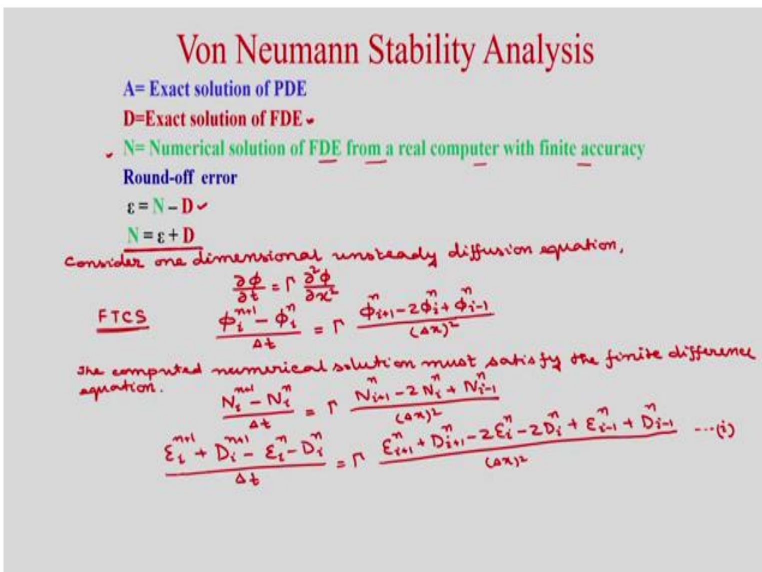

So, you can write the exact solution, D must satisfy the finite difference equation, so it satisfies the finite difference equation, then you will get D i n plus 1 minus D i n divided by delta t is equal to gamma delta x square D i plus 1 n minus 2 D i n plus D i minus 1 n, so if you give the equation number it has 2 and this equation as 1.

Then you subtract the second equation from the first one, so substrate then you will get so substrate equation 2 from equation 1, so what you will get? If you subtract you will get epsilon i n plus 1 minus epsilon i n divided by delta t is equal to gamma epsilon i plus 1 n minus 2 epsilon i n plus epsilon i minus 1 n divided by delta x square.

So, what is this equation? So, that means your error also satisfy the finite difference equation, so the this means that the numerical error that the numerical error and exact solution both posses the same growth property in time, so that means either of these two you can use for studying the stability analysis.

So, for in Von Neumann stability analysis actually we will use the this equation, the error satisfies the finite difference equation and this equation we will use and find the error at time level n plus 1 and we will see the amplification factor, which is G so epsilon i n plus 1 divided by epsilon i n.

So, if mod G is less than equal to 1, so this is the stability criteria. So, if mod G is greater than 1 that means your it will be unstable, but if it is less than equal to 1 then the scheme will be stable. So, today we will consider only the parabolic equations and already we have discretized this parabolic equation using finite difference scheme. So, few of those schemes will consider and we will show the stability criteria, which we have already discussed in earlier modules.

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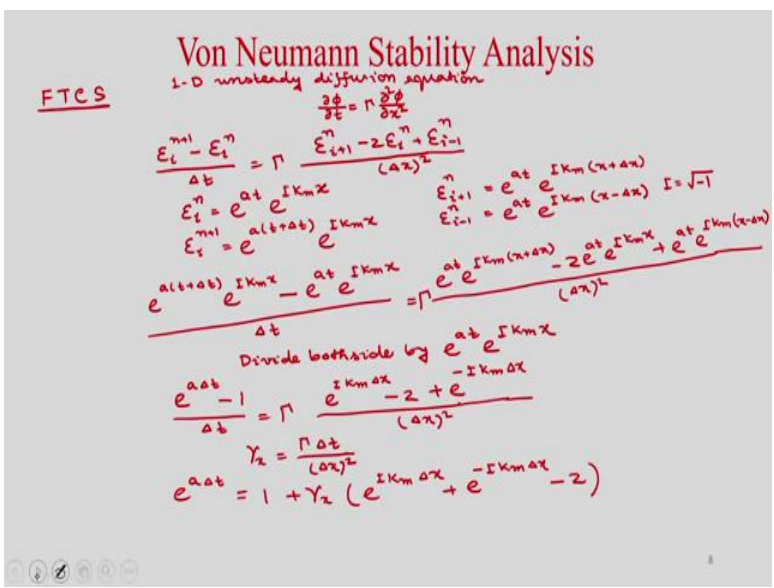

