Computational Fluid Dynamics for Incompressible Flows
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Module 1: Introduction to Computational Fluid Dynamics
Lecture 2: Basic equations of fluid dynamics and heat transfer

Hello everyone, in todays class we will discuss about the basic equations of fluid mechanics and heat transfer in basic fluid dynamics course and basic heat transfer course you might have derived these equations. So we will not derive in this class these equations only will write down the governing equations of fluid flow and heat transfer. First will write down the fluid flow equations for incompressible and Newtonian fluid flow.

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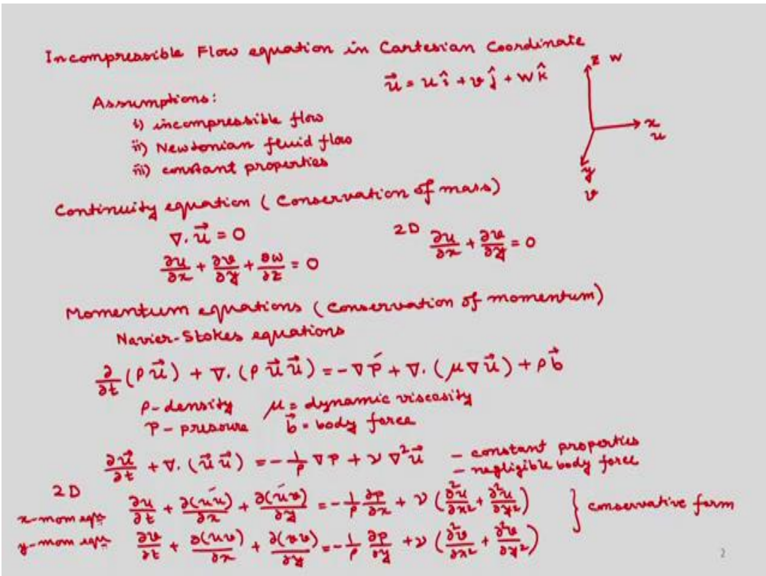

So incompressible flow equations in Cartesian coordinate, okay so if we consider a Cartesian coordinate, let us say this is your x, this is your y and this is your z, so respectively will define the velocities in x direction u, in y direction v and in z direction w. So, if you consider a velocity vector u then this is we can write u is the velocity in the x direction then the unit vector in that direction i, v, j, v is the velocity in the y direction and w.

So will have some assumptions to write the governing equations, so one assumptions already we have told that it is incompressible flows that means density is constant or you can tell that mac number is less than 0.3 Will also consider Newtonian fluid flow and will consider constant properties, so with those assumptions if you write down the governing equations. So the assumptions we have taken one is obviously incompressible flow, second is Newtonian fluid flow and we can write with constant properties.

So in fluid flow equations, so will first write the continuity equation which you can derive from the conservation of mass, right. So will write the continuity equation first, continuity equation okay, so you can derive it from conservation of mass, so it will conserve the mass, right, so it is conservation of mass. So, as it is incompressible flow so it will be we just in vector form divergence of, so we can write this continuity equation for incompressible flow in vector form as divergence of u is equal to 0.

So we can write Del u by Del x plus Del v by Del y plus Del w by Del z is equal to 0. So, if you write this equation for 2 dimensional then it will be just Del u by Del x plus Del v by Del y is equal to 0. So if you write this equation for 2 dimensional case then it will be Del u by Del x plus Del v by Del y is equal to 0. So, this is the continuity equation.

Now, will write the momentum equations, so in momentum equation we conserve the momentum. So, momentum equations, conservation of momentum, so as we are using incompressible flow and for Stokesian fluid, we can write the Navier-stokes equations. So, this is also known as Navier-Stokes equations, Navier-stokes equations okay will write in vector form, first will write in general so del by del t rho u plus divergence of the rho u, u is equal to minus grad p plus divergence of Mu grad u plus rho b, okay.

So where rho is your density of the fluid okay and Mu is the viscosity of the fluid, it is dynamic viscosity okay. So it is dynamic viscosity. So you can write Mu is equal to dynamic viscosity of the fluid. And p, p is your pressure. Okay so p is your pressure and b is the body force, body force. So you can have gravity as body force okay or if you have a multi-physics problem, then electromagnetic force or magnetic force you can have as source term or body force term.

So, this equation now for the constant properties if you write down this equation, it will be Del u by Del t. So, rho you can take it outside and divide both sides then divergence of u-u is equal to minus 1 by rho grad p plus. So, if Mu is constant you can take it outside and Mu by rho will be Mu. So, grad square u, okay this equation what we are writing constant properties and no body force or negligible body force, okay negligible body force.

