Computational Fluid Dynamics for Incompressible Flows
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Lecture 4
Finite difference formulations of Parabolic Equations:
Unsteady Three-Dimensional Equation Diffusion Equations

Hello everyone, so in last lecture we considered two dimensional unsteady diffusion equation and we used different discretization methods and we have seen its stability criteria, an order of accuracy. Today, we will consider three dimensional unsteady diffusion equation and we will try to write the final discretize algebraic equations.

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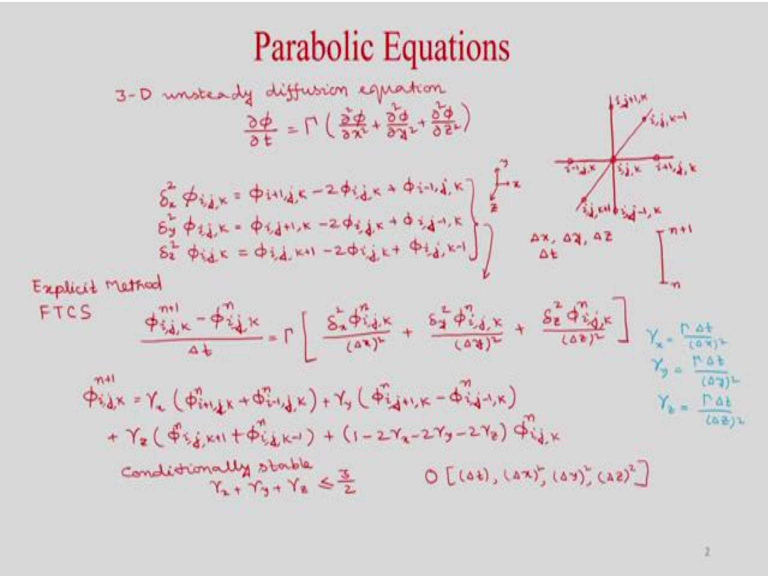

So, we will consider 3 dimensional unsteady diffusion equations, 3 dimensional unsteady diffusion equation. So, obviously you have a for general variable phi, we can write Del phi by Del t equal to gamma which is your diffusion coefficient and you have Del 2 phi by del x square plus Del 2 phi by Del y square and Del 2 phi by Del z square. So, in this case the dependent variable phi is function of 3 space coordinates x, y, z and time t. So, we can write the or we can draw the grid.

So, now your third direction also, so let us say this is your x, this is your y and this is your z. So, you can write this is your point i, j, k. So, k is the index to note the discrete point in z direction and this is your i plus 1 j, k. This is your i minus 1 j k, this is your i, j minus 1 k and this is your i, j plus 1, k. And in z direction now this is your i, j, k minus 1 and this is your i, j, k plus 1. So, we have uniform as step size in x, y, z direction. So, those are delta x, delta y and delta z. And also we have the time step size delta t.

So, we are moving from n to n plus 1. So, n is your previous time level and n plus 1 is your current time level. So, to discretize this equation, we will use discretization operator, which is known as central difference operator. So, if you operate in the direction x, delta x square and for if you operate on phi i, j, k.

So, it will give phi i plus 1, j, k minus 2 phi i, j, k and plus phi i minus 1, j, k. So, this is just operator acting on phi i, j, k and as we are getting the finite central difference formulation. So, it is known as central difference operator. So, this is discretization operator. So, similarly if we apply in y direction, so and z direction then you will get delta y square phi i, j, k, so obviously to be phi i, j plus 1, k minus 2 phi i, j, k and plus phi i, j minus 1, k.

And similarly, delta z square you can write if you operate it on phi i, j, k then you can write phi i, j, k plus 1 minus 2 phi i, j, k plus phi i, j, k minus 1. So, we will use this operator to write these discretized equations, so it will be in compact form because it is in three dimensions, so we can write in compact form. So, you can see now if you use simple FTCS scheme for a time center space which is your explicit method, then we can write the equation as explicit method, we are using FTCS, forward time central space.

So, what you can write? So, you can write as phi i, j, k at n plus 1 minus phi i, j, k, n divided by delta t is equal to gamma, so now we will write the operator. So, delta x square phi i, j, k by delta x square, so what will be the time level of phi because we are using explicit method then obviously, it will be previous time level that means n.

So, let us put n here. Similarly, delta y square phi n i, j, k divided by delta y square and plus delta z square phi i, j, k n by Del z square. So, you can see that in compact form, we have written and if you substitute all these here then you will get final discretize equation. And if you write for this particular FTCS scheme then we can write as, so you can see only one unknown is there in the left hand side phi i, j, k n plus 1 and all other phi at level n.

So, you can write phi i, j, k n plus 1 and let us write gamma x is equal to gamma delta t by Del x square, gamma y as gamma Del t by Del y square and gamma z as gamma Del t, Del z square. So, with this if you write, you will get this discretize equation gamma x phi i plus 1, j, k plus phi i minus 1, j, k at level n.

Then plus gamma y phi i, j plus 1, k minus phi i, j minus 1, k at level n and plus gamma z phi i, j, k plus 1 minus phi i, j, k minus 1 at level n and you have the diagonal coefficient. So, that will be, so you are taking these sets, it will be 1 minus 2 gamma x minus 2 gamma y minus 2 gamma z phi i, j, k n.

