Computational Fluid Dynamics for Incompressible Flows

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Lecture- 1

Finite difference formulations of the first order wave equation: Explicit Method

Hello everyone, so today will start a new module hyperbolic equations. So, will consider first order wave equation which is hyperbolic equation and will discretize using different explicit and implicit methods.

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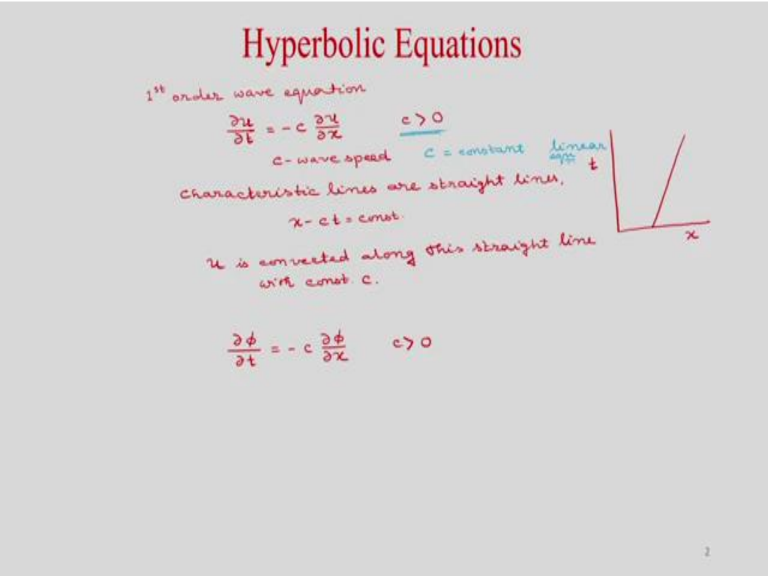

So, let us consider the first order wave equation so you know del u by del t is equal to minus c del u by del x. So, you know that c is your wave speed c is your wave speed also you know that for this simple equation that characteristic lines are straight lines given by the equation, x minus ct is equal to constant.

So, characteristic lines are straight lines given by the equation, x minus ct is equal to constant. So, if you see that if this is your x, x-axis is x if x-axis is x and y-axis is t, then it will have some straight line. So, it is a straight line and the quantity u is convicted along these lines with constant c.

So, u is convicted along this straight line with constant c. So, will consider a model equation in terms of general variable phi so we can write del phi by del t is equal to minus c del phi by del x. So, will consider this equation and will discretize using explicit method where one will be unknown then will consider implicit schemes, where more than one unknown will be there.

So, in this first order wave equation we are considering c greater than 0 so c greater than 0 where your wave speed is greater than 0. So, here this wave equation, what we are considering here c is the wave speed and this wave speed is constant, if c is constant c is constant then this will be a linear equation linear equation. So, if c is constant then it will be linear equation and mostly will consider the discretization whatever will consider in this lecture for c greater than 0, c greater than 0.

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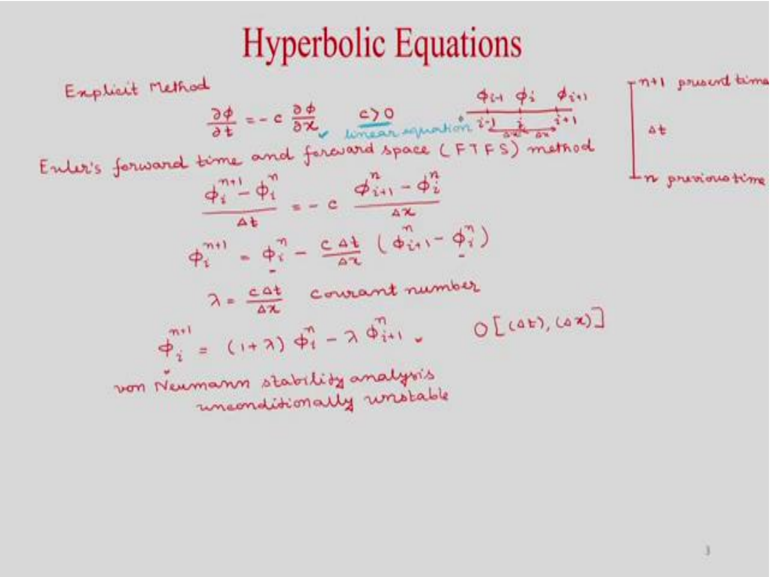

So, first let us consider explicit methods and will consider forward time forward space method. So, first will consider explicit method. Explicit method you know that only one unknown will be there and in this marching time marching will go from n to n plus 1 and n plus 1 is your present time and n is your previous time. And it is one dimensional in space so will consider the grid as i, i plus 1 and i minus 1. So, here phi i is known, here phi i plus 1 is known and here phi i minus 1.