So, for a 2 dimensional situations you can write down these equations u momentum equation and v momentum equation or x momentum equation and y momentum equation. So, for 2D, 2 dimensional situation you can write x momentum equation as Del u by Del t plus Del u-u by Del x plus Del u v by Del y is equal to minus 1 by rho, so obviously in the x direction. So, pressure gradient will be Del p by Del x plus Nu Del 2 u by Del x square plus Del 2 u by Del y square.

Okay similarly, y momentum equation you can write as del v by del t plus del u v by del x plus del v b by del y is equal to minus 1 by rho del p by dal y is the pressure gradient in the y direction plus Nu del 2 v by del x square plus del 2 v by del y square okay. So, you can see the second term and third term we have written the velocities inside the derivative. So, these form are known as conservative form of the governing equations, conservative form, okay.

So, now let us write these equations for steady 2 dimensional flow in non-conservative form. So, if you do so, then if it is steady then this term and this term you can drop down and these you can see how you can write.

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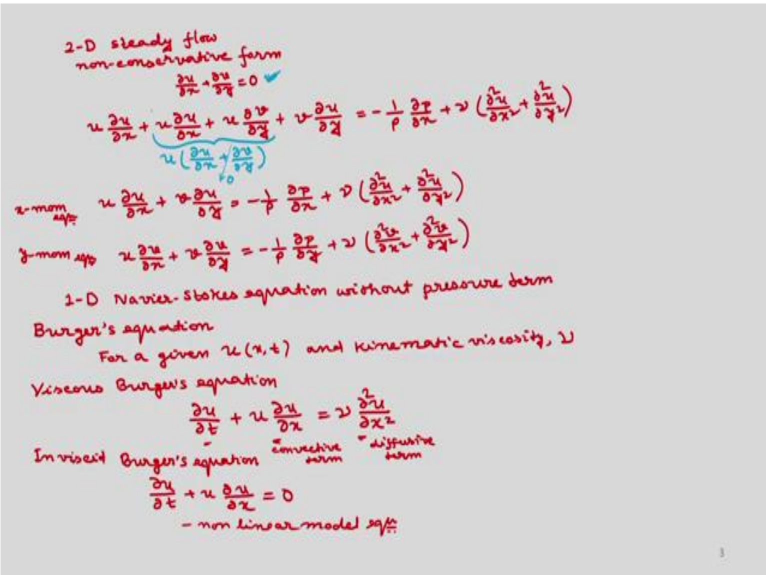

So, if you consider to 2D steady flow okay and in non-conservative form we are writing, in non-conservative form we want to write, conservative form. So, if you write in this form, so, you will actually invoke the continuity equation and you know this for 2 dimensional situation del u by del x plus del v by del y is equal to 0 is continuity equation and you will get del u by del x (okay) plus u del u by del x then you will get u del u by del y plus, 1 minute.

So u del sorry del v by del y plus b del u by del y is equal to minus 1 by rho del p by del x plus Nu del 2 u by del x square plus del 2 u by del y square okay. So, if you do that, so, you can see this 2 terms okay if you take u common then it will be del u by del x plus del v by del y. So, now you can see that it is a continuity equation. So, for incompressible flows obviously, this will become 0, okay. So, this will become 0.

So, these 2 terms will become 0 combinedly, because it is actually continuity equation. So, you can write the equation as u del u by del x plus v del u by del y is equal to minus 1 by rho del p by del x plus new del 2 u by del x square plus del 2 u by del y square. So, this is the x momentum equation in non-conservative form for 2 dimensional steady flow.

Similarly, you can write the y momentum equation Similarly, you can write y momentum equation as u del v by del x plus v del v by del y is equal to minus 1 by rho del p by del y plus Nu del 2 v by del x square plus del 2 v by del y square okay. So, these equations are written in non-conservative form. Now let us write the 1 dimensional okay 1D may be a stoke equation without pressure term. Okay because this is a very simple equation and we consider it as model equation in our CFD course. So that we can discretize this equation using different scheme.

So, it is a model equation 1 dimensional unsteady Navier-stokes equation without pressure term. So, obviously you can write it as, so it is actually known as Burgers equation. So, it is a fundamental partial differential equation from fluid mechanics and it relates to the 1 dimensional Navier-Stokes equation for incompressible flow with the pressure term removed.