So, you can see that in left hand side only phi i, j, k at n plus 1 is unknown. So, obviously it is explicit method and it is conditionally stable and condition for stability is, so it is conditionally stable and condition is gamma x plus gamma y plus gamma z, should be less than equal to 3 by 2 and what is the order of accuracy? Obviously, it is FTCS, so you will get first order accurate in time and second order in space. So, order of accuracy is delta t, delta x square, delta y square and delta z squared.

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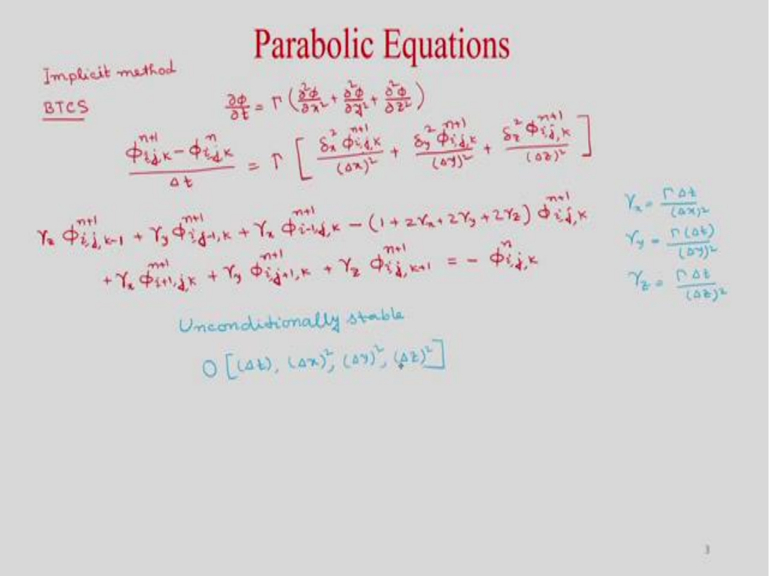

So, now similarly let us use BTCS, Backward Time Center Space for this 3 dimensional unsteady diffusion equation, so what you will get? So, it is implicit, implicit method. So, we are using BTCS, BTCS. So, our governing equation is Del phi by Del t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square and Del 2 phi by Del z square.

So, similarly we will use this central difference discretization operator and we will write the discretized equation as phi i, j, k minus phi i, j, k n plus 1 n divided by Del t equal to gamma. So, it will be delta x square phi i, j, k, so at what time level? As it is implicit method, so we will take at the current time level, so that will be n plus1.

So, you write n plus 1 and delta x square and similarly, we will write Del y square phi i, j, k at time level n plus 1 divided by Del y square plus delta z square, phi i, j, k n plus 1 divided by Del z square. So, you can see that in right hand side, all the terms are unknown. It is at current time n level.

So, if you write the discretize equation, finally you will get gamma z phi i, j, k minus 1, n plus 1 plus gamma y phi i, j minus 1, k n plus 1 plus gamma x phi i minus 1 phi i minus 1 j, k n plus 1. Then you will get minus 1 then you will get plus 2 gamma x plus 2 gamma y plus 2 gamma z, phi i, j, k at n plus 1 plus gamma x phi i plus 1, j, k n plus 1 plus gamma y phi i j plus 1 k n plus 1 plus gamma z phi i, j, k plus 1 n plus 1 is equal to, so minus phi i, j, k at n.

So, this is known from the previous time level and similarly, we define gamma x, gamma y and gamma z as gamma x equal to gamma delta t by Del x square gamma y is equal to gamma Del t by Del y square and gamma z as gamma Del t by Del z square. So, obviously, this is unconditionally stable, this is unconditionally stable. This scheme is unconditionally stable and order of accuracy, first order accurate in time and second order in space.

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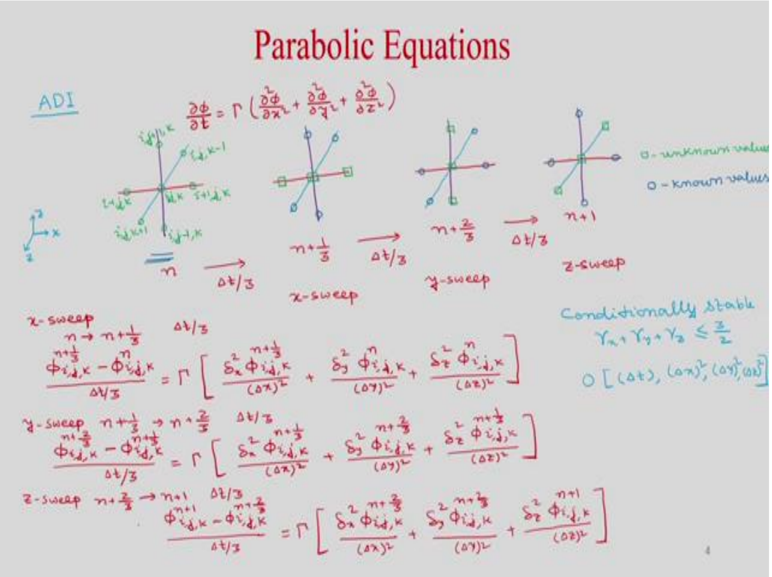

So, now, let us use ADI. So, in 2 dimensional unsteady diffusion equation, already we have used this alternating direction implicit method. Now, let us apply to these 3 dimensional unsteady diffusion equation, so ADI, Alternating Direction Implicit method. So, we have the governing equation Del phi by Del t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square plus Del 2 phi by Del z square.