So, at discrete point i, i plus 1 and i minus 1, phi i, phi i plus 1 and phi i minus 1. So, will consider this equation which is your model equation del phi by del t is equal to minus c del phi by del x and this is one a hyperbolic equation. First will consider, Euler’s Forward Time and Forward Space so which is known as FTFS, Forward Time and Forward Space method.

So, in this method we will consider obviously from the name you know that the time derivative will use forward time discretization and in special derivative del phi by del x will use forward space. So, in both these time derivative and special derivative we will use first order scheme forward difference approximation.

So, you can see this will be your phi i n plus 1 minus phi i n divided by delta t is equal to minus c phi so forward space it will be phi i plus 1 minus phi i divided by delta x and as we are considering explicit scheme. So, all these phi i and phi i plus 1 will be a time label n. So, now if you rearrange it and left hand side only you keep phi i n plus 1 then you will get phi i n minus c delta t by delta x, phi i plus 1 n minus phi i n.

And let us define lambda is equal to c delta t by delta x, here this lambda is known as Courant number. So, now if you rearrange it so you can see here phi i is there, here phi i is there, so together if you take and this minus-minus will be plus. So, you can write phi i n plus 1 will be so it will be 1 plus lambda phi i n minus lambda phi i plus 1 n.

So, the discretize equation for using this forward time and forward space method is this one where, phi i n plus 1 is only unknown, right hand side phi i and phi i plus 1 are known from time label n. And lambda we are defining as wave speed c into the time step delta t and the step size delta x and we are obviously assuming delta x is constant.

Delta x is uniform, delta x is uniform and this is the delta t from n to n plus 1 you are going, so that is the time step size delta t. But, if you do the Von Neumann stability analysis of this method, then we can show that it is unconditionally unstable so, it is unconditionally unstable.

So, if you do a Von Neumann stability analysis, Von Neumann stability analysis then you can show that it is unconditionally unstable. Although this is having the order of accuracy is your delta t and delta x first order accurate in time and first order accurate in space. But, as it is unconditionally unstable then you cannot use this because you will not get a solution.

So, when we are considering this first order wave equation so we are considering c greater than 0 and obviously c is constant because we are considering a linear equation. For, c is equal to constant it is a linear equation and also we are considering c greater than 0 that means your wave speed greater than 0.

Next will consider Euler a forward time central space method so will consider the first derivative using forward time. So, will discretize the time derivative del phi by del t using forward time approximation and the special derivative will use central difference approximation.

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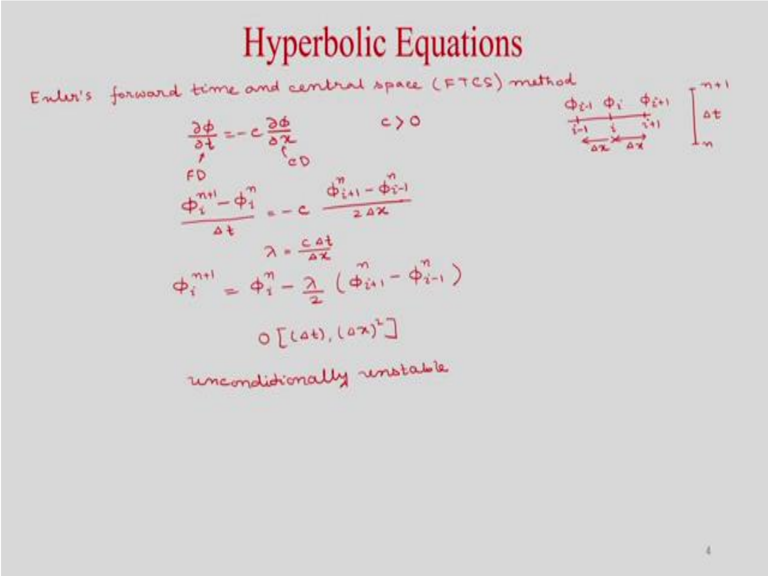

So, will use Euler's forward time and central space, so Forward Time Central Space, FTCS method. So, obviously will use the will use constant step size i, i plus 1 and i minus 1, so this is your delta x and this is also delta x so it is uniform step size and also from n to n plus 1, so this is your time step delta t.

