Computational Fluid Dynamics for Incompressible Flows
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Lecture- 2
Finite difference formulations of the first order wave equation: Implicit Method

Hello everyone, so in last lecture we have considered linear hyperbolic equations and we considered on first order wave equation and we used different explicit discretization method and discretize that equation. In today’s lecture will consider the same equation and will use implicit method. So, today lecture 2 finite difference formulations in the first order wave equation, Implicit Method.

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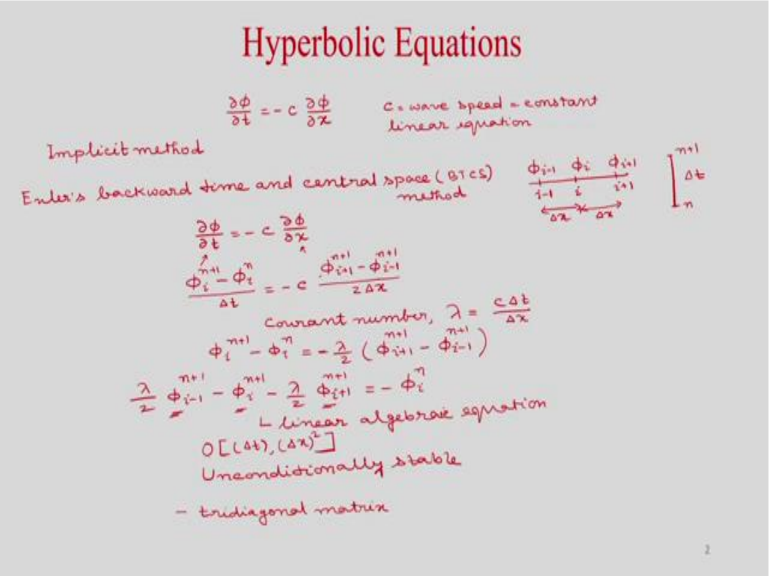

So, will consider the same equation that is your model equation del phi by del t is equal to minus c del phi by del x. So, obviously this is your wave speed and it is constant so this equation is linear, so c is wave speed so this is linear equation. So, will consider implicit method so as you know that in the implicit method there will be more than one unknown.

So, will consider the grid as i, i plus 1, i minus 1 and the dependant variable phi i, phi i plus 1 and phi i minus 1 and uniform step size delta x. And as will march in time so will go from n to n plus 1 with time step delta t.

So, let us consider first Euler's Backward Time and Central Space so BTCS method. So, in this method this time derivative will consider backward difference approximation and for the special derivative will consider central difference approximation. So, the equation is del phi by del t is equal to minus c del phi by del x so if you apply backward difference approximation for this time derivative then you can write, phi i n plus 1 minus phi i n divided by delta t which is your first order accurate in time is equal to minus c.

So, del phi by del x will consider central difference approximation. So, will use phi i plus 1, minus phi i minus 1 divided by 2 delta x and here you can see that as we are using implicit method and backward time so the dependant variable will be at time label n plus 1. So, you can see that there are more than one unknown.

If you rearrange it so all phi i plus 1 at time label n plus 1 you take in the left hand side and define the Courant number as lambda is equal to c delta t by delta x. So, you can see that you can write this equation as so it will phi i n plus 1 minus phi i n is equal to minus lambda by 2, phi i plus 1 n plus 1 minus phi i minus 1 n plus 1.

Now, you take all the n plus 1 time label dependant variable in the left hand side and rearrange it, you will get lambda by 2 phi i minus 1 n plus 1 minus phi i n plus 1 minus lambda by 2, phi i minus 1 sorry i plus 1 n plus 1 is equal to minus phi i n. So, for a discrete point i, we have discretize this first order wave equation and we got this linear algebraic equation, linear algebraic equation.

So, what is the order of accuracy? Because we have seen that we have used first order accurate in time. And central difference we have used, so second order accurate in space so order of accuracy is delta t delta x square. And if you do the stability analysis then, we can show using Von Neumann stability analysis that this method is unconditionally stable so this is your unconditionally stable.

So, you can see that in the left hand side there are three unknowns phi i minus 1 n plus 1, phi i n plus 1 and phi i plus 1 n plus 1. And at each grid point if you discretize this equation you will get a system of linear algebraic equation and if you write in a matrix format then you will get a tridiagonal matrix. So, you will get a tridiagonal matrix and you will be able to solve using Thomas algorithm.

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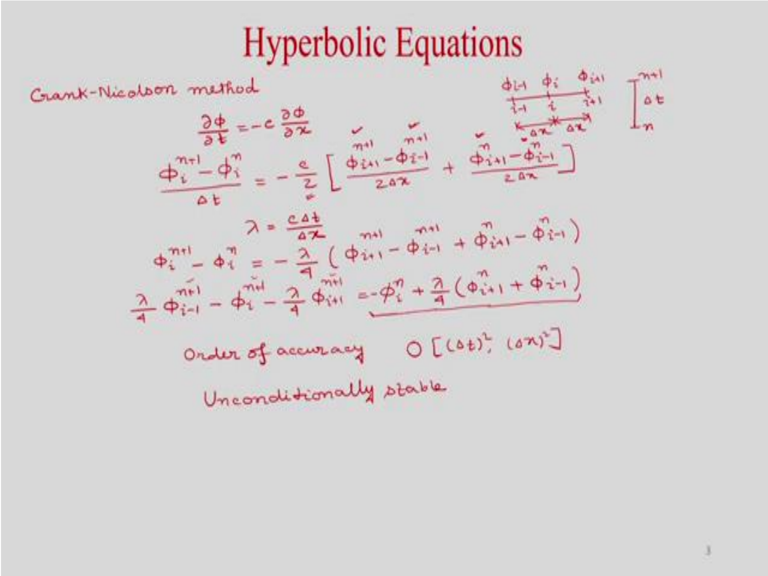

Next, let us consider another method which you know, Crank-Nicolson method so which is second order accurate in time and space. So, will use Crank Nicolson method which is your implicit method Crank-Nicolson method so will use this grid uniform step size and n to n plus 1 we are marching so delta t is your time step.