So, in this case, now we will use 3 step method, so we will go from n to n plus 1 by 3 where we will do the x sweep n plus 1 by 3 to n plus 2 by 3 where we will do the y sweep and n plus 2 by 3 to n plus 1, we will do the z sweep. So, if you draw the sweep, if you draw the greed, so you have x in this direction, y in this direction and z in this direction.

So, now at we are moving from n to n plus 1 by 3, then n plus 1 by 3 to n plus 2 by 3 and n plus 2 by 3 to n plus 1. And what is the step size? Obviously Del t by 3, Del t by 3 and Del t by 3, so this is your x sweep. This is your y sweep and this is your z sweep. So, in x sweep, we will have the unknowns in only x direction 3 unknown.

So, if you draw the grid points. So, this is your i, j, k i plus 1, j, k. So, this is your i, j, k. This is your i plus 1, j, k, this is your i minus 1, j, k, this is your i, j minus 1, k, this is your i, j plus 1, k, this is your i, j, k minus 1 and this is your i, j, k plus 1. So, these are the discrete points.

Now, when you are moving from n to n plus 1 by 3, so we are doing the x sweep and we will have 3 unknowns i minus 1, j, k, i, j, k and i plus 1, j, k. So, at these 3 discrete points, phi is unknown. So, unknowns let us denote with square. So, these are the unknowns. In y sweep only in y direction, three points i, j plus 1, k, i, j, k and i, j minus 1, k.

And in z sweep i, j, k plus 1 i, j, k and i, j, k minus 1. So, these are the unknowns and known values will be from here. I am not writing the index I, j, k from the these figure you can see, so these are known values and this square is unknown values. So, if you now write using the central difference discretization operated, let us write the discretized equation in x sweep, y sweep and z sweep, so your x sweep.

So, now you are moving from n to n plus 1 by 3 and time step is delta t by 3. So, you will get phi i, j, k minus phi i, j, k. So, we are moving n plus 1 by 3 and n divided by Del t by 3. So, now in right hand side, if you write the discretize equation using central difference, so you will get delta x square phi i, j, k by delta x square.

So, what will be this time level? So, as we are doing x sweep, so obviously in x direction, all these three points will be at time level n plus 1 by 3. So, it is n plus 1 by 3, rest will be at n. So, delta y square phi i, j, k, so it will be at n divided by Del y square and similarly delta z square phi i, j, k, it will be also at n time level divided by Del z square.

So, now in x sweep, so you will get the value at phi n plus 1 by 3, so now these are known. So, at n plus 2 by 3 when we will move in the y sweep, so in y direction there will be 3 unknowns. So, in when we will discretize the Del 2 phi by Del y square, so there will have the unknown time level that will be n by 2 by 3. So, now we are going y sweep.

So, it is moving n plus 1 by 3 to n plus 2 by 3 and what is the time step? Del t by 3, so you can write phi i, j, k minus phi i, j, k, so now we are moving n plus 2 by 3 from n plus 1 by 3 divided by Del t sorry Del t by 3. So, in diffusion terms, so now you have delta x square phi i. j, k divided by Del x square. So, now, from the previous time level, you have to use so previous time level is n plus 1 by 3 but delta y square phi i, j when we are writing, so it will be at current time level.

So, it will be n plus 2 by 3 divided by delta y square and delta z square again it will be from the previous time level that means, in this case it is n plus 1 by 3 divided by Del z square. Now, you do the z sweep. In z sweep, we are moving from n plus 2 by 3 to n plus 1 with the time step size Del t by 3.

So, let us right phi i, j, k minus phi i, j, k divided by Del t by 3. So, now we are moving from n plus 2 by 3 to n plus 1. So, n plus 1, level the values are unknown. So, in right hand side now, you will write delta x square phi i, j, k. So, it will be from that previous time level, so n plus 2 by 3 divided by Del x square plus delta y square phi i, j, k at n plus 2 by 3 divided by Del y square and you have delta z square phi i, j, k.

Now, it will be at n plus 1 because we are doing the z sweep, so Del z square. I am not going to write the final discretize equation but you can rearrange it and you can write and you need to solve those discretized algebraic equations. So, in this case obviously you can see that although it is ADI method.

But it is not unconditionally stable unlike ADI method in 2 dimensional unsteady diffusion equation. So, it is order of accuracy is first order in time and also it is conditionally stable and condition is same as BTCS. So, this is your conditionally stable and the condition is gamma x plus gamma y plus gamma z will be less than equal to 3 by 2.

And order of accuracy is not second order, it is first order delta t, delta x square, second order in space, delta y square and delta z square. Now, let us write some discretization method for 3 dimensional, unsteady diffusion equation which is second order accurate in time and space and it is unconditionally stable.

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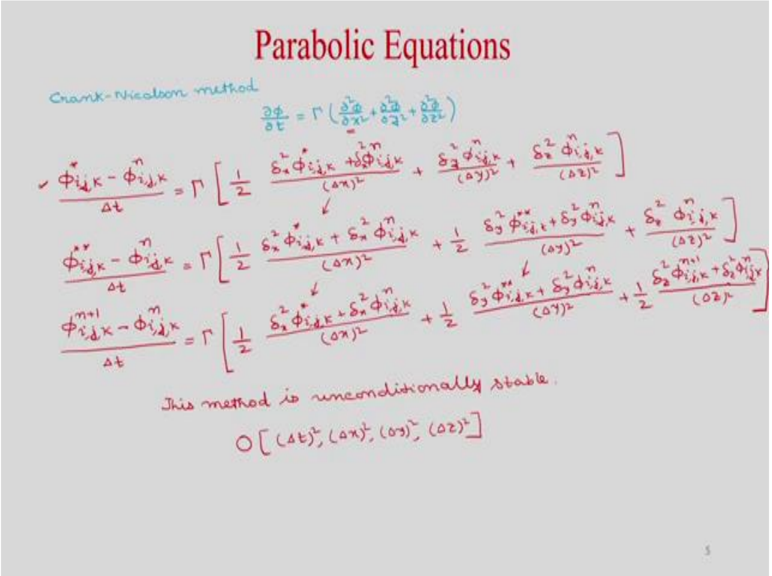

So, we will use Crank Nicolson method in different ways. So you can see, so it will meet Crank Nicolson method, so our governing equation is Del phi by t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square plus Del 2 phi by Del z square. So, this we can write like phi i, j, k.