So, our equation is del phi by del t is equal to minus c del phi by del x, so in this method so will use forward difference approximation for this time derivative. And central difference approximation so forward difference approximation and central difference approximation for this special derivative will use.

So, you can write phi i n plus 1 minus phi i n divided by delta t is equal to minus c. So, you will use central difference so it will be phi i plus 1 minus phi i minus 1 and the distance between these two dependant variables. So, it is here phi i and this is your phi i plus 1 and phi i minus 1 so you can see it is 2 delta x, so will write 2 delta x and as it is explicit method will use the time label n.

Similarly, will define the Courant number lambda is equal to c delta t by delta x and after rearranging so what you can write, phi i n plus 1 is equal to phi i n minus lambda divided by 2, phi i plus 1 n minus phi i minus 1 n. And in all these conditions we are considering c greater than 0.

So, wave speed is greater than 0. So, what is the order of accuracy of this method? So, we can see that first order in time and second order in x space. But, if you use the stability analysis then you can find that this scheme is also unconditionally unstable, so this is your unconditionally unstable.

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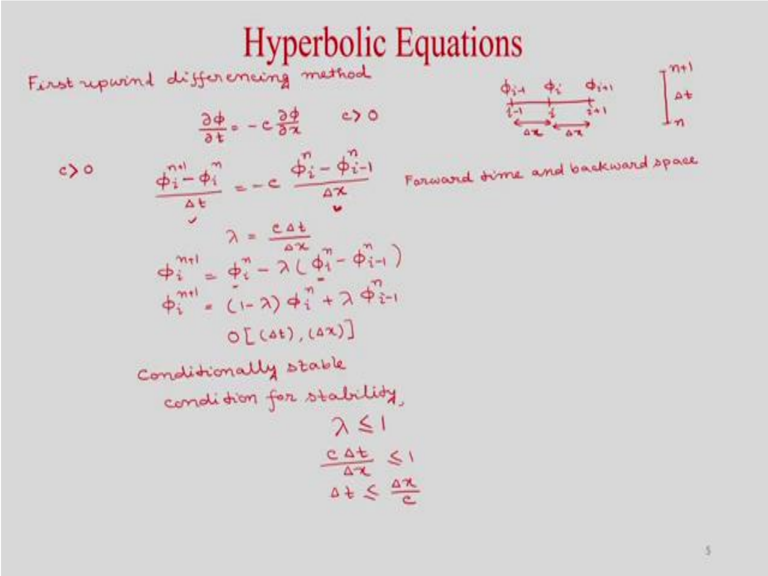

So, now will consider the next method as, first upwind differencing method, first upwind differencing method. So, in this method what will do, depending on the sign of wave speed whether it is c greater than 0 or c less than 0 will choose the scheme. So, same grid will use this is your i, this is your i plus 1 and this is your i minus 1 with uniform step size delta x and this is your delta x, phi i, phi i plus 1 and phi i minus 1.

And your time label is going from n to n plus 1 with time step size delta t. So, if your wave speed c is greater than 0 we have consider for this equation, del phi by del t is equal to minus c del phi by del x where c is greater than 0 in general we have considered. So, if c greater than 0 then will use forward time and backward in space and if c less than 0 then will consider forward in time and forward in space.

So, for c greater than 0 we can use phi i n plus 1 minus phi i n divided by delta t is equal to minus c will use backward in space, so it will be use phi i minus phi i minus 1 divided by delta x with the time label n so, we are using forward time and backward in space. So, you can see said for the time derivative we have used forward in time but in the special derivative we have used backward in space.

And if you write the final algebraic equation and defining the lambda as c delta t by delta x you can write phi i n plus 1, is equal to phi i n minus lambda phi i n minus phi i minus 1 n. And rearranging you will get phi i n plus 1 so this is your phi i n and this is also phi i n. So, you will get 1 minus lambda phi i n and minus-minus plus so it will get lambda phi i minus 1 n.

