Computational Fluid Dynamics for Incompressible Flows
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Lecture 2
Finite Difference Formulation of the First Order Wave Equation: Implicit Method

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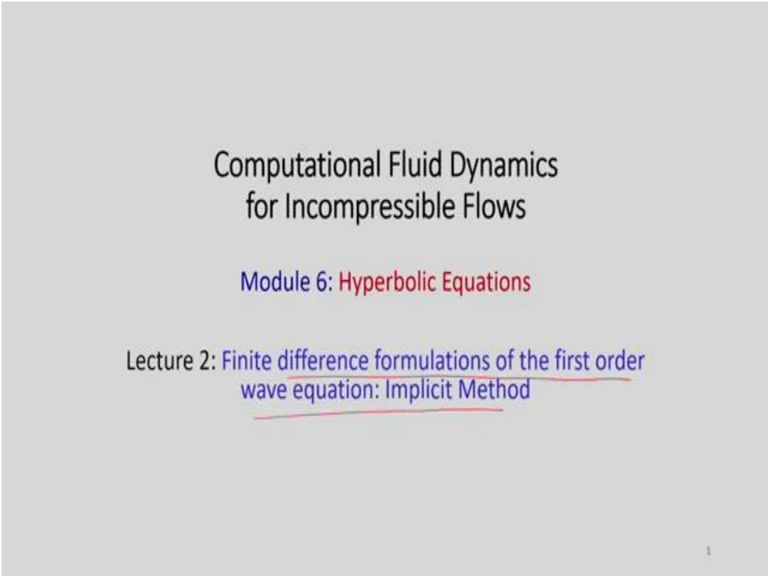

Hello everyone. So, in last lectured, we have considered linear hyperbolic equations and we considered on first order wave equation and we used different explicit discretization method and discretized that equation. In today's lecture, we will consider the same equation and we will use implicit method. So, today lecture 2, finite difference formulation in the first order wave equation implicit method.

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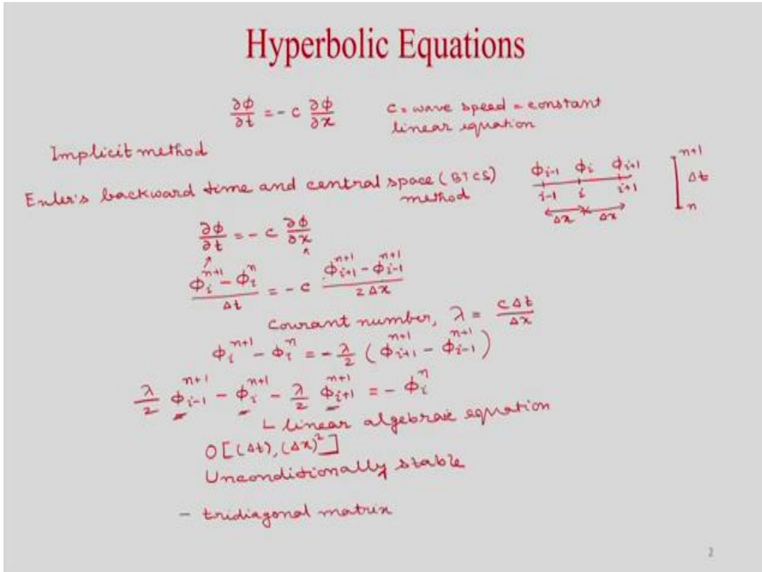

So, we will consider the same equation that is your model equation del phi by del t is equal to minus c del phi by del x. So, obviously this is your wave speed and it is constant, so this equation is linear. So, c is wave speed, so this is linear equation. So, we will considered implicit method, so as you know that in the implicit method there will be more than one unknown.

So, we will consider the grid as i, i plus 1, i minus 1 and the dependent variable ii ii plus 1 and phi i minus 1 and uniform step size delta x. And as you will march in time, so we will go from n to n plus 1 with time step delta t. So, let us consider first Euler's backward time and central space, so BTCS method. So, in this method this time derivative we will consider backward difference approximation and for the special derivative will consider central difference approximation.

So, the equation is del phi by del t is equal to minus c del phi by del x, so if you apply backward difference approximation for this time derivative, then you can write phi i n plus 1 minus phi i n divided by delta t, which is your first order accurate in time is equal to minus c, so del phi by del x will consider central difference approximation. So, we will use i i plus 1 minus phi i minus 1 divided by 2 delta x.

And here you can see that as we are using implicit method and backward time, so the dependent variable will be at time level n plus 1. So, you can see that there are more than 1 unknown, if you rearrange it, so all phi I plus 1 at time level n plus 1 you take in the left hand side and define the courant number, courant number as lambda is equal to c delta t by delta x.

So, you can see that you can write this equation as so it will be phi i n plus 1 minus phi i n is equal to minus lambda by 2 phi i plus 1 n plus 1 minus phi i minus 1 n plus 1, now you take all the n plus 1 time level dependent variable in the left hand side and rearrange it you will get lambda by 2, phi i minus 1 n plus 1 minus phi i n n plus 1 minus lambda by 2 phi i minus 1, sorry i plus 1 n plus 1 is equal to minus phi i n.

So, for a discrete point i we have discretized this first order wave equation and we got this linear algebraic equation, linear algebraic, so what is the order of accuracy? Because you have seen that we have used first order accurate in time and central difference we have used, so second order accurate in space. So, order of accuracy is delta t delta x square and if you do the stability analysis, then we can show using von Neumann stability analysis that this method is unconditionally stable. So, this is your unconditionally stable.

So, you can see that in the left hand side there are 3 unknowns phi i minus 1 n plus 1, phi i n plus 1 and phi i plus 1 n plus 1 and at each grid point if you discretize this equation, you will get a system of linear algebraic equation and if you write in a matrix format, then you will get a tridiagonal matrix. So, you will get tridiagonal matrix and you will be able to solve using Thomas algorithm.