Your equation is del phi by del t is equal to minus c del phi by del x. So, in Crank-Nicolson method what we will do? So, the special derivative we take average of explicit and implicit kind of thing. So, you write the left hand side as phi i n plus 1 minus phi i n divided by delta t is equal to minus c by 2, half we are taking so this is your phi i plus 1 minus phi i minus 1 divided by 2 delta x. So, you can take at n plus 1 label and another you can take phi i plus 1 minus phi i minus 1 divided by 2 delta x at time label n.

So, you can see this is your n plus 1 label and this is your n th label so average of this we are taking so half we have written here, so this is Crank-Nicolson method. So, if you rearrange it and all the n plus 1 time label dependant variable you take in the left hand side. And define the Courant number lambda is equal to c delta t by delta x then you can write this equation as, so phi i n plus 1 minus phi i n is equal to minus lambda by 4.

Minus lambda by 4 two delta x if you take outside then you can write phi i plus 1 n plus 1, minus phi i minus 1 n plus 1 plus phi i plus 1 n minus phi i minus 1 n. Then if you rewrite it so you can write as, lambda by 4 phi i minus 1 n plus 1 minus phi i n plus 1 minus lambda by 4 phi i plus 1 n plus 1 and in right hand side, all the dependant variables at time label n.

So, this is your i, plus lambda by 4 phi i plus 1 n plus phi i minus 1 n. So, we can see in right hand side this is your known term only left hand side at n plus 1 you have so these are unknown and obviously at each grid point if you write this equation then you will get a system of linear algebraic equation.

Which actually will give you a matrix format a tridiagonal matrix and you can solve it using the Thomas algorithm. And what is the order of accuracy? Order of accuracy obviously it is delta t square and delta x square. So, it is second order accurate in time as well as in space. So, order of accuracy is order of delta t square and delta x square and if you carry out this Von Neumann stability analysis of this method, then we can show that it is unconditionally stable, so it is a stable scheme.

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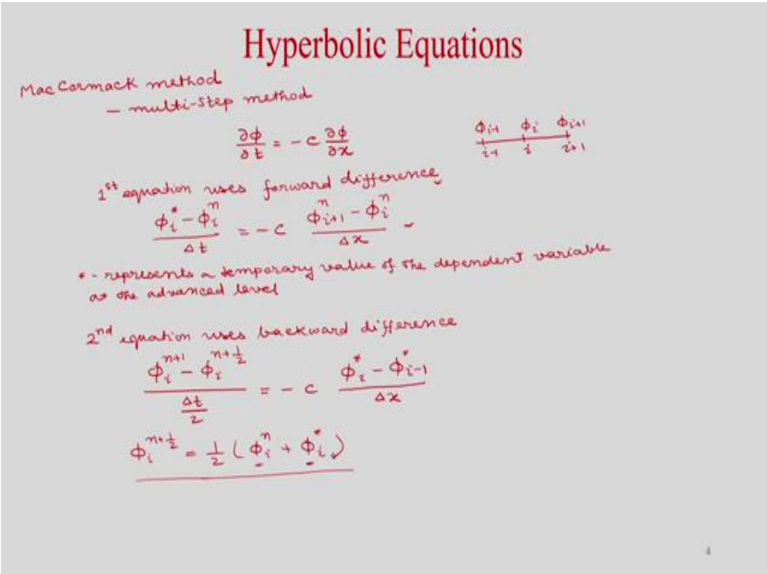

So, now in next will consider on multi-step method, next will consider a multi-step method which is known as Mac-Cormack method, it is a multi-step method. So, in this Mac-Cormack method what we will do? This equation will discretize first using forward difference then will use backward difference.

So, you can see what is our model equation del phi by del t is equal to minus c del phi by del x. So, will use first forward difference, then backward difference in next step will consider backward difference. So, first equation uses forward difference so forward difference in space special derivative, so forward difference in special derivative we will use.

So, we will use phi i star minus phi i n divided by del t is equal to minus c phi i plus 1 n minus phi i n divided by delta x. So, this star actually represents a temporary value of the dependant variable at the advanced level so in the next step will use this value. Now, in second equation will use the backward difference of this special derivative. So, second equation uses backward difference, what you can do vice versa first you can use backward difference, then you can use forward difference.

So, what will use here, phi i n plus 1 minus phi i n plus half divided by del t by 2 is equal to minus c phi i star minus phi i minus 1 star. So, you can see for this special derivative we are using backward difference. So, but phi i at n plus half is not known so that will use the average value of this.

So, phi i n plus half will use the average value of phi i n and phi i star so star you have calculated in the first level so in the second level you calculate phi i n plus 1 but phi i n plus half you take half plus sorry phi i n plus half is average of phi i n and phi i star. So, phi i star you calculate from the first level equation.