So, let us consider Forward Time Central Space, we are considering parabolic equation 1 D unsteady diffusion equation, so the discretization already we have done, so it is del phi by del t is equal to gamma del 2 phi by del x square, so using different scheme you can discretize this equation, but we are going to write in terms of the error because we have already shown that error also satisfies the finite difference equation.

So, if you write the error equation for this forward time central space scheme, then you can write epsilon i n plus 1 minus epsilon i n divided by delta t is equal to gamma epsilon i plus 1 n minus twice epsilon i n plus epsilon i minus 1 n divided by delta x square. So, the error satisfied the finite difference equation. Now, whatever from the Fourier series one term we have considered for the error that will consider now, so it consists of epsilon i n is equal to e to the power a t which is your growth term and e to the power i km x, which is the spatial variation.

So, similarly for epsilon i n plus 1, what you can write? So, it is in place of t will write t plus delta t, so a t plus delta t e to the power i k m x, so in the error equation, we also have epsilon i plus 1 n, so if you write that, so epsilon i plus 1 n, so you are going in the special direction, so x Plus delta x, so you can write? e to the power a t, e to the power i k m x plus delta x, because i plus 1 so x plus delta x, where i is equal to root minus 1.

So, unit imaginary number and epsilon i minus 1 n you can write e to the power a t, e to the power i k m x minus delta x, because i minus 1. Now, you substitute all these in the finite difference equation of the error. So, if you substitute then you will get epsilon i n plus 1, so you will get e to the power a t plus delta t, e to the power i k m x minus e to the power a t, e to the power i k m x divided by delta t is equal to e to the power a t, e to the power i k m x plus del x minus 2 e to the power a t, e to the power i k m x plus e to the power i minus 1 n, so it is a t e to the power i k m x minus delta x divided by delta x square.

So, now what we will do? We will divide both sides by e to the power a t and e to the power i k m x, because in both side divide both side by e to the power a t, e to the power i k m x. So, what you are going to get? You are going to get you see first e to the power a t plus delta t, that means e to the power a t and e to the power a delta t.

So, you can write e to the power a delta t, this will be minus 1 divided by delta t is equal to sorry here gamma is missing you right the gamma here, so gamma then you see this is your e to the power i k m delta x minus 2 plus e to the power minus i k m delta x and divided by delta x square.

So, we have divided both sides by e to the power a delta t sorry e to the power a t and e to the power i k m x. So, now rearrange it, so you can write gamma x is equal to gamma delta t gamma delta t by delta x square, so you can write e to the power a delta t is equal to 1 plus gamma x then you can write e to the power i k m delta x plus e to the power minus i k m delta x minus 2.