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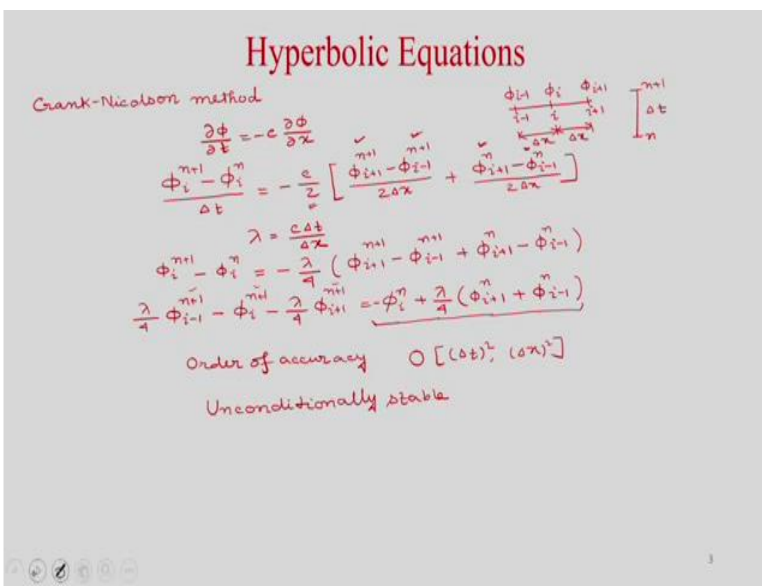

Next let us consider another method which you know Crank-Nicholson method, so which is second order accurate in time and space. So, we will use Crank-Nicholson method, which is your implicit method, Crank-Nicholson method, so we will use this grid uniform step size and n to n plus 1 we are marching so delta t is your time step, your equation is del phi by del t is equal to minus c del phi by del x. So, in Crank-Nicholson method, what we will do? So, the special derivative we take average of explicit and implicit kind of thing.

So, you write the left-hand side as phi i n plus 1 minus phi i n divided by delta t is equal to minus c by 2 half we are taking, so this is your phi i plus 1 minus phi i minus 1 divided by 2 delta x, so you can take at n plus 1 level and another you can take phi i plus 1, phi i minus 1 divided by 2 delta x at time level n, so you can see this is your n plus 1 level and this is your nth level, so average of this we are taking so half we have written here, so this is Crank-Nicholson method.

So, if you rearrange it and all the n plus 1 time level dependent variable you take in the left hand side and define the courant number lambda is equal to c delta t by delta x, then you can write this equation as so phi i n plus 1 minus phi i n is equal to minus lambda by 4, minus lambda by 4, 2 delta x if you take outside, then you can write phi i plus 1, n plus 1 minus phi i minus 1, n plus 1 plus phi i plus 1 n minus phi i minus 1 n.

Then if you rewrite it, so you can write as lambda by 4 phi i minus 1 n plus 1 minus phi i n plus 1 minus lambda by 4 phi i plus 1 n plus 1 and in right-hand side all the dependent variables at time level n. So, this is your i plus lambda by 4 phi i plus 1 n plus phi i minus 1, this you can see in right hand side this is your known term, this is your known term only left hand side at n plus 1 you have so this is these are unknown and obviously at each grid point if you write this equation, then you will get a system of linear algebraic equation, which actually will give you a matrix format at tridiagonal matrix and you can solve it using Thomas algorithm.

And what is the order of accuracy? Order of accuracy obviously it is delta t square and delta x square. So, it is second order accurate in time as well as in space. So, ordered of accuracy is order of delta t square and delta x square. And if we carry out this Van Neumann stability analysis of this method, then we can show that it is unconditionally stable. So, it is a stable scheme.

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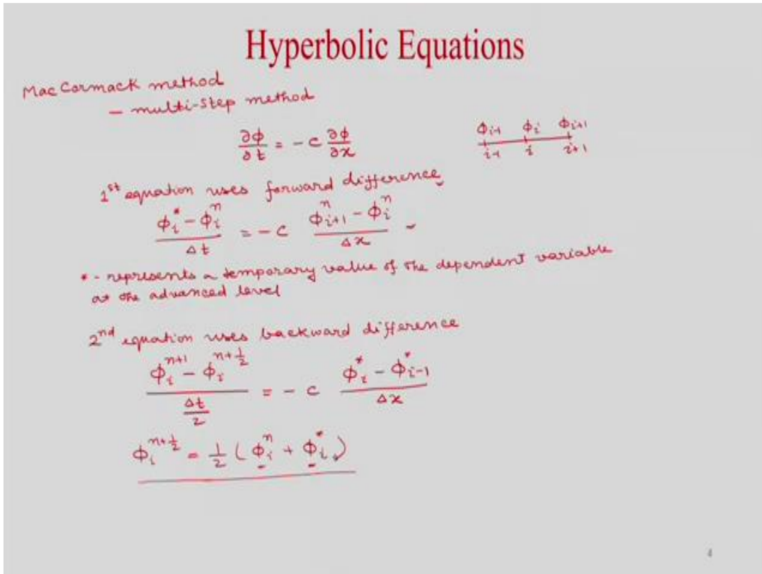

So, now in next we will consider one multi-step method, next we will consider a multi-step method which is known as Mac Cormack method, Mac Cormack method, it is a multi-step method. So, in this Mac Cormack method what we will do this equation we will discretized fast using forward difference, then we will use backward difference. So, you can see what is our model equation del phi by del t is equal to minus c del phi by del x, so we will use fastback forward difference then backward difference in next step will consider backward difference.

So, first equation uses forward difference, so forward difference in space special derivative. So, forward differences in special derivative we will use, so we will use phi i star minus phi i n divided by del t is equal to minus c phi i plus 1 n minus phi i n divided by delta x, so this star actually represent a temporary value of the dependent variable at the advanced level. So, in the next step, we will use this value. Now, in second equation, we will use the backward difference of the special derivative.