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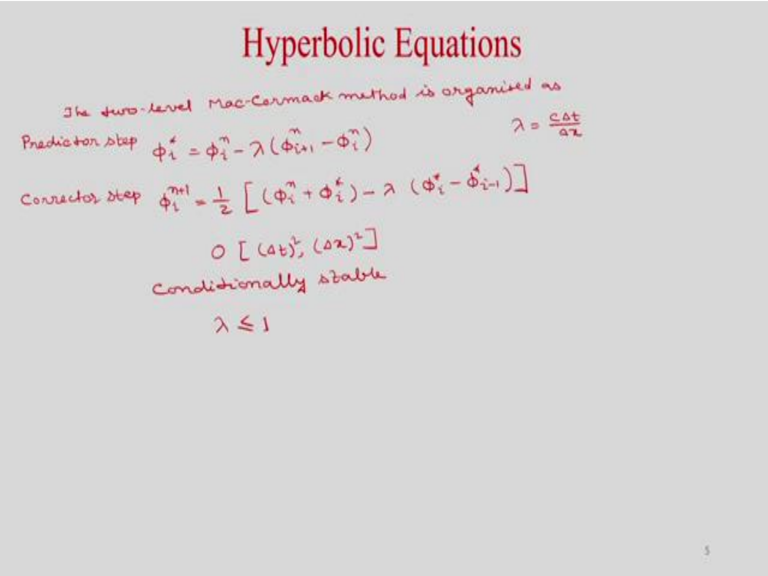

So, this method if you write in two steps so this is known as Predictor-corrector the two level Mac-Cormack method is organised as so first level whatever we have written. So, you write phi i star so this is your Predictor step, you are predicting phi i star then that you are using in the second level.

So, phi i star is equal to phi i n minus lambda where lambda is equal to c delta t by delta x phi i plus 1 n minus phi i n. So, that you calculate phi i star from here, then in the character step you find the value of phi i n plus 1 as half phi i n plus phi i star minus lambda phi i star minus phi i minus 1 star.

So, in this equation you can see that in first equation or first level or in the Predictor level, we have used forward difference in the special derivative. But, in the second level in the corrector step we have used backward difference in the special derivative. But you can do in the vice versa way. So, you can first use backward difference in a first level and in a second level you can use forward difference. So, what is the order of accuracy? Order of accuracy is delta t square and delta x square. So, it is second order accurate in both time and space. So, order of accuracy is delta t square and delta x square.

But, if you do the Von Neumann stability analysis of this method, then you will find that it is conditionally stable and the condition is courant number should be less than equal to 1. So, it is conditionally stable and condition is lambda should be less than equal to. So, we consider linear equation linear hyperbolic equation and use different explicit and implicit methods.

Now, next let us consider non-linear equation and will use just few discretization scheme all what already we have learnt and discretize this non-linear equations. So, when this equation will become non-linear? When c is not constant. So, when c is not constant that time this equation will become non-linear.

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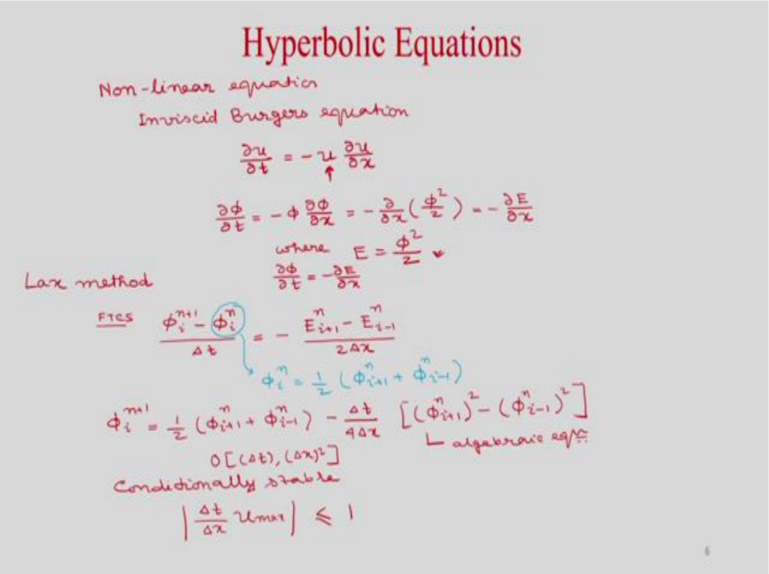

So, will consider Inviscid Burgers equation. So, will consider non-linear equation so first will consider Inviscid Burgers equation so we can represent as, del u by del t is equal to minus u del u by del x. So, you can see this is your wave speed and at different instances this velocity or wave speed is changing. So, you will be because from the from this equation you will find and that you will be used here so it is non-linear in nature.

Now, let us write this equation for general variable phi so that will be our model equation. So, you can write del phi by del t is equal to minus phi del phi by del x. So, this phi if you take inside this derivative then you can write minus del of del x phi square by 2 and this will write del E by del x where, E is equal to phi square by 2. So, now let us use the scheme whatever we have learned before and applied to this non-linear equation, so first will consider let us say lax method. So, in the lax method you know that, we consider the forward time and central space.

Then we phi n plus 1 so first you write forward time and central space phi i n plus 1 minus phi i n divided by del t. So, now governing equation is now, del phi by del t is equal to minus del E by del x. So, this will write as minus so central difference we are using so it is n minus E i n sorry i minus 1 divided by 2 delta x. So, this is your forward time and central space method. But, you know that it is unconditionally unstable so what we did? We used this phi i n as average of two neighbouring points value.

So, it will be phi i n is equal to half of phi i plus 1 n plus phi i minus 1 n, so that is the modification in the lax method and it will be conditionally stable. So, if you put it this value here then you will get phi i n plus 1 is equal to half phi i plus 1 n plus phi i minus 1 n minus del t by so this will be E.

So, it will be and if you substitute E is equal to phi square by 2, then you will get 4 delta x phi i plus 1 at label n square minus phi i minus 1 at label n square so this is your discretized equation. And you need to solve to this and the order of accuracy is delta t delta x square so first order accurate in time and second order accurate in space.

And the it is conditionally stable and condition for stability is, del t by del x u max should be less than equal to 1. So, we consider non-linear equation and applied this lax method which you learnt during discretization of this linear hyperbolic equation. Now, the final algebraic equation is this one using lax method for this non-linear equation. So, now similar way you can use Mac-Cormack method, so just now we have learnt for today so in two step method. Predictor and corrector that we can also use for this non-linear problem.