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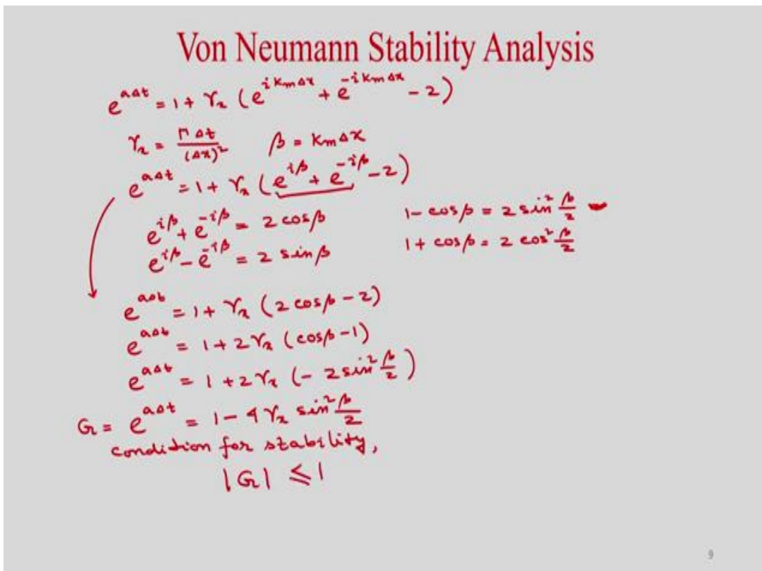

So, now we will use some trigonometric function our equation is now you got e to the power a delta t is equal to 1 plus gamma x, e to the power i k m delta x plus e to the power minus i k m delta x minus 2, so where gamma x is gamma delta t by delta x square and now we will write beta is equal to km into delta x, km into delta x and now you know that you can write e to the power a delta t is equal to 1 plus gamma x, e to the power i beta plus e to the power minus i beta minus 2.

So, now we can write you see e to the power i beta plus e to the power minus i beta is 2 cos beta and e to the power i beta minus e to the power minus i beta will give you 2 sin beta. And also we know that 1 minus cos beta is equal to 2 sin square beta by 2 and 1 plus cos beta is 2 cos square beta by 2, so that you know. So, now if we write it here, so you can write to the power a delta t is equal to 1 plus gamma x, so these together what we can write? 2 cos beta minus 2, then you can write e to the power a delta t is equal to 1, 2 you take common, so 2 gamma x cos beta minus 1.

So, 1 minus cos beta you see here 2 sin square beta by 2, so cos beta minus 1 will be minus 2 sin square beta by 2, so you can write e to the power a delta t is equal to 1 plus 2 gamma x into minus 2 sin square beta by 2. So, you are going to get e to the power a delta t is equal to 1 minus 4 gamma x sin square beta by 2.

So, now we have found the expression of the growth factor because it is the growth factor e to the power a delta t which is your G, G is the growth factor e to the power a delta t is equal to 1 minus 4 gamma x sin square beta by 2. For stability, what is the condition? Condition for stability is mod G should be less than equal to 1.

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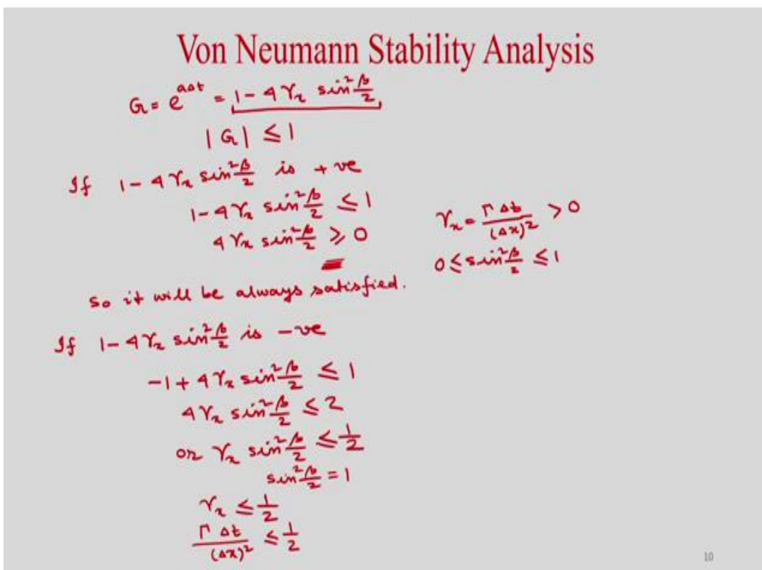

So, G is equal to e to the power a delta t is 1 minus 4 gamma x sin square beta by 2, so mod G should be less than equal to 1, so if now you can see that this term whole term can be positive or negative, so if 1 minus 4 gamma x sin square beta by 2 is positive, if it is positive then you can write 1 minus 4 gamma x sin square beta by 2 should be less than equal to 1 and less than equal to 1, so you can see it will be so if you take this side, so it will be 4 gamma x sin square beta by 2 should be greater than equal to 0.