So, for a given u which is function of one space coordinate x and time and kinematic viscosity, kinematic viscosity Nu you can write this equation viscous burgers equation, viscous burgers equation as del u by del t plus u del u by del x is equal to Nu del 2 u by del x square okay. So, you can see here you have the temporal term del u by del t okay you have the convective term as well as you have the diffusion term okay.

So, it represents a model equation, where you have all the terms involved and it is simple equation where pressure term is removed. And similarly, you can write in burgers equation just dropping the conductive term sorry you can write the inviscid burger equation dropping the diffusion term. So, this is your convictive term and this is your diffusive term.

Now, you can write inviscid burgers equation dropping a diffusive term, so, you can write simple del u by del t plus u del u by del x is equal to 0 okay. So, this is kind of a wave equation 1 dimensional wave equation with a varying speed u. So, this u is actually wave speed which is varying, so obviously this is known as nonlinear model equation, nonlinear model equation.

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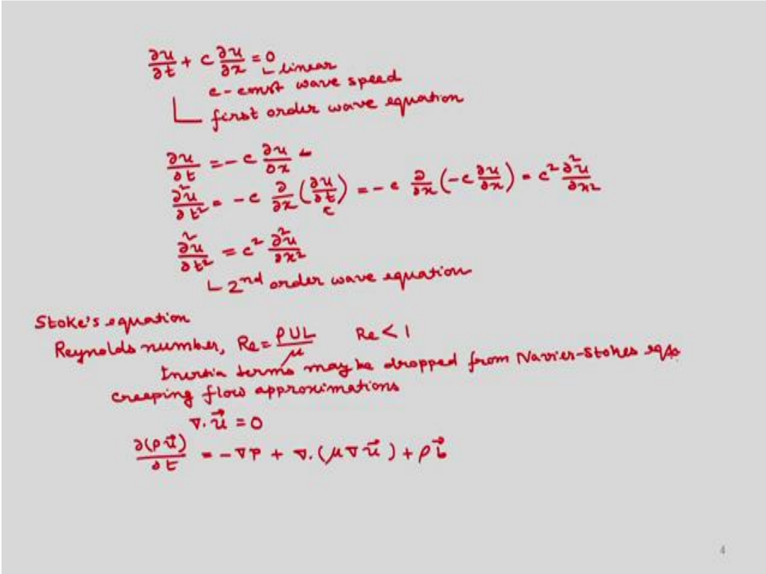

Now, if you consider u as a constant speed some way speed c then the equation you can write as Del u by Del t plus c Del u by del x is equal to 0, if c is constant okay, wave speed then this equation is known as first order wave equation okay and you can see this is linear because c is constant, so it is a linear equation.

And now, we can write Del u by Del t is equal to minus c Del u by Del x. If you take the time derivative, in both sides, then you can write Del 2 u by Del t square is equal to minus c, c is constant. So you can take it outside, you change the variable so you can write Del of del x of Del u by Del t because we are taking the time derivative.

Now, you can substitute Del u by Del t with this equation, so you can write c Del of del x minus c Del u by Del x. So, you can see now it will be minus-minus plus and c square Del 2 u by Del x square. So, you got this equation del 2 u by del t square is equal to c is the speed c square del 2 u by del x square and this is known as second order wave equation, second order wave equation.

So, now, if you consider the full Navier-Stokes equations and if it is very low Reynolds number flow, Reynolds number, what is the Reynolds number? Reynolds number is the ratio of inertia force by viscous force. If the Reynolds number is very low, then your inertia force is very low, so viscous force is dominant.

In that case, you can drop the convective term from the Navier-Stokes equation and flow is known as creeping flow. So, this equation is known as stokes equations, stokes equation. So stokes equation, so here Reynolds number, okay Reynolds number is very small, okay Re is equal to rho some characteristic velocity u, characteristic length L divided by viscosity Mu and rho is the density.

So if Reynolds number is less than 1 then inertia term you can drop because viscous force will dominate. Okay so inertia, inertia terms may be dropped from Navier-Stokes equation, from Navier-Stokes equation. So, this is a creeping flow approximation, Reynolds number less than 1, so this is your creeping flow approximation. So, in that case you can write the governing equations as obviously the continuity equation will be as it is and momentum equation will be just del rho u by del t.

So, inertia term may not be there, you just write the pressure gradient term plus the viscous term and obviously you can have the body force term. So, now if you have a high Reynolds number flow, okay so in high Reynolds number flow inertia force will be dominating. So, in that case you can drop the viscous term and that equation is known as Euler equation.