So, second equation uses backward difference or you can do vice versa fast you can use backward difference then you can use forward difference. So, what we will use here? phi i n plus 1 minus phi i n plus half divided by del t by 2 is equal to minus c phi i start minus phi i minus 1 star, so you can see for the special derivative we are using backward difference. So, but phi i at n plus half is not known.

So, that we will use the average value of this, so phi i n plus half we will use the average value of phi i n and phi i star, so star you have calculated in the first level. So, in the second level you calculate phi i n plus 1 but phi i and plus half you take half plus sorry phi i n plus half is average of phi i n and phi i plus i star phi i star. So, phi i star you calculate from the first level equation.

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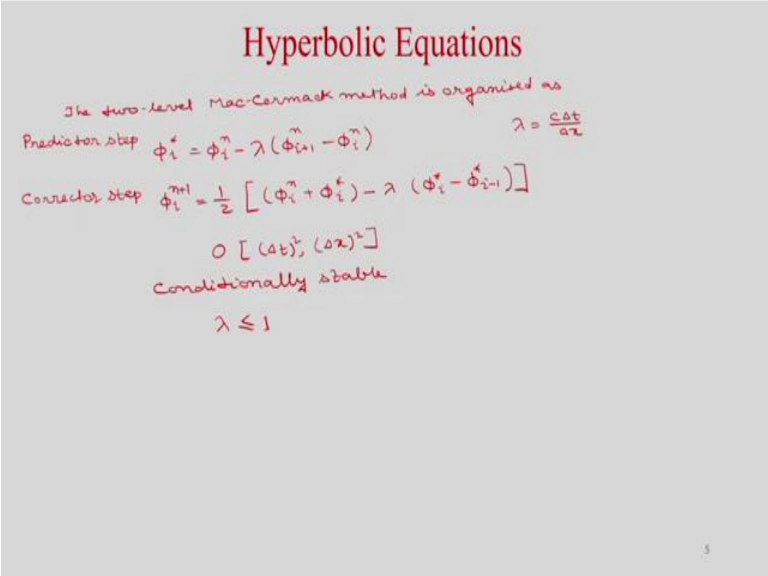

So, this method if you write in 2 steps, so this is known as predictor corrector step the 2 level Mac Cormack method is organized as so first level whatever we have written, so you write phi i start, so this is your predictor step, you are predicting phi i star then that you are using in the second level, predictor step so phi i star is equal to phi i n minus lambda, where lambda is equal to c delta t by delta x phi i plus 1 n minus phi i n.

So, that you calculate phi i star from here, then in the corrector step you find the value of phi i n plus 1 as half phi i n plus phi i star minus lambda phi i star minus phi i minus one star. So, in this equation, you can see that in first equation or first level or in the predictor level, we have used forward difference in the special derivative. But in the second level in the corrector step, we have used backward difference in the special derivative, but you can do in the vice versa way, so you can first use backward difference in the first level and in the second level you can use forward difference.

So, what is the order of accuracy? Order of accuracy is delta t square and delta x square. So, it is second order accurate in both time and space, so order of accuracy is delta t square and delta x square. But if you do the von Neumann stability analysis of this method, then you will find that it is conditionally stable and the condition is courant number should be less than equal to 1. So, it is conditionally stable, conditionally stable and condition is lambda should be less than equal to.

So, we considered linear equation linear hyperbolic equation and use different explicit and implicit methods, now next let us consider nonlinear equation and we will use just few discretization scheme or what already we have learned and discretize this nonlinear equations. So, when this equation will become nonlinear? When c is not constant, so when c is not constant that time this equation will become nonlinear.

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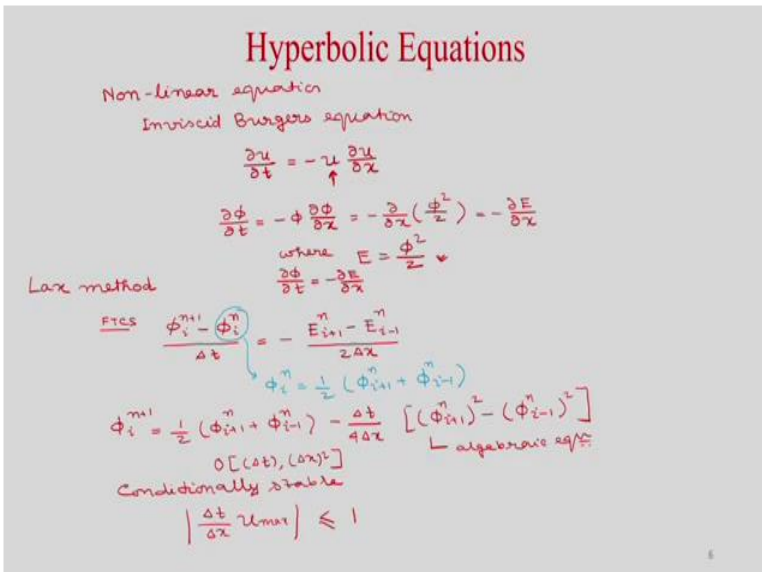

So, we will consider in Inviscid Burgers equation. So, we will consider nonlinear equation, so first we will consider Inviscid Burgers, Inviscid Burger’s equation, so we can represent as del u by del t is equal to minus u del u by del x, so you can see this is your wave speed and at different instances these velocity or wave speed is changing, so you will be because from the from this equation you will find and that you will be used here, so it is nonlinear in nature.

Now, let us write this equation for general variable phi, so that will be our model equation, so we can write del phi by del t is equal to minus phi del phi by del x. So, this phi if you take inside this derivative then you can write minus del of del x phi square by 2 and this we will write del E by del x square where E is equal to phi square by 2. So, now let us use the scheme whatever we have learnt before and apply to this nonlinear equation.