And you can see from here that gamma x, what is gamma x? Gamma x is equal to gamma into delta t by delta x square. Now, you see whether it is a positive or negative quantity, it is always positive, because gamma is the diffusion coefficient it is positive, delta t is the time step it is positive and delta x is the grid size, so it is also positive, so gamma x always greater than 0. So, and sin square beta by 2, what is the value? It is between 0 to 1, so it is between greater than sorry so sin square beta by 2 will be greater than equal to 0 and less than equal to 1, because its value will lie between 0 and 1.

So, we can see that if it is 0 to 1 and gamma is always positive, so it is it will be always greater than equal to 0, so always it will be satisfied, so it will be satisfy always satisfied, it will be always satisfied. Now, you consider if 1 minus 4 gamma x sin square beta by 2 is less than 0, so is negative then what will happen?

So, if 1 minus 4 gamma x sin square beta by 2 is less than 0 negative and if it is negative, it is mod G, so it will be then you can write minus 1 plus 4 gamma x, because it is mod G we are taking so if it is a negative, then you will have to take minus x, so it is minus 1 plus 4 gamma x sin square beta by 2 should be less than equal to 1.

So, if minus 1 taking the right hand side, then you will get 4 gamma x sin square beta by 2 should be less than equal to 2 or gamma x sin square beta by 2 should be less should be less than equal to half, 2 by 4, so it should be less then equal to half. So, you can see, sin square beta by 2, so sin square beta by 2 the values lies between 0 and 1, 0 and 1, so if lies between 0 and 1, so if the maximum value if you take sin square beta by 2 is equal to 1, because for 0 sin square beta by 2 is equal to 1 if you take then you can see that gamma x should be less than equal to half.

So, that means the stability criteria for this forward time and central space scheme for this one dimensional unsteady diffusion equation that this gamma x should be less than equal to half that we have already discussed while discretizing this equation in earlier modules, but now we have shown here using Von Neumann stability analysis that the stability criteria for forward time and central space scheme is gamma x should be less than equal to half, that means, gamma delta t by delta x square should be less than equal to half.

So, you have to choose the time step delta t such a way that gamma delta t by delta x square should be less than equal to half. So, gamma is diffusion coefficient, delta x if your grid is fixed then delta is delta x is fixed, then accordingly you can choose the value of delta t for a stable solution. So, today we considered Von Neumann stability analysis, so this is also known as Fourier stability analysis, because we consider the Fourier series and this Von Neumann stability analysis is applicable to linear equation with periodic boundary conditions.

So, as it is a linear equation, so the error whatever we are considering in the Fourier series, if you consider only one term and see the behaviour of it, so it will give the behaviour of the whole series will be the same of the considered term in the Fourier Series. So, in that case, we considered only one term e to the power a delta t, which is your got term and e to the power i k m x. And we have then from the numerical solution we will satisfy the finite difference equation, so from there, we have shown that error also satisfies the finite difference equation.

So, from there we are trying to see the growth of the error, so we have seen the amplification factor epsilon i n plus 1 divided by epsilon i n whether it is bounded or it is less than equal to 1. So, from that we have considered the stability criteria. So, for the stability condition is that your amplification factor should be less than equal to 1.

So, we considered today one dimensional unsteady diffusion equation and discretize it using forward time and central space and we have shown that the stability requirement is gamma x is equal to gamma x should be less than equal to half. So, in next class, we will choose other schemes and we will show the stability criteria. Thank you.