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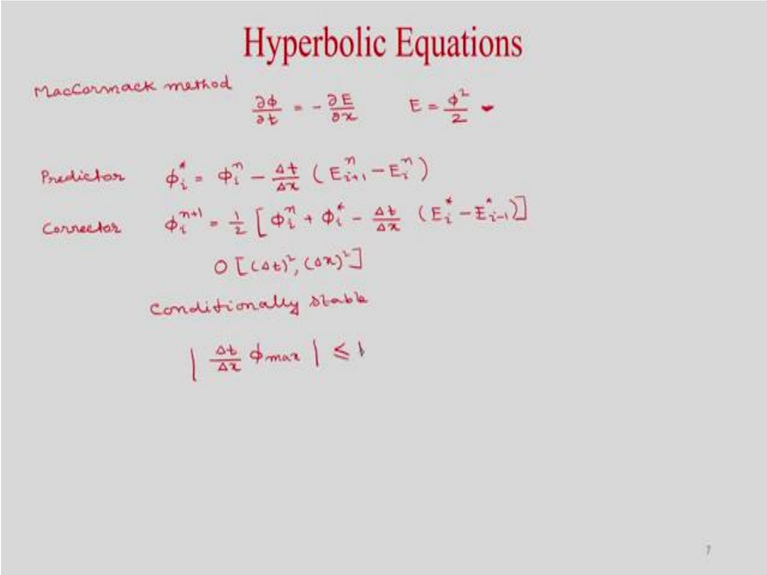

So, if you apply this Mac-Cormack method for this non-linear equation where we consider the Burgers equation, Inviscid Burger equation, del phi by del t minus del E by del x where, E is equal to phi square by 2. So, now in Mac-Cormack method two steps will do so one is, Predictor step. So, what will be that, you will find phi i star so phi i star is equal to phi i n minus del t by del x, E i plus 1 n minus E i n.

And in the corrector step you will find, phi i n plus 1 is equal to half phi i n plus phi i star minus, del t by del x E i star minus E i minus 1 star. So, you can see that forward difference we have used in the Predictor level, for this special derivative and backward difference we have used in the corrector step for the special derivative and E you can replace later with phi square by 2.

So, obviously the order of accuracy is second order in both time and space so order of accuracy is del t square, del x square and it is conditionally stable so it is conditionally stable and it is del t by del x phi max should be less than equal to 1. Sorry here this lax method is conditionally stable and the condition is del t by del x phi max mod should be less than equal to 1.

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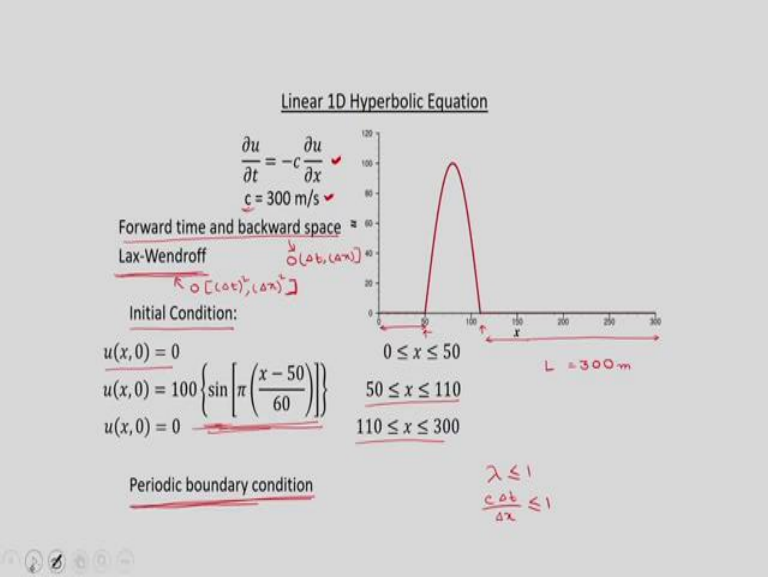

Now, let us consider one linear hyperbolic equation and will use two discretization methods, one is your forward time and backward space which is first order accurate in time and space. And will use Lax-Wendroff method will apply these two methods to solve this equation for the given initial condition and the boundary condition. And will discuss about the computer programme and later will see the results.

So, this is the equation where c is we have taken 300 meter per second so you can see, that this is your linear equation, linear hyperbolic equation. So, this is the domain 0 to 300 so this is your meter, so domain is the length of the domain is 300 meter and we are applying periodic boundary condition in the special direction.

So, what does it mean? That whatever flow is going out that is coming in so that will apply periodic boundary condition for this problem and what is the initial condition is given? So, you can see that at t is equal to 0, u is 0 for 0 to 50. So, up to this you have u is equal to 0 and again 110 to 300 this value is also 0.

So, you can see from this, this is the initial condition so this is also 0. But, from 50 to 110 so you can see 50 to 110 it varies with this function. So, this is the initial condition so obviously with the wave speed c 300 meter per second u will be convected and we have periodic boundary condition. So, whatever will going out so again it will start at the near to the inlet. So, this we have used forward time and backward space and we have used also Lax-Wendroff method.

So, you can see that forward time and backward space so this is your order of delta t and delta x first order accurate in time and space and Lax-Wendroff is order of accuracy is delta t square and delta x square. And both are having the condition for stability is lambda should be less than equal to 1 so lambda is, c is the wave speed into delta t by delta x should be less than equal to 1. So this is the condition for stability.