So, first we will consider let us say Lax method, so in the Lax method, you know that we will considered the forward time and central space, then we phi n plus 1, so first you write forward time and central space phi i n plus 1 minus phi i n divided by del t, so now governing equation is now del phi by del t is equal to minus del E by del x, so this we will write as minus so central difference we are using so it is n minus E i n sorry E minus i minus 1 divided by 2 delta x.

So, this is your forward time and central space method but you know that it is unconditionally unstable, so what we did we use this phi i n, phi i n as average of 2 neighbouring points value, so it will phi i n is equal to half of phi i plus 1 n plus phi i minus 1 n, so that is the modification in the Lax method and it will be conditionally stable.

So, if you put it this value here, then you will get phi i n plus 1 is equal to half phi i plus 1 n plus phi i minus 1 n minus del t by so this will be E so it will be and if you substitute E is equal to phi square by 2, then you will get 4 delta x phi i plus 1 at level n square minus phi i minus 1 at level n square, so this is your discretized equation.

And you need to solve this and the order of accuracy is delta t delta x square, so first order accurate in time and second order accurate in space and the it is conditionally stable, conditionally stable and condition for stability is condition for stability is del t by del x u max should be less than equal to 1.

So, we considered nonlinear equation and apply this Lax method which you learned during discretization of these linear hyperbolic equation, now the final algebraic equation is this one using Lax method for this nonlinear equation. So, now similar way you can use Mac Cormack method, so just now we have learned for today, so in 2 step method predictor and corrector that we can also use for this nonlinear problem.

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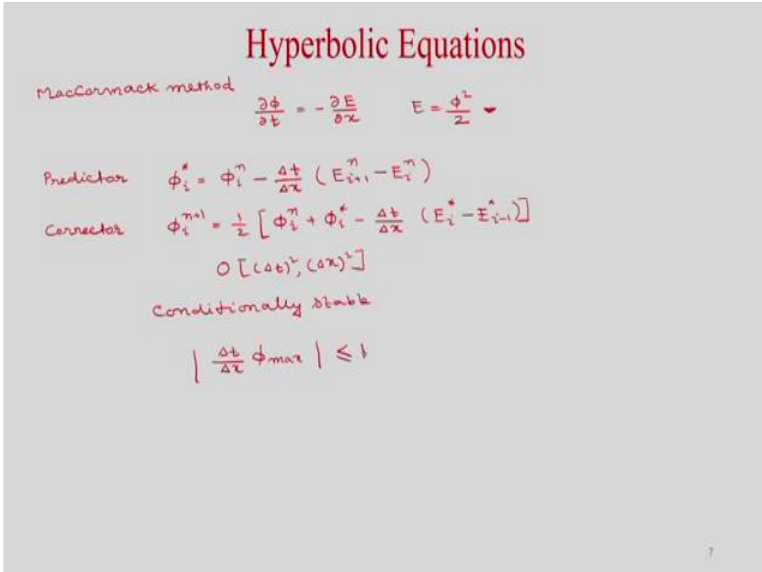

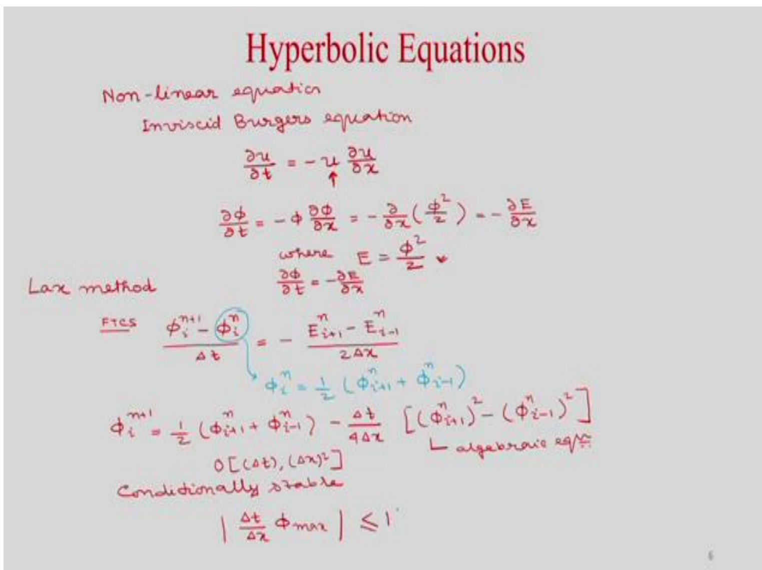

So, if you apply this Mac Cormack method, Mac Cormack method for this nonlinear equation where we considered the Burgers equation emissive Burgers equation del phi by del t minus del E by del x where E is equal to phi Square by 2. So, now in Mac Cormack method 2 steps we will do, so one is predictor, predictor step, so what will be that?

You will find phi i star, so phi i star is equal to phi i n minus del t by del x, E i plus 1 n minus E i n and in the corrector step you will find phi i n plus 1 is equal to half phi i n plus phi i star minus del t by del x E i star minus E i minus 1 star.

So, you can see that forward difference we have used in the predictor level for this special derivative and backward difference we have use in the corrector step for the special derivative and E you can replace later with phi square by 2. So, obviously the order of accuracy is second order in both time and space, so order of accuracy is del t square, del x square and it is conditionally stable,.

So, it is conditionally stable, conditionally stable and it is del t by del x phi max should be less than equal to 1. So, here this Lax method is conditionally stable and the condition is del t by del x phi max mod should be less than equal to 1.