So, some provisional phi you have to calculate from the value of phi i, j, k n divided by Del t is equal to gamma. So, we will use only the average of BTCS and FTCS for the term Del 2 phi by Del x square first and we will find the provisional phi star. So, it will be Del x square phi i, j, k, it will be at star plus phi i, j, k n divided by delta x square.

And these as usual, we will write delta y square phi i, j, k from n divided by delta y square plus delta z square phi i, j, k at n divided by Del z square. So, after finding this star quantity using that now you solve the, this equation phi i, j, k, so star and phi i, j, k n divided by delta t. So, now you use half delta x square, it will be same as this, same as this. So, it will be phi i, j, k star plus sorry here it will be delta x square plus delta x square phi i, j, k at n divided by delta x square.

But now, in this we will use half of BTCS and FTCS for this, so it will be half delta y square, so it will be phi. So, unknown will be double star i, j, k plus delta y square phi i, j, k from n divided by delta y square plus delta z square it will be phi i, j, k n divided by delta z square. So, now next step, now you find the value of phi n plus 1 i, j, k divided by phi i, j, k n divided by Del t is equal to gamma. So, now, this first term will be as it is, so it will be we half delta x square phi i, j, k star plus delta x square phi i, j, k n divided by Del x square plus half it will be again same as these.

So, delta y square phi double star i, j, k plus delta y square phi i, j, k n divided by del y square plus now we will use half delta z square phi i, j, k at n plus 1 plus delta z square phi i, j, k at n divided by Del z square. So, if you use this method then you will get second order accurate in time and also if you do the stability analysis you will find that this method is unconditionally stable. So, this method is unconditionally stable, stable and order of accuracy is delta t square delta x square delta y square and delta z square.

So, for this 3 dimensional unsteady diffusion equation if you use Crank Nicolson method then only it will become unconditionally stable and you will get second order accurate in time and space. So, now next we will show some example problems and we will show some results of these 1 D unsteady diffusion equation which will consider this Couette flow you have already learnt in your fluid mechanics problem analytical solution.

You know that the velocity linearly varies if the bottom all is stationary and affordable is moving with some constant velocity. Next, we will consider 2 dimensional unsteady heat conduction equation and we will show some results.

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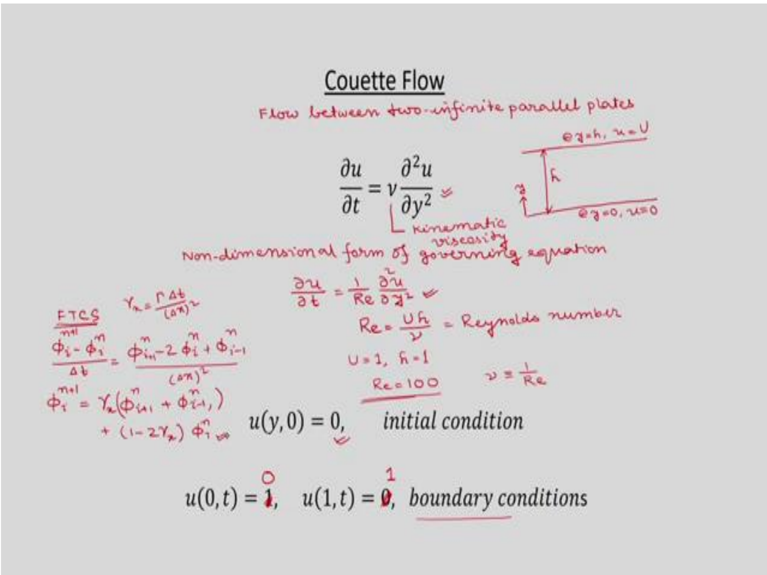

So, now let us consider Couette flow, so Couette flow you know say flow between 2 infinite parallel plates. So, flow between 2 infinite parallel plates. So, let us say that this is your y, so at y is equal to 0, the velocity is 0 and at y is equal to h, let us say this is your h. So, u is some velocity u.

So, this is the governing equation. So, if you write the Navier stoke equation and with proper assumptions, you can write this equation Del u by Del t is equal to nu Del 2 u by Del y square where nu is your kinematic viscosity. I hope that you have solved this equation analytically in your fluid mechanics course.

Now, let us write these governing equation in non-dimensional form with proper non-dimensional parameter, you can write this equation as Del u by Del t is equal to 1 by Re, Del 2 u by Del y square. Here u, y, t in non-dimensional form and obviously they know Re is equal to, so it will some UL by nu. So, here U in this case particular case Uh by nu and it is known as Reynolds number. You know that Reynolds number is the ratio of inertia force to the viscous force.