And obviously the order of accuracy is first order in time and first order in space because backward difference approximation you have used for the special derivative. So, it is order of approximation is del t and del x, first order accurate in both space and time. So, if you use this scheme then you will find using Von Neumann stability analysis that it is conditionally stable.

And the condition is that Courant number should be less than or equal to 1. So, it is conditionally stable and condition for stability is condition for stability is, courant number should be less than or equal to 1. So, that means c delta t by delta x should be less than or equal to 1.

So, now the delta t you have to choose depending on the value of wave speed and the step size delta x. So, delta t should be less than or equal to delta x by c. Now, if you consider that if c less than 0 for the same equation, if wave speed is less than 0 then you use forward time and forward space.

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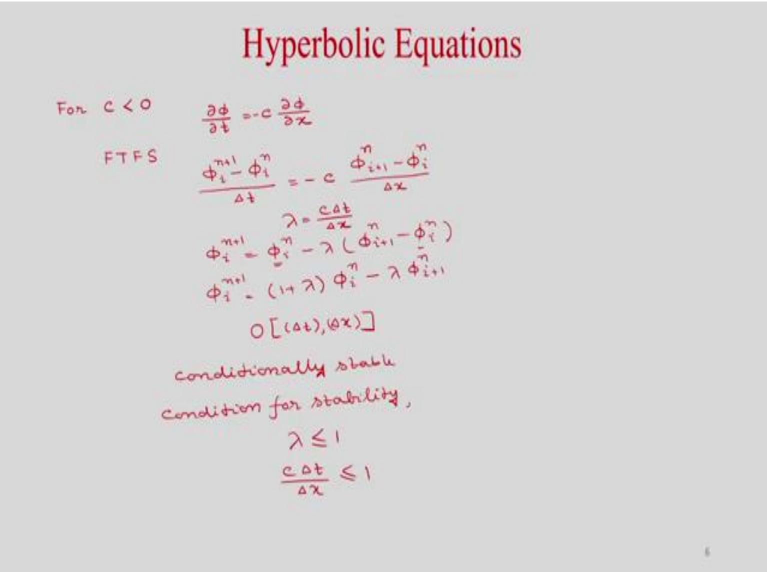

So, now for c less than 0 del phi by del t is equal to c minus c del phi by del x and here now c less than 0 so you use forward in time and forward in space. So, you will get phi i n plus 1 minus phi i n divided by delta t is equal to, minus c phi so forward in space so it will be phi i plus 1 minus phi i divided by delta x with the time label n.

So, if you rearrange it c delta t by delta x you are defining as lambda then you will get phi i n plus 1 is equal to phi i n minus lambda phi i plus 1 n minus phi i n. And you will get phi i n plus 1 so this phi i n and phi n you take together. So, it will be 1 plus lambda phi i n minus lambda phi i plus 1 n.

So, obviously you can see that we have used forward in time and forward in space, so it is first order accurate in both time and space. So, order of accuracy is delta t, delta x and if you do the Von Neumann stability analysis for this scheme then you will be able to show that, that it is conditionally stable with the condition that Courant number should be less than or equal to 1.

So, it is conditionally stable and condition for stability is lambda should be less than or equal to 1. That means, c delta t by delta x should be less than or equal to 1. So, now we have seen that for this first order wave equation depending on c greater than 0 or c less than 0. We have used either, forward time and backward space or forward time forward space.

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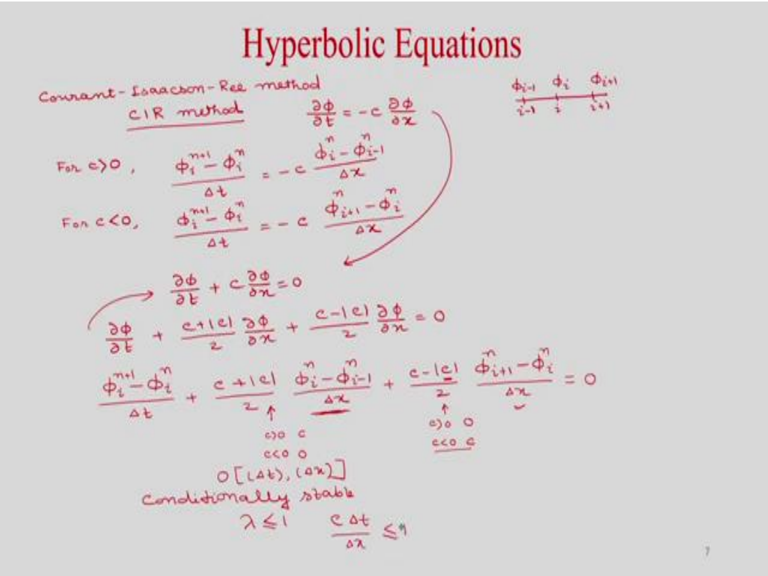