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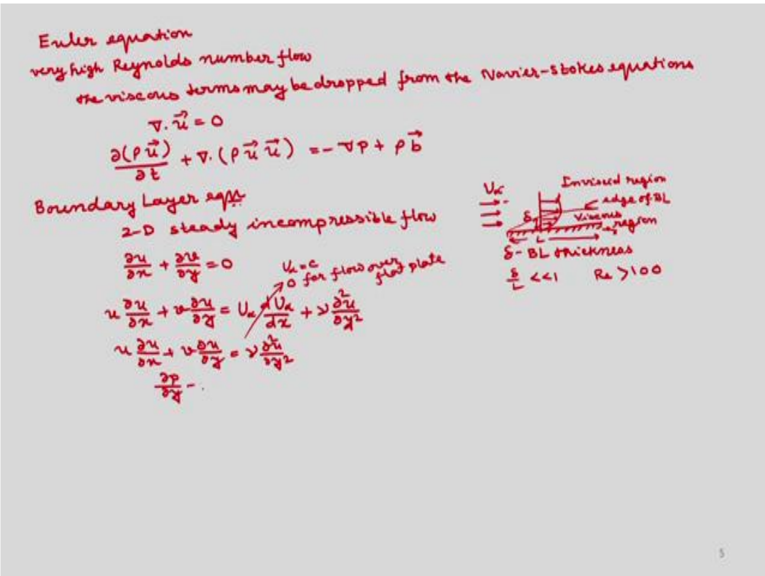

So, will write now Euler equation where you have a high Reynolds number flow, so Reynolds number is very high okay. So, in that case the viscous terms can be or so, Euler equation we can write for high Reynolds number flow, very high Reynolds number flow, Reynolds number flow. So, in that case you can drop the viscous term, so the viscous term may be dropped from the Navier-Stokes equation. So we can write the continuity equation will be as it is, only the momentum equation you can drop the viscous term.

So, you can write as del rho u by del t plus divergence of rho u-u is equal to minus grad p. So, there will be no viscous term just rho b. Now, if you consider a fluid flow, over a flat plate then you can have the boundary layer flow. So, already you have started the boundary layer equations, so will write down the boundary layer equation.

So, obviously boundary layer flow you can see that due to the no slip condition because velocities of those fluid particles sitting on the solid wall will be 0. So, from 0 to the free stream velocity there will be velocity gradient and you can distinct two different regions, one is viscous region okay where velocity variation will be there and you can have the inviscid region outside these boundary layer, edge of the boundary layer that is known as inviscid region.

So, with these certain assumptions, you can write the Navier stokes equation in a droppings you can write the Navier stokes equation dropping some term. So, that is known as boundary layer equations and the assumptions for writing this boundary layer equation is that there is no flow separation and the boundary layer thickness is much-much smaller than the characteristic length and if it is a high Reynolds number flow.

So, for that you can write the boundary layer equation and you can write for 2D steady incompressible flow. So, if you consider a flat plate you have free stream velocity at u infinity. So, when it will flow over this flat plate there will be one region which is known. So, there will be a distinct region where you will have viscous region and another is inviscid region.

So, in inviscid region your velocity will be at free stream velocity. So, whatever free stream velocity you have that will be your free stream velocity and inside this viscous region there will be change in the velocity from 0 to free stream velocity okay.

So, this is known as the edge of boundary layer and the distant at any location x, if it is x the distance of this edge of boundary layer is known as delta which is your boundary layer thickness, boundary layer thickness. So with delta and L is the length of the plate then delta by L is much-much smaller than 1 and it is valid when you have a Reynolds number greater than 100 okay.

So for that you can write the equation, continuity equation del u by del x plus del v by del y is equal to 0 and you can have the boundary layer equation del 2 u by del x plus v del u by del y is equal to free stream velocity u infinity du infinity by dx that you can write the pressure gradient in terms of the free stream velocity plus Nu del 2 u by del y square, so here from the viscous term del 2 u by del x square is dot because that is much-much smaller than the del 2 u by del y square.

So, you can have this del 2 u by del y square and for flow over flat plate as u infinity is constant. So, this will be 0 for flow over flat plate. Okay so, you can have the equation as u del u by del x plus v del u by del y is equal to Nu del 2 u by del y square and you have del p by del y is 0. These are the equations of fluid flow. So, now, another transport equation will derive from the Navier stokes equation.