So, now after writing in non-dimensional form, so you will get this equation. Now, as it is a parabolic equation so you need initial condition as well as boundary conditions. So, at t is equal to 0, you need to specify the velocity. So, here now in non-dimensional form now we have taken that U is equal to 1, h is equal to 1. So, obviously if you put nu is equal to 1, h is equal to 1 then automatically this will be satisfied. And let us say that Reynolds number for 100 we are calculating.

So, here you can see that from the here, so nu is equivalent to 1 by Re. So, the initial condition now we are putting non-dimensional value of U as 0 and boundary condition as I told, so U sorry here it is wrongly written so, this will be 0. So, at y is equal to 0 it is 0 and y is equal to 1, it is 1.

So, at y is equal to z, non-dimensional (())(32:57), so it is U is equal to 1 and this is your 0. So, now this you write the computer program, you can discretize this equation already we have done in first lecture that discretization of this 1 dimensional unsteady diffusion equation. So, using FTCS method if you do, so what you will get so in the FTCS method?

So, it will be phi i minus phi i n plus 1 n divided by Del t is equal to phi i n minus this i plus 1, 2 phi i at n plus phi i minus 1 n divided by Del x square. So, you will get phi i n plus 1 is equal to, so gamma x, gamma x, gamma x just we are writing gamma in to delta t by Del x square. So, it will phi i plus 1 n plus phi i minus 1 n and this if you take in this side, so it will be 1 minus 2 gamma x phi i. So, this is your FTCS method. So, now we will show the computer program to solve this discretized equation and will show the results.

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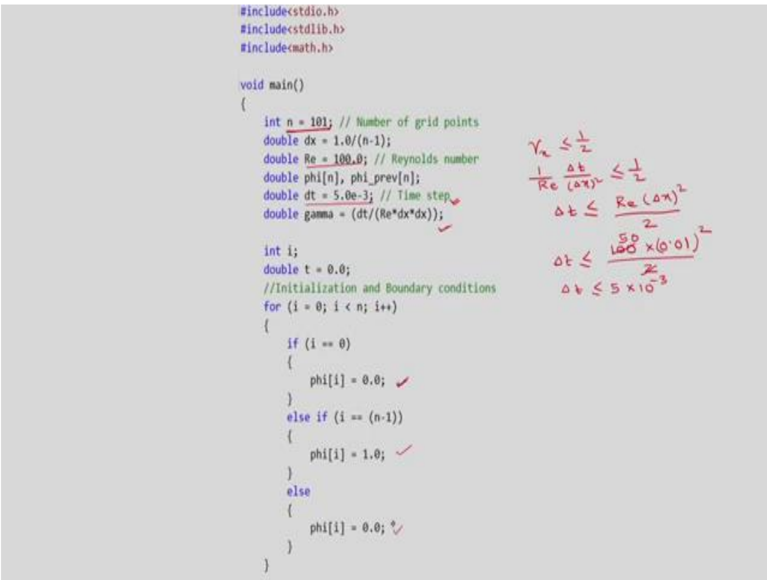

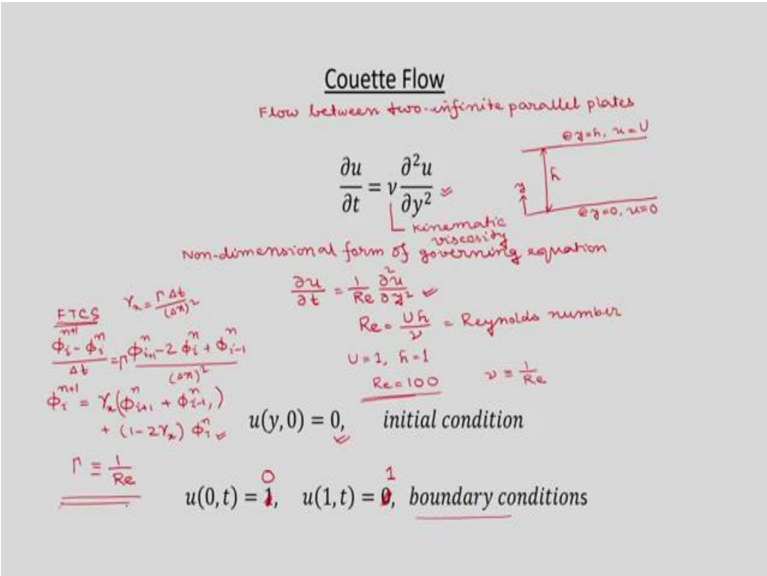

So, you can see in the main function, we have taken total 101 grid points in y direction, so the dx, the distance. The step size we have taken 1 by n minus 1, Reynolds number we have considered 1 and delta t, we have taken 5 into 10 to the power minus 3. So, you can see that what is the condition for stability gamma x is less than equal to half. So, gamma x is you can see, so it is we have different gamma. So, in this case sorry here your gamma will be there.

So, here gamma is nothing but equivalent to 1 by Re, gamma is nothing but equivalent to 1 by Re for the non-dimensional equation. So, in this case, now it will be 1 by Re, gamma delta t by delta x square should be less than equal to half. So, this maximum delta t you can take from this relation. So, delta t should be less than equal to Re into Del x square by 2. So, Re is 100, Re is 100, delta x you can see 1 by n minus 1, so it will be 0.01 divided by 2. So, delta t is less than equal to these, so it will be 50. So, delta t will be less than equal to, so it is sorry this is square, this is square. So, it will be 5 into 10 to the power minus 3.