So, these two methods we can combine and we can write in general form which is known as Courant-Isaacson-Ree method, so Courant-Isaacson so these are scientist name Ree method, commonly known as CIR method. So, these two methods will combine and will write in general form using CIR method.

So, already we have seen that for c greater than 0, so we have used them del phi by del t is equal to minus c del phi by del x, i, i plus 1 and i minus 1. So, this is phi i, phi i plus 1 and phi i minus 1. So, for c greater than 0, it is phi i n plus 1 minus phi i n divided by delta t so for c greater than 0 we have written minus c phi so it is backward in space.

Phi i minus phi i minus 1 delta x n and for c less than 0 we have written, phi i n plus 1 minus phi i n divided by delta t is equal to minus c. So, we have written so we have used forward difference approximation for the special derivative. Phi i plus 1 minus phi i divided by delta x.

So, will just combine these two and will write the so you have del phi by del t plus c del phi by del x equal to 0 so, this is the equation. So, now we can write it as del phi by del t plus c plus mod c by 2, del phi by del x plus c minus mod c by 2 del phi by del x is equal to 0. So, what we have done?

We have just added mod c by 2 here and we have subtracted here mod c by 2 del phi by del x. So, mod c by 2 del phi by del x minus mod c by 2, del phi by del x and here c by 2 del phi by del x and here c by 2 del phi by del x combine it will give c del phi by del x. So, you can get back this equation so, we have written in this way.

Now, you discretize this equation phi i n plus 1 minus phi i n divided by delta t, plus c plus mod c by 2, now del phi by del x so this will use for the wave speed greater than 0. So, will use forward time backword space, so we will get phi i minus phi i minus 1 n n divided by delta x.

And for the second special derivative will use as we use the discretization for c less than 0. So, will use c minus mod c by 2, phi i plus 1 minus phi i n n divided by delta x equal to 0. So, now you see this equation, if your c is greater than 0, so what will happen if c is greater than 0? Then it will written only c. So, this term will written only c.

So, c plus c divided by 2 it will written c and if you see this equation then if it is c greater than 0, so this will written only c. And if it is c greater than 0, so it will written c and c minus c will 0 so it will written 0 and now if you see less than 0 if c less than 0, then you can see if c is less than 0 so obviously it will written minus c.

So, it is c minus c so it will become 0 and if this c less than 0 then it will written minus c, this will written minus c. And minus c and this will become minus plus so 2c by 2 so it will written c. So, you can see for c less than 0 so this will be valid and for c greater than 0 this approximation will be valid.

So, this way in general form we are writing this CIR method so depending on the wave speed, whether it is greater than 0 or less than 0. So, you will get either forward time backward space or forward time forward space.

So, this scheme if you do the Von Neumann stability analysis then this is also conditionally stable and condition for stability is this courant number will be less than or equal to 1. So, this is your first order accurate del t, del x and it is conditionally stable. And condition for stability is that lambda should be less than or equal to 1 that means, c delta t by delta x should be less than or equal to 1.