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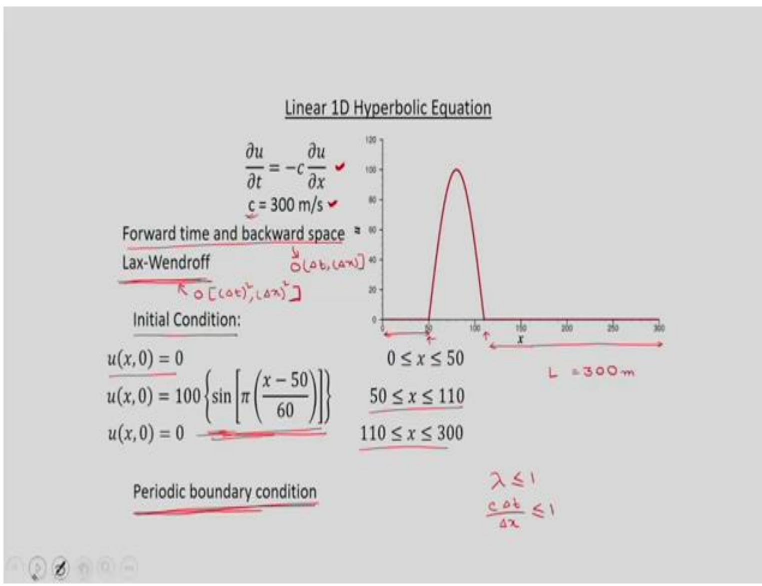

Now, let us consider one linear hyperbolic equation and we will use two discretization methods one is your forward time and backward space, which is first order accurate in time and space and we will use Lax-Wendroff method we will apply these two methods to solve this equation for the given initial condition and boundary condition and we will discuss about the computer program and later we will see the results.

So, this is the equation where c is we have taken 300 meter per second, 300 meter per second, so you can see that this is your linear equation, linear hyperbolic equation, so this is the domain 0 to 300, so this is your meter, so domain is the length of the domain is 300 meter and we are applying periodic boundary condition in the special direction. So, what does it mean? That whatever flow is going out that is coming in, so that we will apply periodic boundary condition for this problem.

And what is the initial condition is given? So, you can see that at t is equal to 0, u is 0 for 0 to 50, so up to this, you can see up to this you have u is equal to 0. And again 110 to 300 this value is also 0, so you can see from this, this is the initial condition, so this is also 0. But from 50 to 110, so you can see 50 to 110, it value is with this function, with this function, so this is the initial condition, so obviously with the wave speed c 300 meter per second, u will be convicted, you will be convicted and we have periodic boundary condition, so whatever will going out. So, again, it will start at the near to the inlet.

So, this we have used forward time and backward space and we have used also Lax-Wendoff method, Lax-Wendroff method. So, you can see that forward time and backward space, so this is your order of delta t and delta x first order accurate in time and space and Lax-Wendroff is order of accuracy is delta t square and delta x square. And both are having the condition for stability is lambda should be less than equal to 1. So, Lambda is c is the wave speed into delta t by delta x should be less than equal to 1. So, this is the condition for stability.

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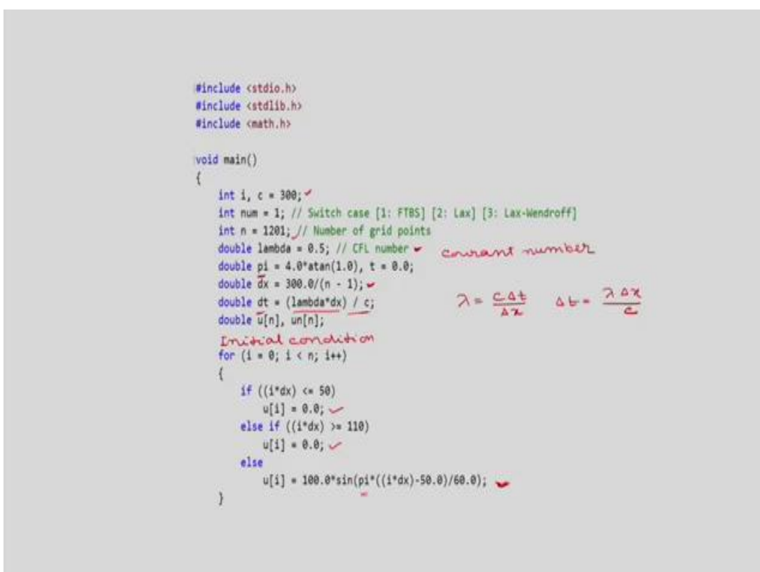

So, now if you write this equation, so first you consider that constant wave speed c is equal to 300 number of grid points taken as 1201 so you can find what is delta x, so delta x is 300 is the domain size divided by n minus 1 by 1200 and lambda is equal to CFL number, so that courant number, so this courant number you can take different value, we have taken different value which is less than equal to 1, so this is the courant number and pi just calculated to use it as a initial condition and delta t is lambda into delta x divided by c.

So, delta t is because lambda is your c delta t by delta x, so from here for a given delta x and c we are calculating delta t as lambda into delta x by c. And lambda value we have considered different value in the computation and now you can see here we have applied the boundary conditions, sorry this is your initial condition you have applied initial condition here, so you see up to 50 it is 0, from 110 to 300 it is 0 and from 50 to 110 this is the function whatever we have used. So, we have written here and this phi we have calculated here.

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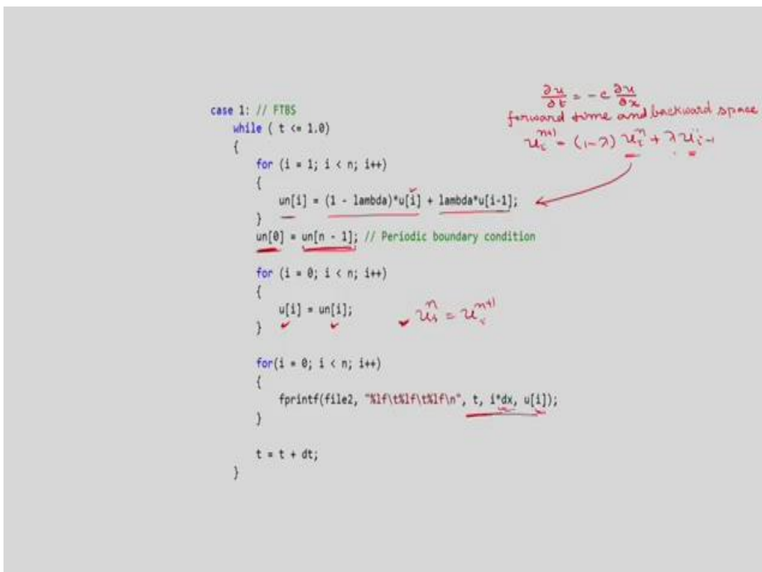

So, first we have considered forward time backwards space, because c greater than 0, so this will be conditionally stable and if you apply so del u by del t is equal to minus c del u by del x del u by del x and if you apply this FTBS team Forward Time, Forward Time and Backward Space, so what you will get? So, you will get finally you will get u i n plus 1 is equal to 1 minus lambda u i n plus lambda into u i minus 1 n.