So, delta t you have to choose, less than equal to phi into 10 to the power minus 3, so maximum delta t you can take 5 into 10 to the power minus 3. So, that we have taken here and gamma, we have defined, gamma we have defined 1 by Re Del t by Del x square, so we are calculating here. So, initial condition, so here initial and boundary conditions we are putting here. So, you can see that i is equal to 0 is your bottom wall.

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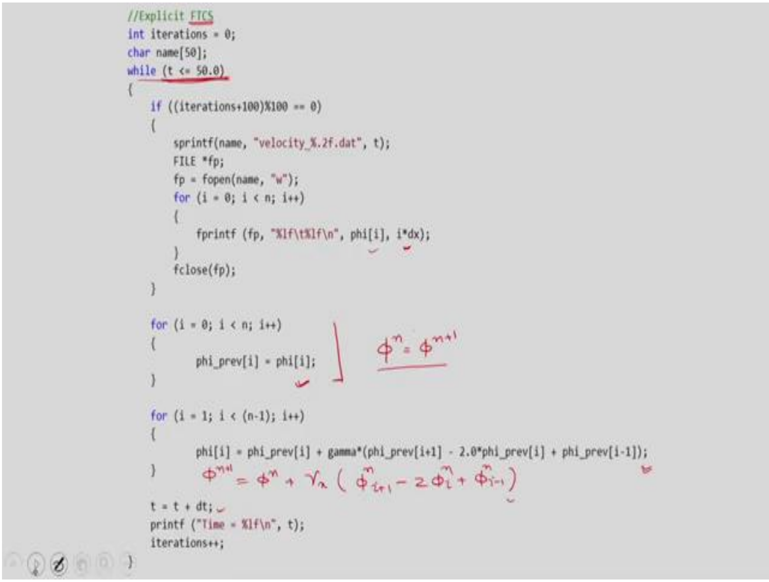

So, you can see all these you define then you come to the FTCS method. So, now we are giving that some condition, t less than equal to 50 we are given, so here as it is parabolic equation, sometime after sometime you have to iterate it, so that we are giving. So here just for a writing, we are writing the phi with the different y direction y coordinate, we are writing these value of phi in a file.

Now, you see the main program here. So, first we are storing these phi n is equal to n plus 1 sorry n is equal to phi n plus 1. So, whatever you are calculating and getting the new value n plus 1, so we need to store as a previous time level. So, phi n is equal to phi n plus 1, we are doing here. And now, you are calculating the phi n plus 1. So, this is your phi n plus 1.

So, obviously it is phi previous plus gamma into, so it is written as phi n plus gamma x into phi i plus 1 n minus 2 pi i n plus phi i minus 1 n. So, you can see that this is your, this way we have written this equation. That is explicit, so only one unknown is there in right hand side all unknown terms. So, you solve this equation and go to t plus delta t and you iterate it up to sometime level then you will get the solution at different time instances, at different y location, the U velocity. So, if you plot it, so what you will get?

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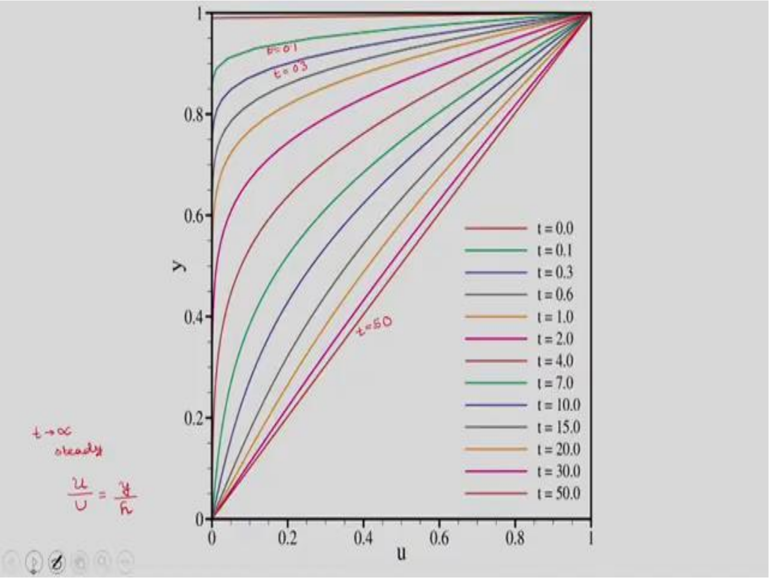

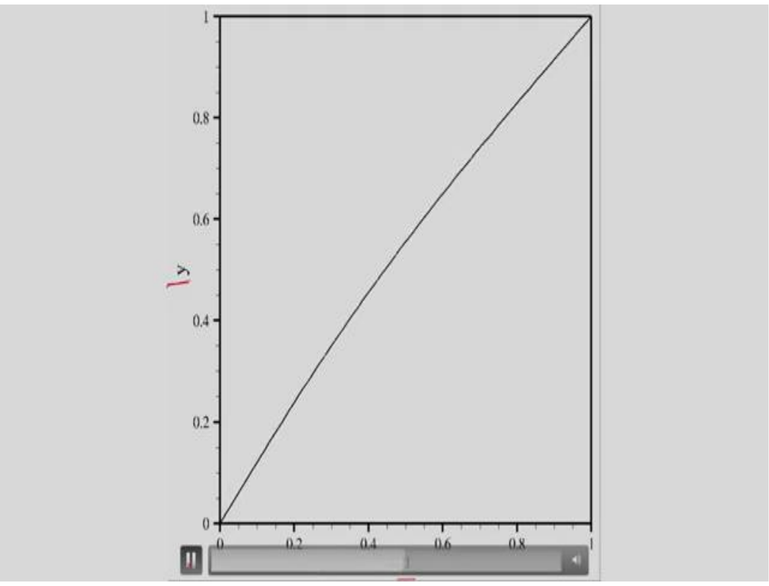

So, you can see at t is equal to 0, everywhere 0 almost. Then as time progresses, so what happens if suddenly your upper plate has started moving right, from upper velocity this will be propagated towards the bottom wall. And velocity profile will be developed. So, you can see with different time instances, so it is t is equal to 0.1.