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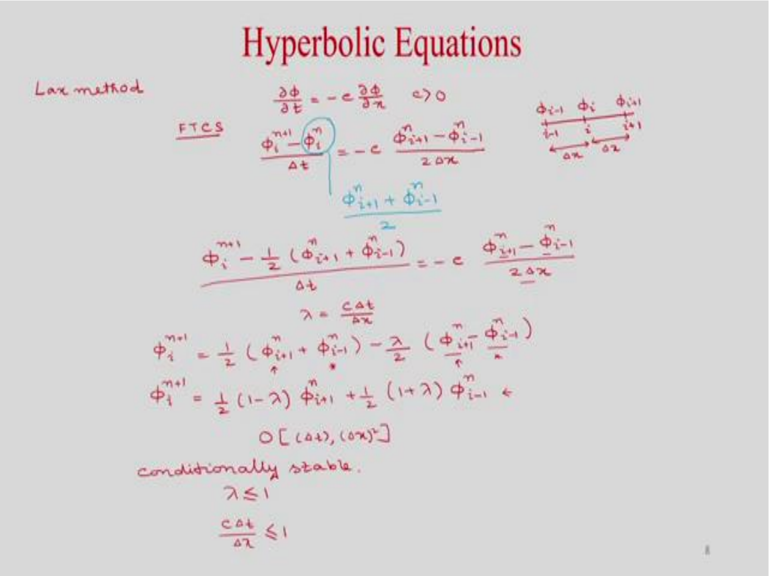

Now, will use Lax method so will start with the forward time and central space. So, with that discretization method and in the time derivative, whatever phi i at time label n will be there that will take the average of phi i n plus 1 and phi i n minus 1. So, if you use FTCS method for the equation del phi by del t is equal to minus c del phi by del x where c greater than 0.

Then we have used already forward time so phi i n plus 1 minus phi i n divided by delta t, is equal to minus c phi i plus 1 minus phi i minus 1 n n divided by so it will be 2 delta x. So, this is your i, i plus 1, i minus 1 so this will be your delta x and this will be your delta x, this is your phi i, phi i plus 1, phi i minus 1.

So, these phi i n which is in FTCS method will use as average of phi i plus 1, plus phi i minus 1 divided by 2 at time label n. Then that scheme is known as Lax method okay so that scheme is known as Lax method. Now, let us write this scheme then so it will be phi i n plus 1, minus we are using the average value.

So, half into phi i plus 1 n plus phi i minus 1 n, divided by delta t is equal to minus c phi i n plus 1, minus phi i minus 1 n divided by 2 delta x. So, let us define lambda as c delta t by delta x, then you can write phi i n plus 1 is equal to half phi i plus 1 n plus phi i minus 1 n. Minus lambda by 2 phi i sorry this is your phi n so phi i n minus phi i minus 1 n. So, if you rearrange it so you can see that one unknown is there in left hand side phi i n plus 1.

So, this is your phi i plus 1, it is only sorry this is your phi i plus 1 and this is your i plus 1 because central difference we have used. So, phi i plus 1 n minus phi i n divided by 2 delta x here also you correct phi i plus 1 n minus phi i minus 1 n. So, now this term and this term and this term and this term you write together.

So, you will get half 1 minus lambda phi i plus 1 n and you will get half 1 plus lambda phi i minus 1 n. So, you can see that, this is your final algebraic equation and what is the order of accuracy, what is the order of accuracy? Because we have used first order accurate approximation in the time derivative.

So, it will be order of delta t but central difference approximation we have used for the time derivative, so it will be delta x square. So, it will be delta t and delta x square and if you do the Von Neumann stability analysis for this scheme, then you will find that it is conditionally stable and condition for stability is lambda less than or equal to 1.

So, conditionally stable and condition for stability is lambda less than equal to 1 where c delta t by delta x should it would be less than equal to 1. So, next will consider another scheme where will use central difference in both in time derivative as well as in special derivative.

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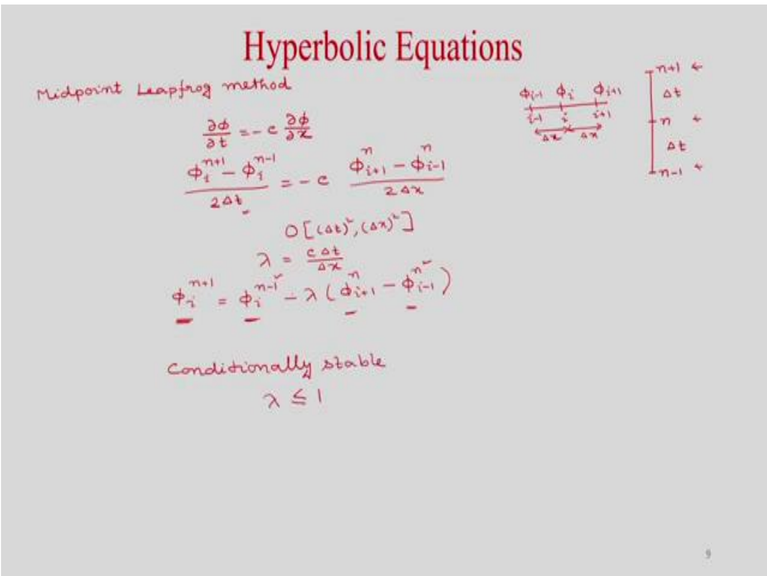