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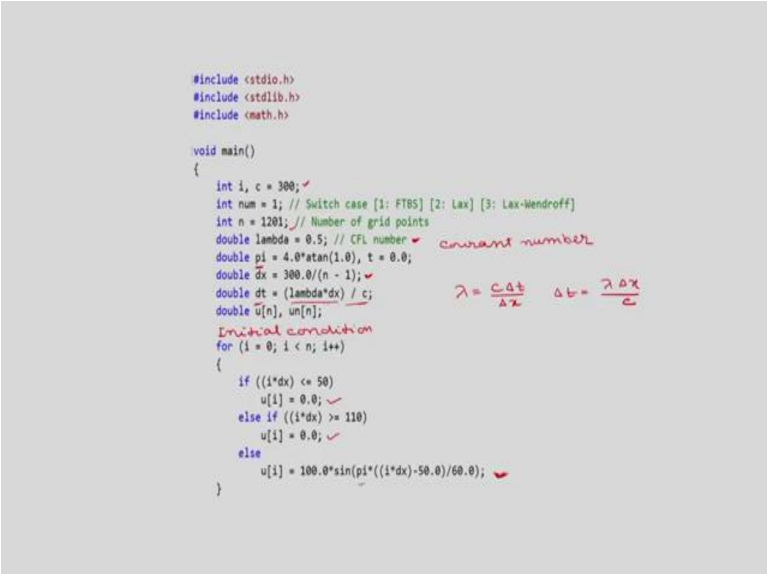

So, now if you write this equation so first you consider that constant wave speed c is equal to 300. Number of grid points taken as 1201 so you can find what is delta x. So, delta x is 300 is the domain size divided n minus 1 by 1200 and lambda is equal to CFL number so that courant number so this courant number you can take different value.

We have taken different value which is less than equal to 1, so this is the courant number and pi we have just calculated to use it as an initial condition. And delta t is lambda into delta x divided by c, so delta t is because lambda is your c delta t by delta x. So, from here for a given delta x and c we are calculating delta t as lambda into delta x by c.

And lambda value you have considered different values in the computation and now you can see here, we have applied the boundary conditions sorry this is your initial condition. We have applied initial condition here so the uc up to 50 it is 0 from 110 to 300 it is 0 and from 50 to 110 this is the function whatever we have used. So, we have written here and this Pi we have calculated here.

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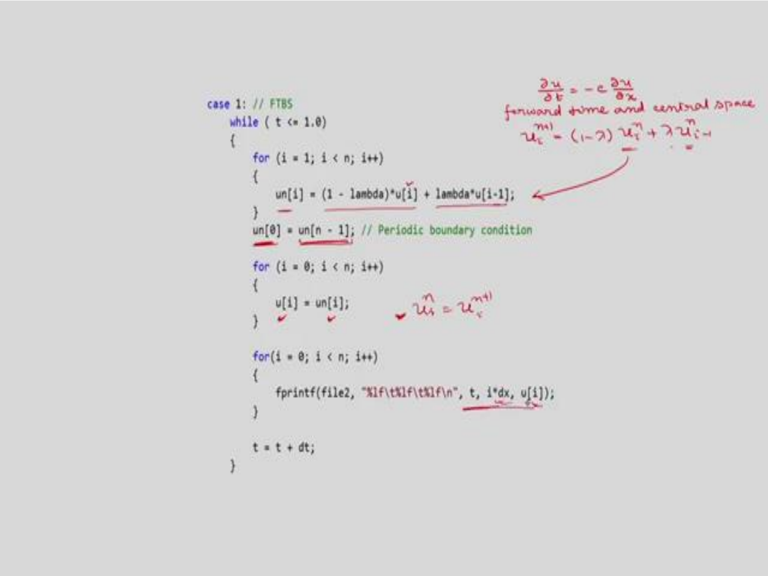

So, first we have consider forward time forward space because c greater than 0 so this will be conditionally stable and if you apply. So, del u by del t is equal to minus c del u by del x and if you apply this FTBS scheme, Forward Time and Central Space, so what you will get? So, you will get finally you will get, u i n plus 1 is equal to 1 minus lambda u i n plus lambda into u i minus 1 n.

So, these are all from previous time step value you can take so that we have written u c. u n i this is the new value at n plus one is 1 minus lambda u i, u i is your previous value in plus lambda into u i minus 1. So, this we have written here. And now periodic boundary condition we have applied that at x equal to 0, un is equal to 0 is equal un n minus 1.

So, previous whatever is going out that we have taken as inlet velocity and this now we have saved the new value in the old value u i because for the next time step we need to use it. So, un this new value of un we have saved as u n minus 1 sorry n plus 1 i because that will be used in the next time step and we have printed the value t and the space and the velocity in a file.

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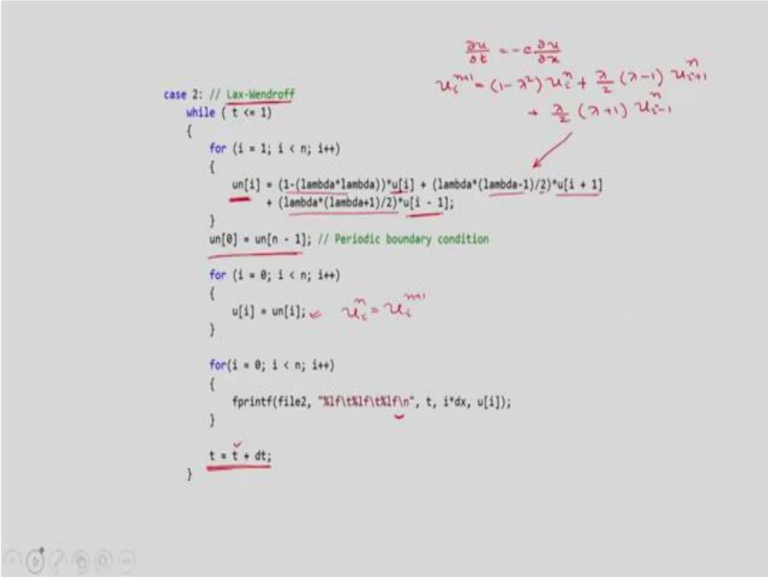