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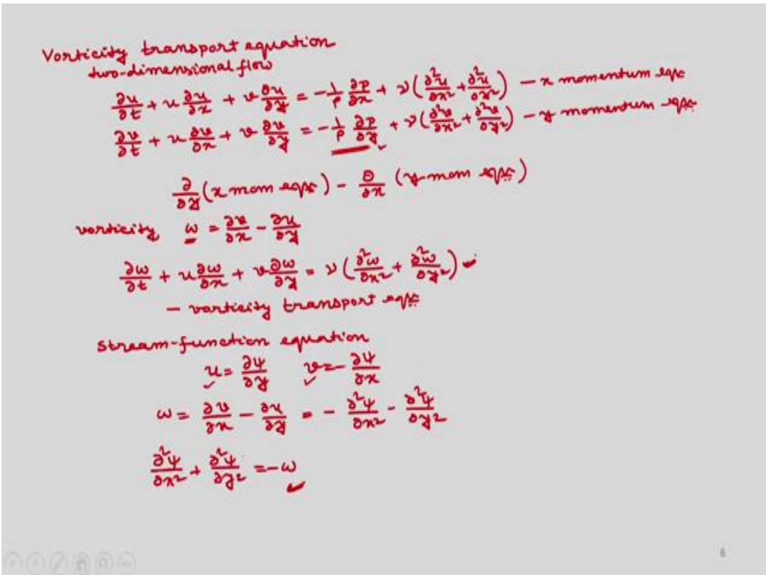

So, if you have 2 dimensional unsteady Navier stokes equation, then we can write the, derive the vorticity transport equation, vorticity transport equation okay. So, in vorticity transport equation, how can you derive? So, you have the momentum equations, so you have momentum equations del u by del t, so 2 dimensional flow okay.

So, we are writing del u by del t, u del u by del x conservative form we are writing u del u by del x plus v del u by del y is equal to minus 1 by rho del p by del x plus Nu del 2 u by del x square plus del 2 u by del y. So, this is the x momentum equation. Now, if you write the y momentum equation you will get del v by del t plus u del v by del x plus v del v by del y is equal to minus 1 by rho del p by del y plus Nu del 2 v by del x square plus del 2 v by del y square.

So you can see that this is your x momentum equation and this is your y momentum equation okay. So, now, if you do this mathematical algebra del of del y of x momentum equation minus del of del x of y momentum equation then you can write down this equation as del rho sorry. So, you can write this equation with the vorticity if you define as del v by del x minus del u by del y. So, vorticity we are defining omega sometime in book you will get this as a angular velocity, but in this case we are defining as the vorticity.

Okay so, vorticity is equal to 2 into angular velocity. So, vorticity is del v by del x minus del u by del y, if you do this mathematical algebra, you can write this equation as del omega by del t del omega by del t plus u del omega by del x plus v del omega by del y and what you are doing actually, you are actually canceling this pressure term. So, doing this mathematical algebra you can see Del of del y so it will be Del 2 p by Del x Del y and this term will be De l2 p by Del x Del y, so it will be canceled.

So, you are actually canceling this pressure term and you will have Mu del 2 omega by del x square plus del 2 omega by del y square okay and this equation is known as Vorticity transport equation. So, this equation along with the stream function equation if you solve then you can have the, you can solve the fluid flow equation.

So stream function equation now you define the stream function. So, u is defined as del Psi by del y and v as minus del Psi by del x. So, you have the omega which is your del v by del x minus del u by del y. So, you can see that if you do the del v by del x it will be minus del 2 Psi by del x square and minus del u by del y it will be minus del 2 Psi by del y square, then you can solve del 2 Psi by del x square plus del 2 Psi by del y square is equal to minus omega. So, this is your stream function equation.

So, if you solve this equation you can get the value of Psi and if you solve the Psi you can find the velocity u and v. Once you solve the u for the u v, you can solve this vorticity transport equation because u v are known. So, in that way you can also solve the fluid flow equations.

Now, let us consider the energy equation, will now have the conservation of energy and will write down the energy equation in terms of temperature.