So, these are all from previous time step value we can take, so that we have written you see, u i, u n i, this is the new value at n plus 1 is 1 minus lambda u i, u i is a previous value n plus lambda into u i minus 1, so these we have written here. And now periodic boundary condition we have applied that at x equal to 0 u n is equal to 0 is equal to u n n minus 1, so previous whatever is going out that we have taken as inlet velocity.

And these now we have saved the new value in the old value u i, because for the next time step we need to use it, so u n this new value of u n we have saved as u n minus 1, sorry n plus 1 i, because that will be used in the next time step and we have printed the valued t and the space and the velocity in a file.

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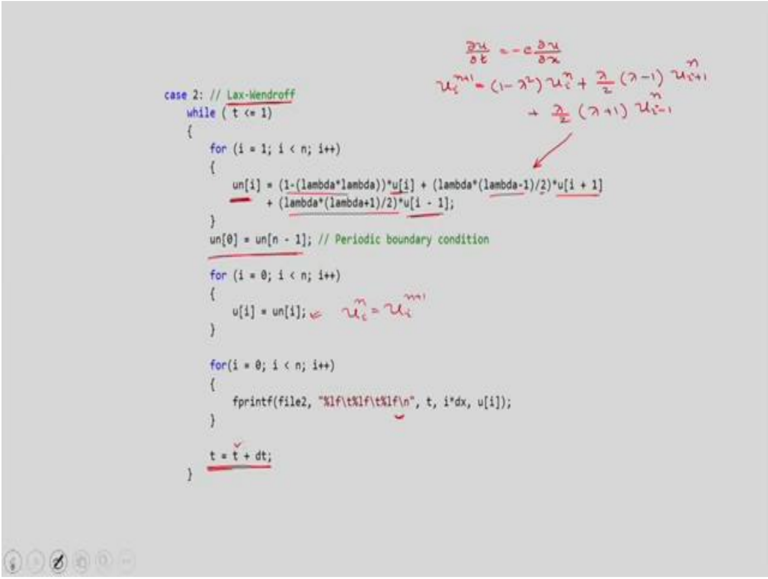

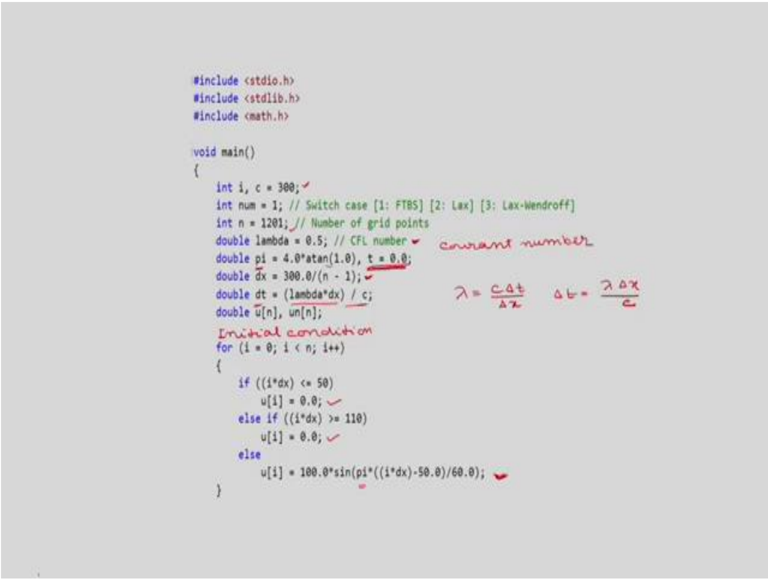

And similarly, we have used Lax-Wendroff method and Lax-Wendroff method, you know that it is second order accurate in time and space and if you discretize this equation what you will get so del u by del t is equal to minus E del u by del x, so if you use Lax-Wendroff method, Lax-Wendroff method, then what you will get? u i n plus 1 already we have discretized this equation in last lecture, so it will be 1 minus lambda square u i n minus lambda by 2, 1 minus lambda you can write plus then lambda you can take in the (())(34:18) lambda minus 1, u i plus 1 n plus lambda by 2, lambda plus 1, u i minus 1 n.

So, this is your Lax-Wendroff method and these we have written here, so this we are solving now you can see. So, u n i, so at n plus 1 we are now calculating at this current time level, 1 minus lambda square u i n, lambda we have already defined plus lambda by 2, lambda minus 1, u i plus 1 so it is previous value previous time step value plus lambda by 2 into lambda plus 1 and u i minus 1 TBS value.

Then we have applied the periodic boundary condition to update the u n at Inlet 0. So, for then we have saved again u i n is equal to u i n plus 1, because it will be used in the next time level and we have saved in the file and we are just increasing the time with t is equal to t plus delta t. And t should be 0 before starting the program, you can see here we have written t is equal to 0, t is equal to 0 you have written, t is equal to 0 we have written. So, you can see that this is the these two methods are explicit and we have solved this equation with the given initial condition.