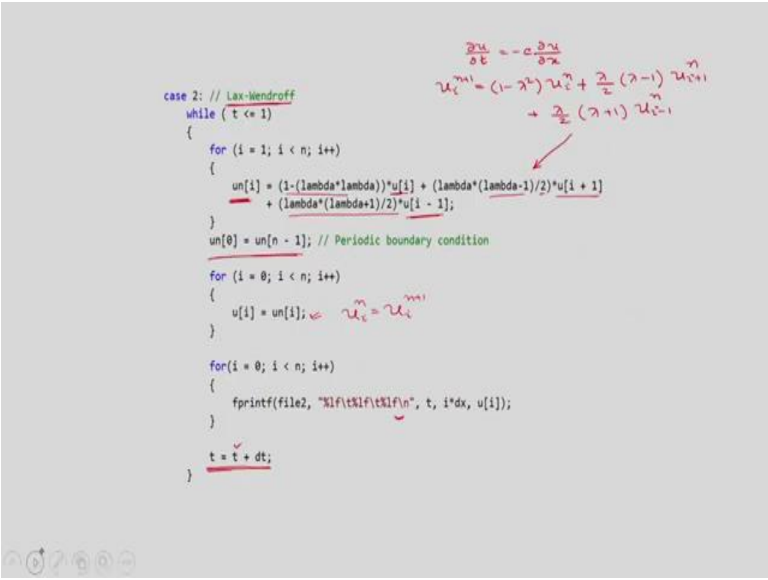

And similarly we have used Lax-Wendroff method and Lax-Wendroff method you know that it is second order accurate in time and space and if you discretize this equation, what you will get? So, del u by del t is equal to minus c del u by del x. So, if you used Lax-Wendroff method then what you will get? u i n plus 1 already we have discretize this equation in last lecture.

So, it will be 1 minus lambda square u i n minus lambda by 2, 1 minus lambda you can write plus then lambda you can take in the so it will be lambda minus 1, u i plus 1 n plus lambda by 2 lambda plus 1, u i minus 1 n. So, this is your Lax-Wendroff method and this we have written here so this we are solving now you can see.

So, un i so at n plus 1 we are now calculating at this current time label, 1 minus lambda square u i n lambda we have already defined, plus lambda by 2, lambda minus 1, u i plus 1. So, it is previous value, previous time step value plus lambda by 2 into lambda plus 1 and u i minus 1 previous value. Then we have applied the periodic boundary condition to update the un at inlet 0.

So, for then we have saved again u i n is equal to u i n plus 1 because it will be used in a next time level and we have saved in a file and we are just increasing the time with t is equal to t plus delta t. And t should be 0 before starting the programme, you can see here we have written t is equal to 0, t is equal to 0 we have written. So, you can see that this the, these two methods are explicit and we have solved this equation with the given initial condition.

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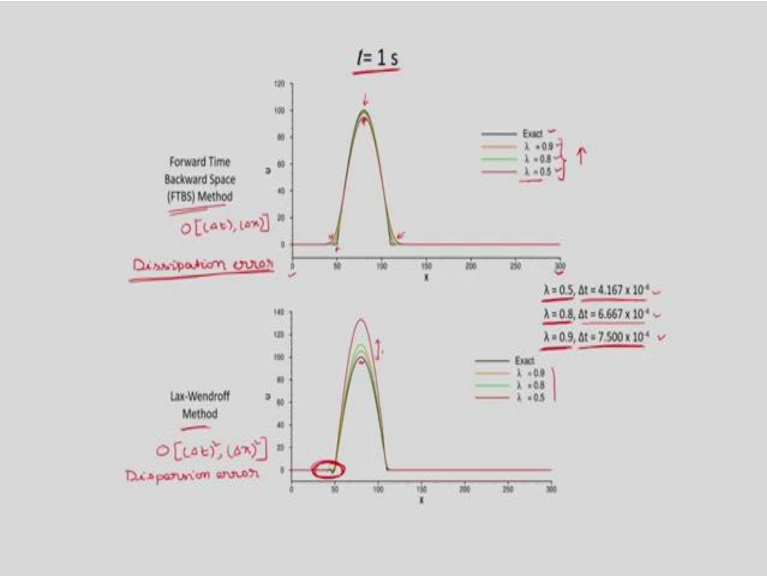

So, if you see the results so you can see this is the forward time backward step, FTBS method we have used. We have plotted the u at time t is equal to one second, so this is the domain 0 to 300 this is the domain and we have consider 3 different values of lambda. Lambda is equal to 0.5 now delta x is fixed because we have consider the grid as 1201.

So delta t if you calculate, you will get this 4.167 into 10 to the power minus 4. So, if you increase the lambda then delta t also will increase so lambda is equal to 0.8 delta t is this. And lambda is equal to 0.9 which is closed to the lambda is equal to 1 so this is the delta t.

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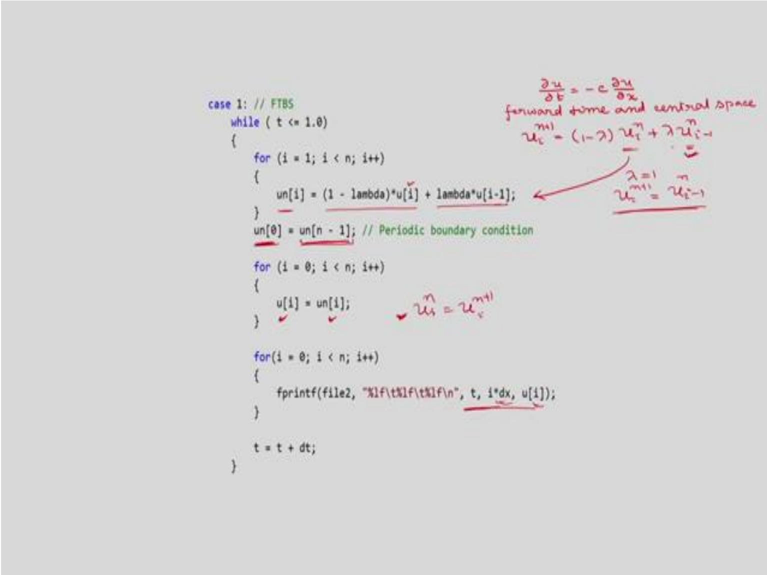

Now, if you go back and see the equation this is the discretize equation, if lambda is closed to 1 or equal to 1, what you will get? You see, u i n plus 1 is equal to u i minus 1. So, this is the exact solution you will get, u i n plus 1 is equal to u i minus 1 n. So, it will be exact solution.