So, this method is known as, Midpoint Leapfrog method so we again will use constant step size. So, phi i, phi i plus 1 and here phi i minus 1 and here we up to now central difference will use in time. So, it will be n minus 1 to n-to-n plus 1 so this is your time step delta t. So, your model equation is del phi by del t is equal to minus c del phi by del x. So, now we will use central difference in time derivative so what will you can write? It will be phi i n plus 1 minus phi i n minus 1 divided by delta t, is equal to minus c phi i plus 1, minus phi i 1 phi i minus 1.

So, this is your central difference we are using divided by 2 delta x sorry here it will be 2 delta t. Because we are using central difference in time derivative as well so it will be phi i n plus 1, minus phi n minus 1 divided by 2 delta t. So, obviously it is order of accuracy you can see it will be delta t square and delta x square because we have used central difference approximation.

So, now you rearrange it so we are defining the Courant number as c delta t by delta x then you can write phi i n plus 1, is equal to phi i n minus 1 minus lambda phi i plus 1 n minus phi i minus 1 n. So, you can see when you are solving for phi at time label n plus 1, you need two initial conditions.

At the starting point you need two initial conditions because when you are calculating at n plus 1 you need the value of phi at time label n as well as n minus 1. So, here you can see that phi i n plus 1 to find it you need the value of phi i n minus 1 and again phi i plus 1 and phi i minus 1 at time label n, n minus 1 and n, to calculate the value at n plus 1 you need two values at time label n and n plus 1.

So, you will find difficulty at the starting point because at the starting point you need two initial conditions. So, if you do not give good initial conditions then obviously you will not get correct or accurate solution, you will not get accurate solution. So, at the starting point at least you can use some method which you need only one step value previous step value.

So, that you can use at the initial condition and you can solve for n plus 1. But, the accuracy will depend on that how accurate you are giving the initial condition at time label n and n minus 1. And if you do the von Neumann stability analysis for this scheme then you will find that it is conditionally stable and the condition is lambda should be less than or equal to 1. So, it is conditionally stable and condition for stability is lambda is less than or equal to 1.

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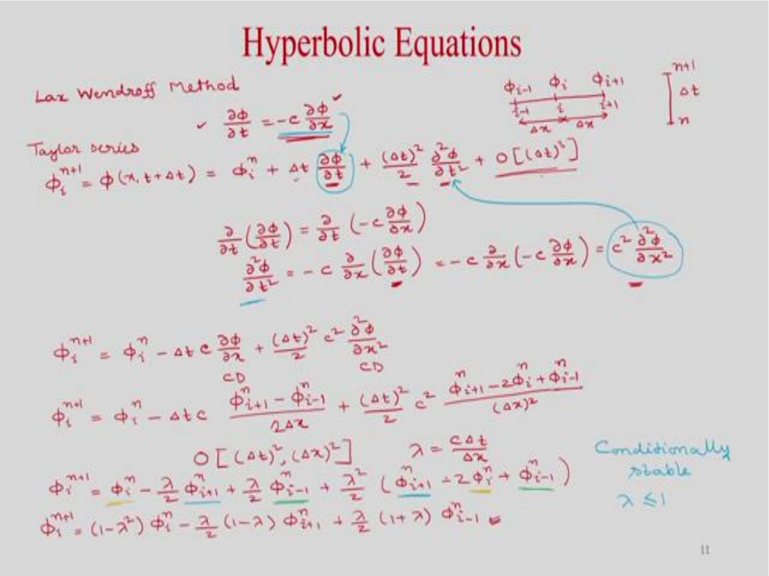

So, next will use Lax-Wendroff method so in Lax-Wendroff method what will use? Will use the Taylor series expansion to expand phi i n plus 1. So, Lax-Wendroff method where will use i, i plus 1, i minus 1 constant step size delta x and this is your delta x, this is your phi i, this is your phi i plus 1 and phi i minus 1.