This is t is equal to 0.3. So, as time progresses, you can see inside this pre domain the velocity is progressing. So, after that at t tends to infinity obviously it will be steady because it is a steady state problem but we are moving in a similar tangent way. So, obviously you can see that as time almost it is 50, you can see that it is a linear labeling.

So, that you know, your analytical solution is u by U is equal to y by L. So, it is linearly varying. So, L means in this case it is h y by h. So, now you can see how that velocity profile looks like and at very long time it will become steady state and you will get a linear profile, after that there will be no variation because it has become the linear profile. So, this way you can plot and you can see the animation, so with time how velocity is propagating. So, you can see it is becoming linear.

So, at t tends to infinity it is becoming linear, so it is starting from 0 then there velocity is becoming linear. So, in this you see x direction is velocity and y direction is y. So, obviously you can see that how it looks. So, you can actually from your data file, you can make this animation from your solution.

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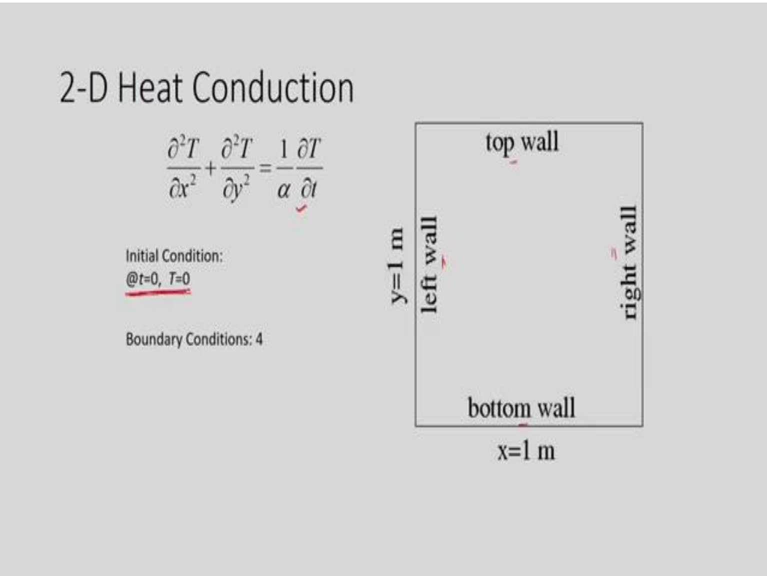

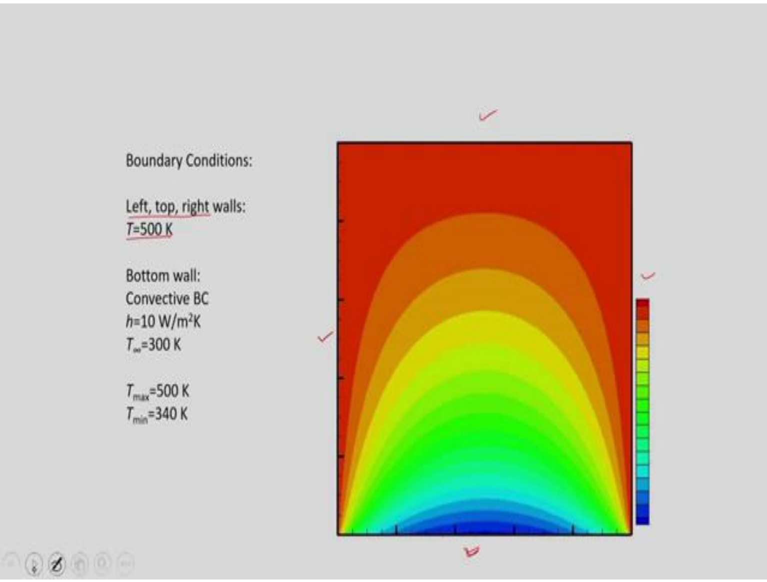

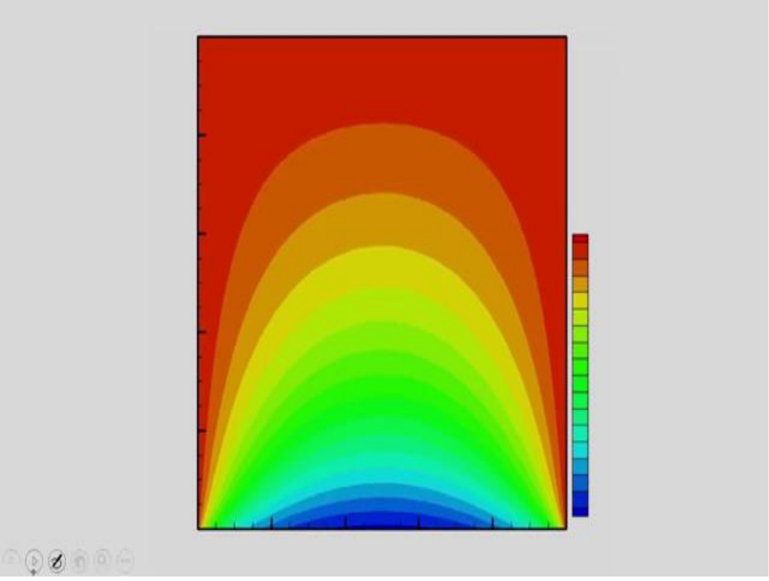

Next, let us consider 2 dimensional unsteady heat conduction equation. So, you can see this is the equation, Del T by Del t is equal to alpha Del 2 T by Del x square plus Del 2 T by Del y square. And let us consider one plate, so this is your bottom wall, this is your top wall, left wall and right all.