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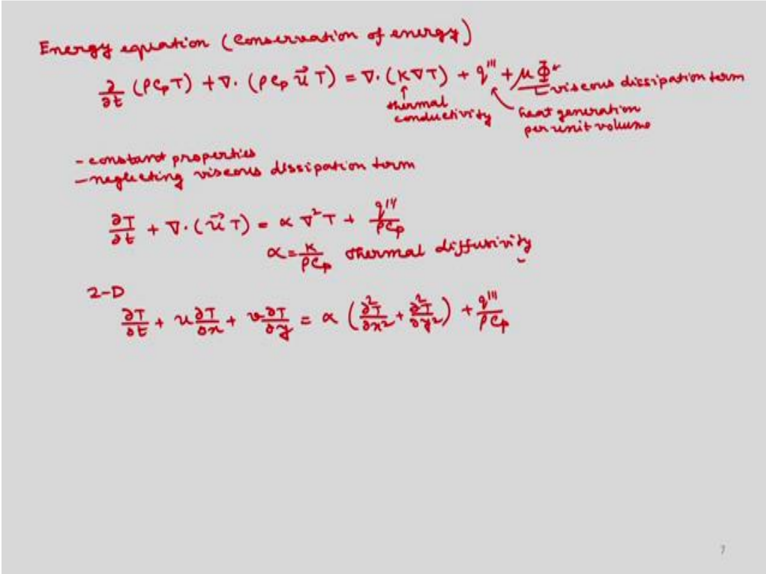

So, now we are writing energy equation, conservation of energy and we are going to write this equation in terms of temperature. So, it will be del of del t rho c p t plus divergence of rho c p ut is equal to divergence of K grad T plus q triple prime plus Mu Pi okay.

So, here K is the thermal conductivity of the fluid thermal conductivity and q is the heat generation per unit volume, heat generation per unit volume and this is the viscous dissipation term, viscous dissipation term okay due to the share there will be convection of mechanical energy to intermolecular energy which in turn will rise the temperature and pies the viscous dissipation function, so, that I am not going to write.

So, if you can neglect this viscous dissipation term and write this equation for a constant properties, then you can write constant properties, constant properties and neglecting viscous dissipation term. We can write this equation as del t by del t constant properties, so rho c p you can take it outside. So, del t by del t t plus divergence u t is equal to alpha is, alpha is the thermal diffusivity grad square T plus q triple prime by rho c p okay.

So, you see alpha is nothing but K by rho c p which is known as thermal diffusivity. So it is constant properties we have considered, so row c p you have taken outside and we have divided. So, alpha is equal to K by rho c p thermal diffusivities. So, in 2 dimensional situation you can write this equation as del t by del t plus u del t by del x plus v del t by del y is equal to alpha del 2 t by del x square plus del 2 t by del y square plus q triple prime rho c p.

Okay so this is the energy equation in 2 dimensions neglecting the viscous dissipation term. So, now if you consider that velocity is 0, so if you consider this heat conduction in solid, then you can make this velocity as 0 and you can write this energy equation as heat conduction equation in a solid.

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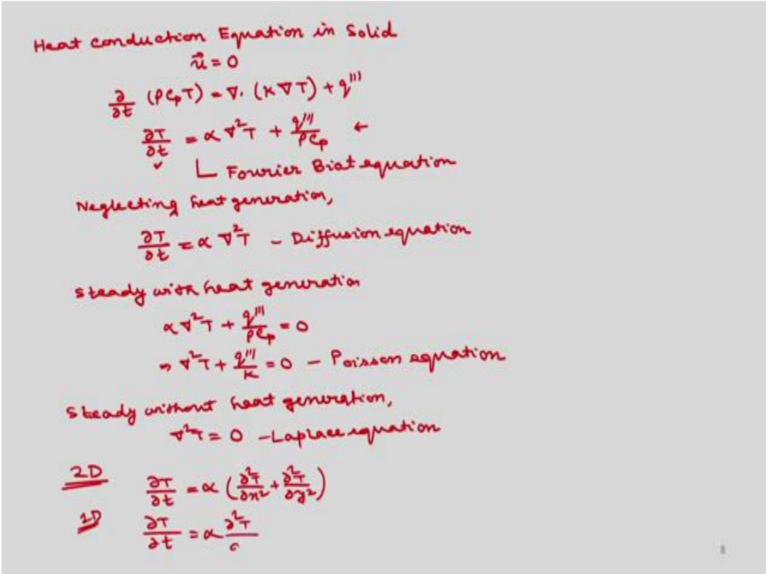

So now will write this equation as now will write heat conduction equation, heat conduction equation in solid okay, so we are considering solid, so obviously velocity you can make 0, okay you can make 0. So your convective term will become 0 and you will have the equation as del of del t rho c p T. So, this your convective is 0 because u is 0 is equal to divergence of K grad T plus q triple prime okay.

So now, if you write for constant properties then you can write del t by del t is equal to alpha grad square T plus q triple prime by rho c p. So, alpha is your thermal diffusivity and this equation is known as Fourier biot number, a biot equation, Fourier biot equation okay. So, q triple prime is the heat generation per unit volume in the solid okay.