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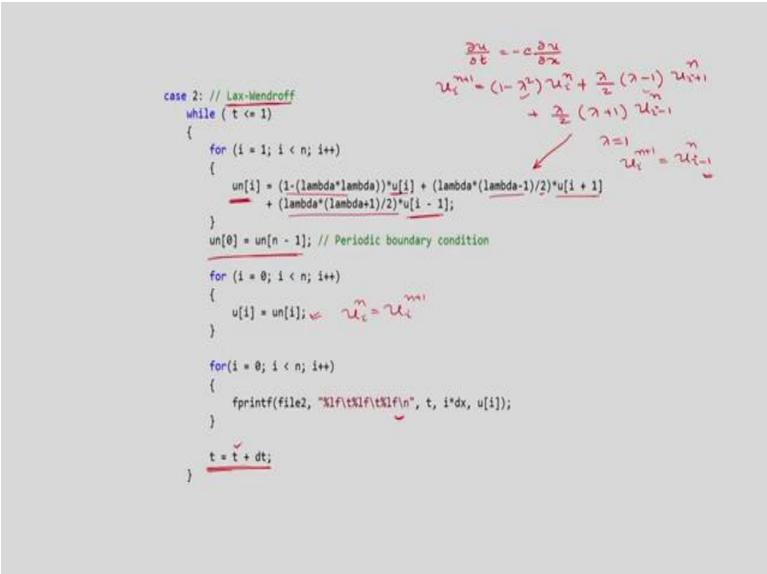

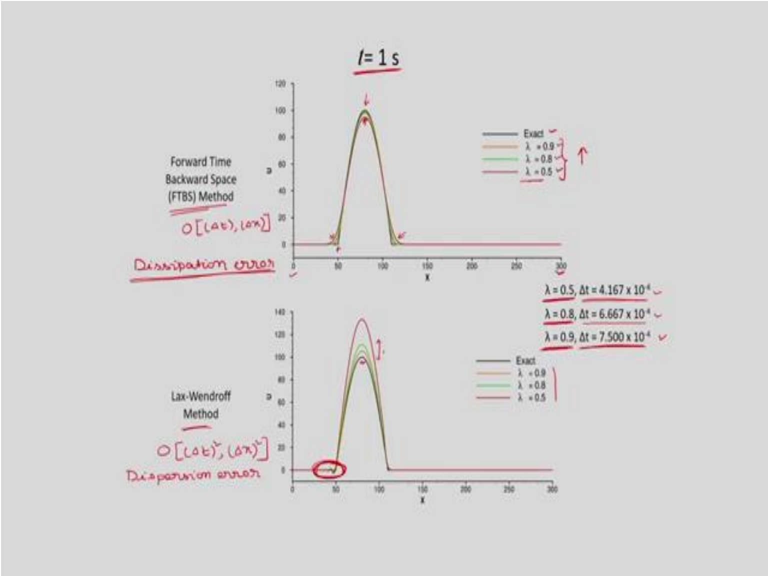

Similarly, in the Lax-Wendroff method you can see lambda is equal to 1 if you put then this too will vanish, this will vanish here also you will get the exact solution. u i n plus 1 is equal to u i minus 1 n because this will become 1 plus 1 2 and 2 by 2 1, so it will be u i minus 1 n. So, if lambda value is closed to 1 it will give the exact solution and obviously if you decrease the value of lambda then accordingly you will get some error in the solution.

So, you can see here so now we are considering FTBS, Forward Time Backward Space which is your accuracy is delta t delta x first order accurate in time and space. And this is the exact solution in black colour we have shown, this is the black colour so this is the black colour. So, you can see here and here top so here it is exact solution.

Now, these are the computed solutions for different values of lambda, lambda is equal to 0.9, 0.8 and 0.7 and corresponding delta t I have written here. And if you can see that if you are decreasing their lambda this is giving mod error and this error will be dissipation error. Because what you have neglected that is order of delta t and that will actually contribute in the dissipation error.

So, you can see the solution exact solution from exact solution there is a some dissipation errors at there and you can see there is a deviation. So, for lambda is equal to 0.5 this is the lambda is equal to 0.5 and you can see that there is some error and when you increasing the lambda it is going to the exact solution.

Similarly, for Lax-Wendroff method you can see which is order of delta t square, delta x square. So, in a leading order term is delta t square, delta x square so it will give dispersion error it will give dispersion error. So, some wiggles will be there in the solution and you can see here, there is some oscillations.

So, this is your dispersion error and obviously your exact value is lambda this black colour and for different value of lambda these are the deviation and you are getting the deviation in the results.