We have taken x is equal to 1, y is equal to 1 and initial condition in every case, we are taking at t is equal to 0, temperature is 0. So, with different boundary condition you can see, so if you left top and right, if it is t 500. So, this wall, this wall and it is 500 and here bottom was convicted boundary conditions.

And if you are marching in time with the initial condition temperature as 0 then finally you will get this temperature when it will reach to the state but in the tangent way you can see how the temperature will vary with time inside the domain. You can see initially it is 0 and slowly it is propagating. So, it is diffusing with time, so it is diffusing with time. Then at t tends to infinity, it will become steady state. So, almost it has steady state you can see. So, from here conviction is taking place and three walls are having reached let boundary conditions.

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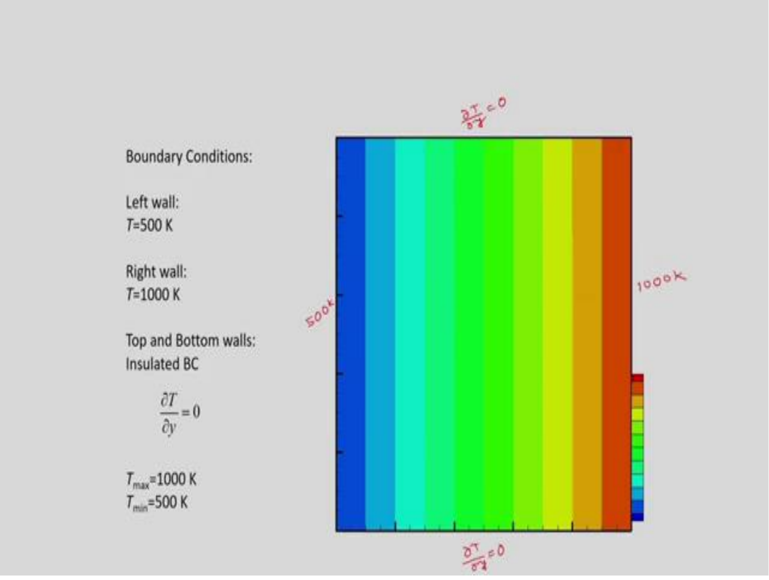

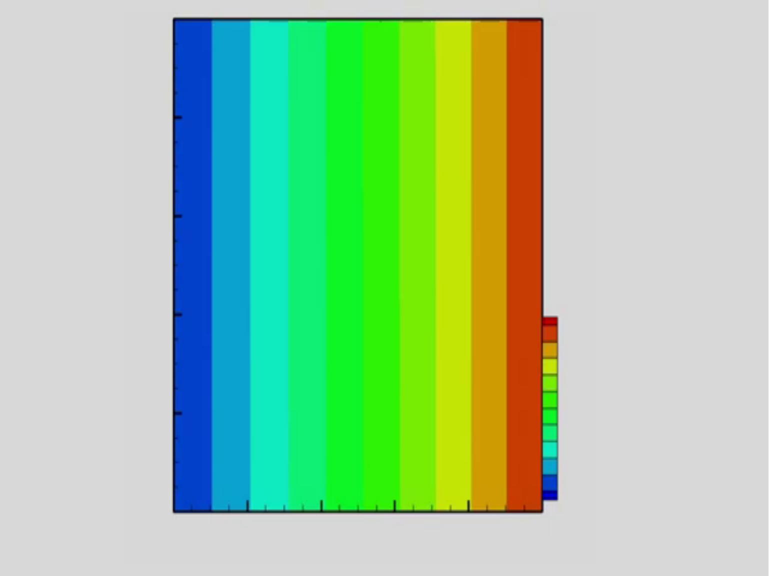

Another problem you consider. So, here left wall is 500 Kelvin, right wall is 1000 Kelvin and these are insulated. So, Del T by Del y is 0 and Del T by Del y is 0. You know it is steady state solution will be linear like this, temperature will vary linearly. But if you start from T at time T is equal to 0 if temperature is 0 everywhere inside the domain and if you solve using unsteady 2 dimensional diffusion equation then how the temperature will propagate that we will see from the animation. So, you can see, so top and bottom walls are insulated and left and right walls are having reach let boundary conditions.

So, you have 1000 Kelvin here, so you can see how the heat is propagating and finally it will reach steady state. So, you can see how it is propagating, so almost it has reached steady state, you can see the variation very less. It is very less. So, we can solve few problems writing the computer program and you try to plot your data file and visualize the result and also you try to write the computer program for different cases.

Then you will learn, how to write the program and post process the result because you will get data points right, after solving the discretize equation but you have to visualize some way. So, you can use some post processing software and you can plot like this. So, we will stop here. Thank you.