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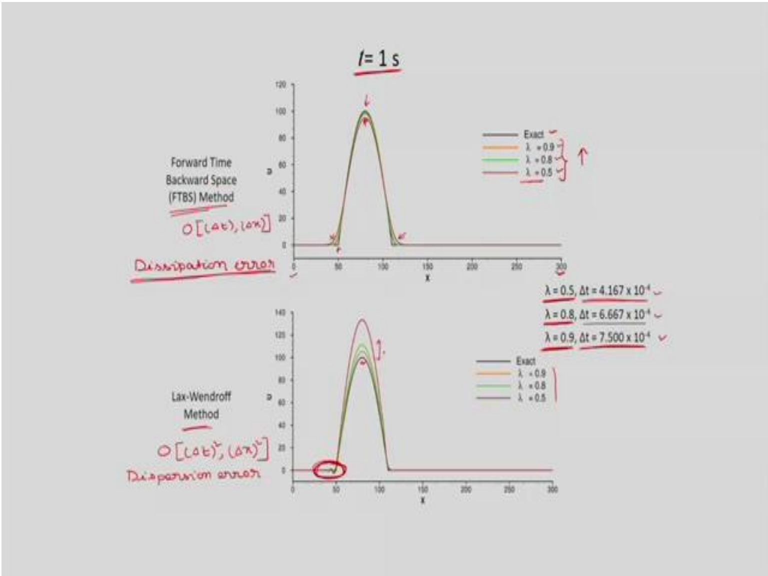

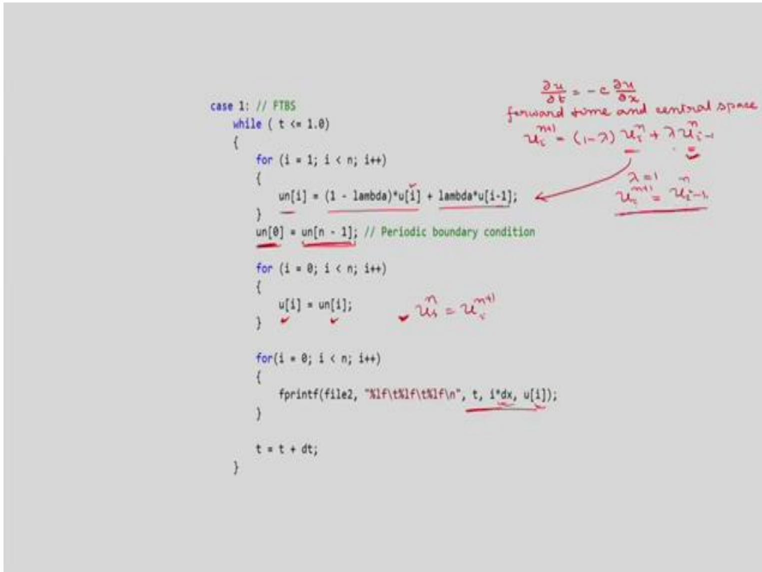

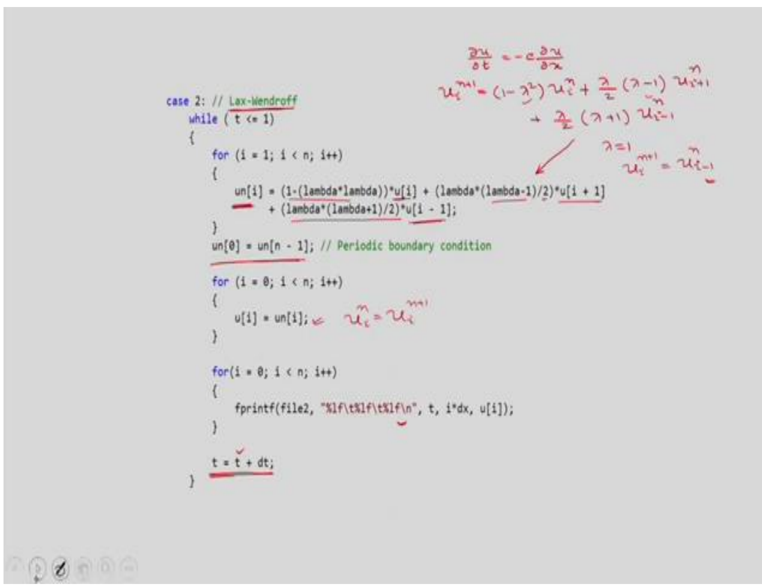

So, if you see the results, so you can see this is the forward time backwards space FTBS method we have used, we have plotted the u at time t is equal to 1 second, t is equal to 1 second. So, this is the domain 0 to 0 to 300, this is the domain and we have considered 3 different values of lambda, lambda is equal to 0.5, now delta x is fix because we have considered the grid as 1201, so delta t if you calculate you will get this 4.167 into 10 to the power minus 4.

So, if you increase the lambda then delta t will also increase so lambda is equal to 0.8 del t is this and lambda is equal to 0.9 which is close to the lambda is equal to 1, so this is the delta t. Now, if you go back and see the equation, this is the discretized equation, if Lambda is close to 1 or equal to 1 what you will get? You see u i n plus 1 is equal to u i minus 1, so this is the exact solution, this is the exact solution you will get, u i n plus 1 is equal to u i minus 1 n. So, it will be exact solution.

Similarly, in the Lax-Wendroff method you can see lambda is equal to 1 if you put then these 2 will vanish is this will vanish, here also, you will get the exact solution, u i n plus 1 is equal to u i minus 1 n, because this will become 1 plus 1 2 and 2 by 2 1, so it will be u i minus 1 n. So, if lambda value is close to 1 it will give the exact solution and obviously if you decrease the value of lambda then accordingly you will get some error in the solution.

So, you can see here, so now we are considering a FTBS, Forward Time Backwards Space, which is your accuracy is delta t delta x first order accurate in time and space and this is the exact solution in black colour we have shown this is the black colour, so this is the black colour, so you can see here and here top, so here it is exact solution. Now, these are the computed solutions, for different values of lambda, lambda is equal to 0.9, 0.8 and 0.7 and corresponding delta t I have written here.