Now, if you neglect this heat generation term then you can write this equation as, so neglecting heat generation okay you can write this equation as del T by del t is equal to alpha grad square t and this equation is known as diffusion equation, heat diffusion equation okay. Now if you consider steady heat conduction equation then you can drop the temporal term, then you can write this equation steady with heat generation, steady with heat generation.

So, this equation you consider, so if it is a steady then the first term you can make 0 then you will get alpha grad square T plus q triple prime by rho c p is equal to 0 or you can write grad square t plus q triple prime by K is equal to z0 and this equation is known as Poisson equation.

Now, if you have steady and negligible heat generation, so steady and without heat generation, so this is a simple one because you have no heat regeneration and you have steady state so del T by del t is equal to 0. So, you will have grad square T is equal to 0 which is known as Laplace equation okay.

And in 2 dimension if you want to write these equations, then you can write del T by del t is equal to alpha del 2 t by del x square plus del 2 t by del y square neglecting the heat generation term and obviously in 1D you can write del t by del t is equal to alpha del 2 t by del x square. Similarly, if you have species transport in the with the fluid flow then you can write the species transport equation okay and we can write in terms of the mass fraction of the species.

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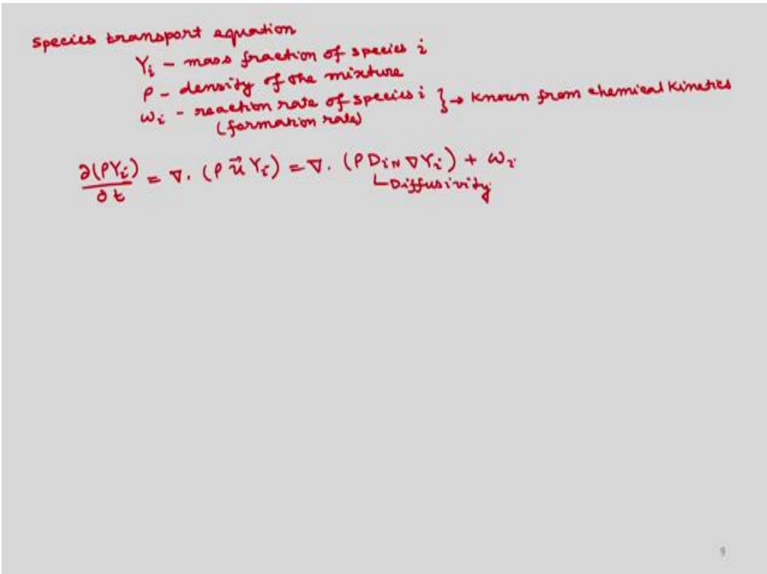

So, will write species transport equation, species transport equation, so now will write in terms of mass fraction which will denote with yi, i is the n species i, so that is your mass fraction of species i and rho is the density of the mixture, density of the mixture. And will consider Wi which is your reaction rate or formation rate of species i okay or formation rate. So, this is known from the chemical kinetics, known from chemical kinetics.

So, if you write the governing equation then you can write it as del of rho yi divided by del t is equal to divergence of rho u, yi is equal to the diffusion term divergence of rho di, n is the diffusion coefficient di to n, so i to n species and grad yi plus omega. So, this is your diffusivity, okay. So, you we can see that all these equations Navier-Stokes equations, the energy equation, the vorticity transport equation and the species transport equation you can write this equation in a general form, okay convective diffusive equation.

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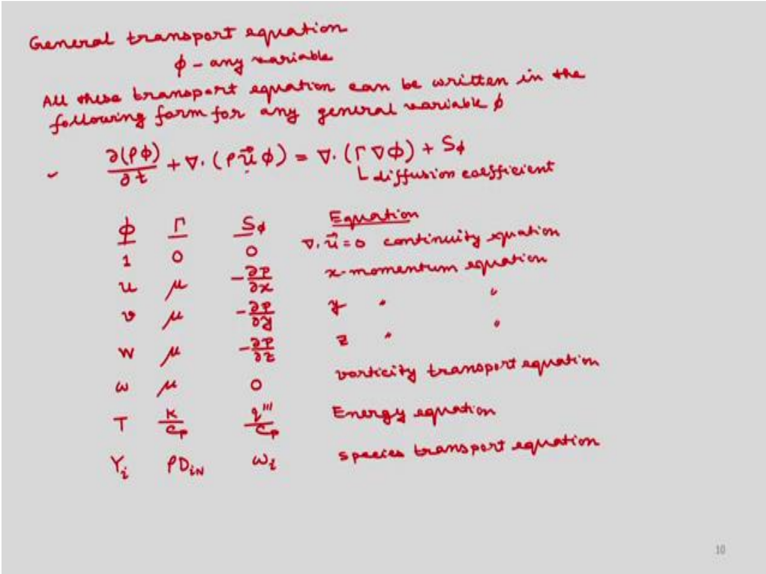