We are going from time label n to n plus 1 with a time step size delta t. So, our model equation is del phi by del t is equal to minus c del phi by del x which is your first order wave equation. Now, first will expand this phi i n plus 1 so will expand phi i n plus 1. So, this is nothing but phi x t plus delta t, n plus 1 is there.

So, now you expand it so what you will get if you expand it? You will get phi i n plus delta t del phi by del t plus del t square by factorial 2 is to del 2 phi by del t square and high order term. So, now what we will use that del phi by del t this term del phi by del t will substitute as minus c del phi by del x.

So, del phi by del t this will substitute with the minus c del phi by del x. Similarly, now if you take the time derivative of this first order wave equation, then what will you get? Del of del t del phi by del t is equal to del of del t minus c del phi by del x. So, now you will get left hand side del 2 phi by del t square is equal to so c is constant.

So, it will be minus c and you change the variable then you can write del of del x del phi by del t. So, now this del phi by del t now you can use substitute minus c del phi by del x. So, this is del phi by del t and this del phi by del t now you substitute it with minus c del phi by del x. So, what will you get? Minus c del of del x minus c del phi by del x. Then c is constant so you can take it out then you can write minus-minus plus c square del 2 phi by del x square.

So, this equation is known as second order wave equation. Now, in the Taylor series expansion what you have del t phi by del t square, you substitute it with c square del 2 u del x square. So, now this term del 2 phi by del x square so you substitute with this term, del 2 phi by del t square so this is your del 2 phi by del t square.

We are substituting with c square del 2 phi by del x square. So, now if you substitute and rearrange it, what will you get? So phi i n plus 1 is equal to phi i i n so now we are writing del phi by del t we are substituting with minus c del phi by del x. It will be minus delta t c del phi by del x.

And this term you substitute with c square del 2 phi by del x square so you will get del t square by 2 c square del 2 phi by del x square and so now it will be high order term. So, now in right hand side we have space derivative, del phi by del x and del 2 phi by del x square. You use central difference method which is second order accurate in space.

So, you use central difference method here and central difference method here, so what you can write phi i n plus 1 is equal to phi i n minus delta t c. So, del phi by del x you are using central difference so phi i plus 1 minus phi i minus 1 at time label n divided by 2 delta x. Plus delta t square by 2 c square now u second order approximation so you will get phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square at time label n.

So, now what is the order of accuracy if you see of discretization from the special derivative you can see it is delta x square. And here you can see that it was when we discretize using Taylor series expansion, we got order of delta t cube. So, now if you see del phi by del t actually we are trying to model it so that if you divide it by delta t, it will be order of delta t square.

So, this is the order of accuracy of this scheme will be second order in time and second order in space. So, now if you rearrange and Courant number is equal to c delta t by delta x if you write then you can rearrange it as, phi i n plus 1 so it will be phi i n. So, it will be c delta t by delta x, it will be minus lambda by 2 phi i plus 1 n then is minus-minus plus.

So, it will be plus lambda by 2 phi i minus 1 n so it will be lambda square by 2 lambda square by 2 phi i plus 1 n minus 2 phi i n plus phi i minus 1 n. So, now you rearrange it so this term phi i plus 1, phi i plus 1 you take together then phi i minus 1, phi i minus 1 you take together and you have phi i n and this phi i n you take together.

So, finally you can write it as phi i n plus 1 is equal to so phi so you can see 1 minus lambda square, so it will be 1 minus lambda square phi i n then if you write i plus 1. So, it will be i plus 1 i plus 1 so it will be minus lambda by 2, 1 minus lambda phi i plus 1 n and phi i minus 1 similarly, you can write lambda by 2, 1 plus lambda phi i plus 1 n sorry phi i minus 1 which will be phi i minus 1 n.

So, this is the final algebraic equation for this Lax-Wendroff method and the order of accuracy is delta t square and delta x square. So, second order accurate in space and time and if you do the Von Neumann stability analysis you can show that it is conditionally stable and the condition for stability is lambda should be less than or equal to 1.

So, it is conditionally stable and condition for stability is lambda less than or equal to 1. So, in the next class will take up some implicit method and will discretize this first order wave equation. And also will see some representative code or programme, computer programme for solution of these first order wave equation and will show some results. Thank you.