And if you can see that if you are decreasing here lambda, this is giving more error and this error will be dissipation error, dissipation error because what you have neglected that is order of delta t and that will actually contribute in the dissipation error, so you can see the solution exact solution from exact solution there is a some dissipation errors at there and you can see there is a deviation.

So, for lambda is equal to 0.5, this is the lambda is equal to 0.5 and you can see that there is some error and when you increasing the lambda it is going to the exact solution. Similarly, for Lax-Wendroff method you can see, which is order of delta t square delta x square, so in the leading order term is delta t square and delta x square, so it will give dispersion error, it will give dispersion error.

So, some wiggles will be there in the solution and you can see here, there is some oscillations. So, this is your dispersion error and obviously your exact value is lambda this black colour and for different value of lambda, these are the deviation and you are getting the deviation in the results.

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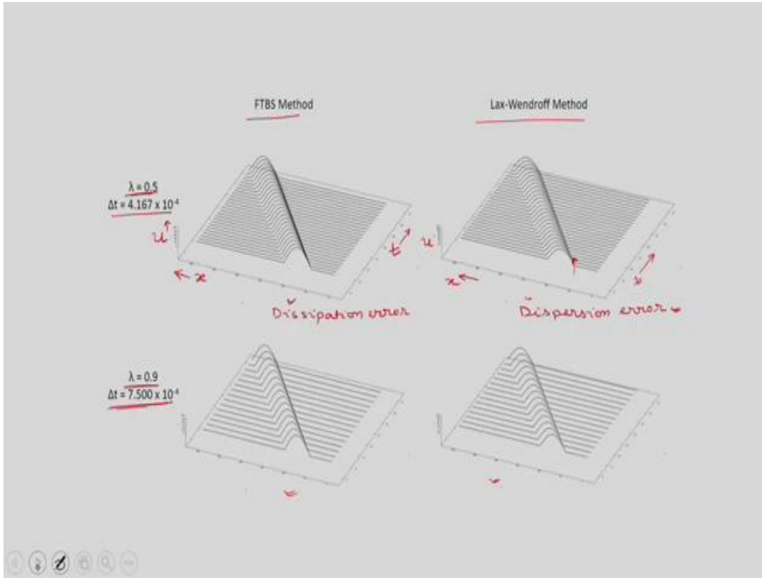

So, it is a 3D plot, so you can see how so this is your FTBS method and this is your Lax-Wendroff method for different 2 different lambda value, lambda is equal to 0.5, where delta t is given as these and for lambda is equal to 0.9, the delta it is given like this and you can see this is your u, so the this axis is u, this axis is u, this axis is x and this axis is t.

So, at different time now what is happening? So, as t increases, it is propagating the velocity is convicted, so you can see so it is in the decision positive x direction is this way and t is in this way, so you can see it is propagating in this direction. And here for Lax-Wendroff method also you can see same thing, so this is your u this is your x and this is your t time.

So, at different time instances you can see and you can see closely if you see here you will find some wiggles, which is coming from the dispersion error and it is for different lambda. So, lambda is equal to 0.9, so it will give almost exact solution for these 2 cases, but here the errors are more in this case, you have more dispersion error and in this case you have sorry in this case you have dissipation error and in this case you have dispersion error.

So, it is contributing from the leading order term, whatever in the truncation error you have, so this is order of delta t and delta x it is order of delta t square and delta x square, so accordingly you will get the errors. So, you can write some simple programs for different boundary conditions and you can plot this u versus says x plot at different time level.

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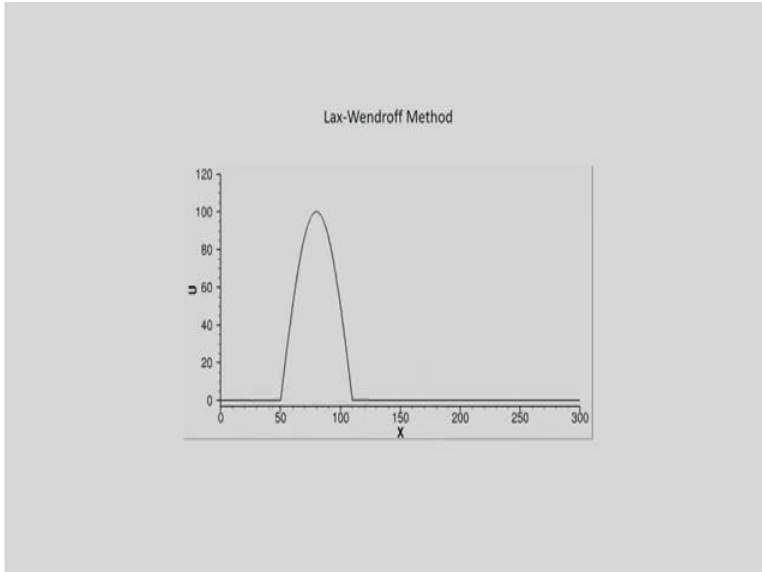

And in the next slide, I am going to show one animation how it is moving this the constant speed u u is convicted u is convicted with the wave speeds, constant wave speed c, again I am you can see how it is propagating. So, u is convicted with the wave speed c is equal to 300 meter per second. So, you can see it is moving it is going out, so it is periodic boundary conditions. So, with this today will conclude that we have considered first linear hyperbolic equation, and we considered explicit and implicit method and discretize that equation.

For explicit method in most of the cases it is conditionally if it is conditionally stable then the condition for stability is lambda is less than equal to 1 and in implicit method we consider the backward time central space and Crank-Nicholson method which is which are actually unconditionally stable. And then we considered nonlinear hyperbolic equation and we discretized using 2 different methods Lax-Wendroff and Mac Cormack method.

And then we have shown the computer program for linear hyperbolic equation and we use 2 different explicit methods one is your FTBS method Forward Time and Backward Space, which is faster and accurate in time and space. And then we considered Lax-Wendroff method which is second order accurate in time and space. Thank you.