So will write the general transport equation, general transport equation okay, so will write for any variable phi, phi is any variable okay it may be u, it may be v or it may be t okay. So, all these transport equations can be written in the following form for any general variable phi. So, this you can write as del rho phi by del t which you have the temporal term plus you have convective term, divergence of rho u phi.

Okay so phi is convicted by the velocity u is equal to divergence of gamma grad phi where gamma is your diffusion coefficient, so this is your diffusion terms, so gamma is your diffusion coefficient, diffusion coefficient and plus some source term S phi. Okay it may be pressure gradient term for the Navier-Stokes equations okay or heat generation term for the energy equations.

So, these all these equations whatever we have written in todays class okay we can write in this form. So here you can see you have a temporal term, convective term and diffusive term and with some source term is phi okay. So, you can see that if you write phi is equal to 1 okay and gamma and this is your S phi, then what your equation you are going to get? Okay so you see, so if you put phi is equal to 1, it is incompressible force over density obviously, is constant. So, this term will become 0.

And if you put these terms, so phi is equal to 1 and density is constant, so gamma dot rho u is equal to now, if you gamma if you put 0 then this diffusion term will become 0 and S phi if you put 0 then you are going to get the equation divergence of u is equal to 0, so that means it is your continuity equation okay.

So, from this general transport equation you can see if you put phi is equal to 1, gamma is equal to 0 and S phi is equal to 0 you are going to get continuity education. Similarly, if you put phi is equal to u and gamma is equal to Mu and del phi S phi as del p by del x, S phi as minus del p by del x, then you are going to get x momentum equation okay without the body force term or if you write body force term then you can add it here.

Or for velocity v if you put gamma as Mu, S phi as minus del p by del y, you are going to get y momentum equation. Similarly, if you write for w and gamma as Mu and S phi as minus del p by del z then you are going to get z momentum equation and similarly for vorticity transport equation you can see.

So, if you write omega phi is equal to omega and gamma is equal to Nu, okay gamma is equal to Nu sorry gamma is equal to Mu because rho is there in the left hand side and if it is source stem is 0 then you will get vorticity transport equation, vorticity transport equation. And for energy equation if you put phi is equal to t and gamma as K by cp and S phi as q triple prime by cp then you will get energy equation.

And if you put mass fraction phi is equal to mass fraction yi and gamma as rho Di and this is as omega i then you will get mass fraction equation or species transport equation, species transport equation. So, why I have written this equation? Because it is a general transport equation which represent any of these equations okay putting the values of phi or diffusion coefficient gamma or the source term S phi you can write any equations from this general transport equation.

So, when we discretize these equations, okay first will discretize in general transport equation, so that you have a convective term, temporal term and the diffusion term and if you can discretize this equation using some discretization method may be finite difference method or finite volume method then you can innovate discretize the other equations like Navier-Stokes equations or the energy transport equation. Only thing is that separately you have to discretize the pressure term.

So, for that reason we have written this equation and when we will use finite volume method specifically that time will consider this general transport equation and will discretize. So, in todays class we have started with writing the basic fluid flow equations, conserving the mass, conserving the momentum, we have written the continuity equation and the Navier stokes equations and from this general Navier-Stokes equation we have written the burgers equation which is your 1 dimensional Navier-Stokes equation without the pressure term.

So, we have written for viscous burgers equation and inviscid burger equation dropping the viscous term and from there we have written the first order wave equation and second order wave equation for a constant speed c, constant wave speed c. Then, we have also written the vorticity transport equation and from there we have written stream function equation, so that these vorticity transport equation and the stream function equation combinedly if you solve you will be able to solve a fluid flow problem.

Then we consider the energy equation, so we from this energy equation in general we have written for a, with the fluid flow, then putting the velocity as 0 we have written the heat conduction equation in a solid and after that we have written the species transport equation in terms of the mass fraction yi.

And at last we have written all these equations, transport equations in a general transport equation for any variable, general variable phi and where you have the temporal term, convective term then diffusive term as well as you have a source term S phi. Thank you.