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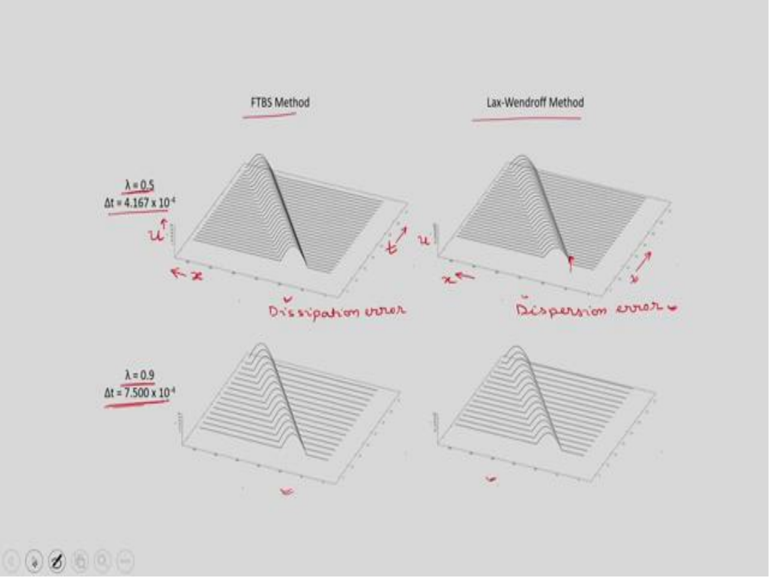

So, it is a 3-D plot, so you can see how so this is your FTBS method and this is your Lax-Wendroff method for different two different lambda value, lambda is equal to 0.5. Where delta t is given as this and for lambda is equal to 0.9, the delta it is-is given like this and you can see this is your u.

So, the this axis is u, this axis is x and this axis is t. So, at different time now what is happening so as t increases, it is propagating the velocity is convected. So, you can see so it is in a this is your positive x direction is this way and t is in this way so you can see it is propagating in this direction.

And here for Lax-Wendroff method also you can see something so this is your u, this is your x and this is your t time. So, at different time instances you can see and you can see closely if you see here you will find some wiggles, which is coming from the dispersion error and it is for different lambda.

So, lambda is equal to 0.9, so it will give almost exact solution for these two cases but here the errors are more. In this case you have mod dispersion error and in this case you have dissipation error and in this case you have dispersion error. So, it is contributing from the leading order term, whatever in the truncation error you have so this is order of delta t and delta x. It is order of delta t square and delta x square so accordingly you will get the errors. So, you can write some simple programs for different boundary conditions.

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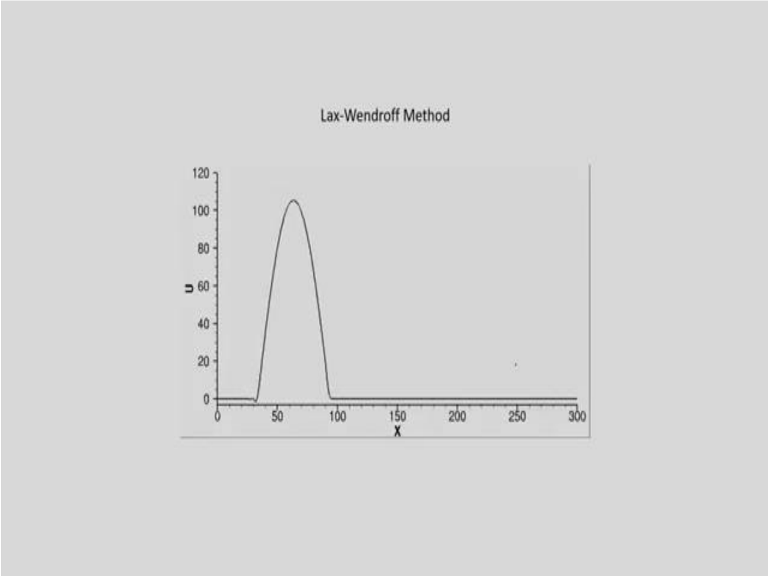

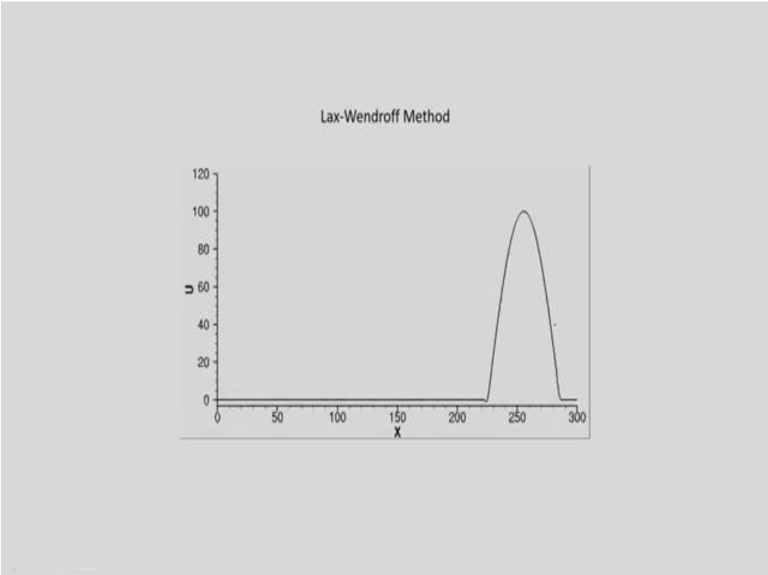

And you can plot this u versus x plot at different time level and in a next slide I am going to show one animation, how it is moving this the constant speed u is convicted u is convicted with the wave speed constant wave speed c. Again you can see how it is propagating, so u is convected with the wave speed c is equal to 300 meter per second.

So, you can see it is moving, it is going out so it is periodic boundary conditions. So, with this today will conclude that we have considered a first linear hyperbolic equation and we considered explicit and implicit method and discretize that equation. For, explicit method in most of the cases, if it is conditionally stable then the condition for stability is lambda is less than equal to 1.

And in implicit method we considered backward time central space and Crank-Nicolson method which is which are actually unconditionally stable. And then we consider non-linear hyperbolic equation and we discretized using two different methods, Lax method and Mac-Cormack method and then we have shown the computer programme for linear hyperbolic equation.

And we used two different explicit methods, one is your FTBS method, Forward Time and Backward Space which is first order accurate in time and space and then we considered Lax-Wendroff method which is second order accurate in time and space. Thank you.