Computational Fluid Dynamics for Incompressible Flows
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Lecture 3
Finite difference formulations of Parabolic Equations:
Unsteady Two-Dimensional Diffusion Equation

Hello everyone, so in last 2 lectures we have learnt the discretization methods for 1 dimensional unsteady parabolic equations, where the dependent variable is function of time and 1 space coordinate. Today, we will consider 2 dimensional unsteady equation, like if you have heat conduction equation in 2 dimensions and if it is unsteady then unsteady 2 dimensional heat conduction equation is a parabolic equation where your dependent variable temperature is function of 2 space coordinates x and y and time t.

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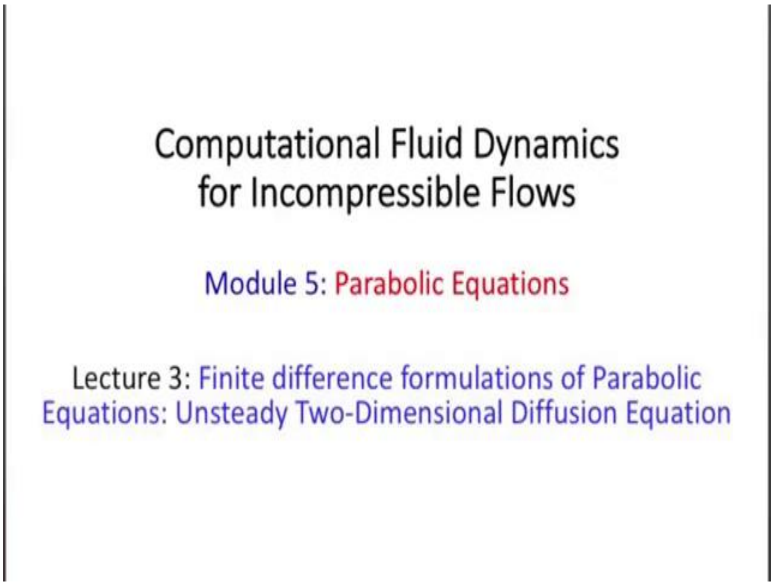

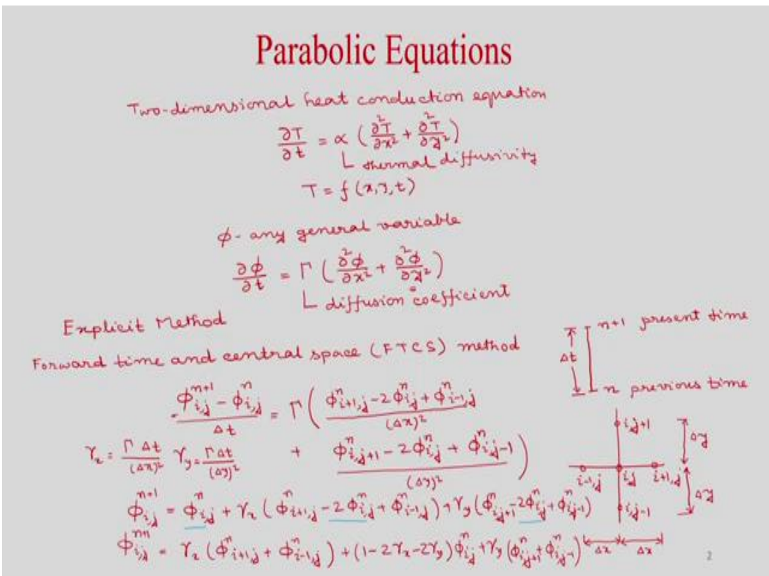

So, today we have lecture 3 of module 5. So, if you considered, two dimensional heat conduction equation. So, what is this equation? So, Del T by Del t is equal to alpha Del 2 T by Del x square plus Del 2 T by Del y square. So, if you consider a heat conduction in a plate, obviously in third direction the temperature given if you neglect or if it is infinite then you can have the assumptions of 2 dimensional heat conduction. And if it varies with time then obviously it is unsteady heat conduction equation.

So, you know that alpha is your thermal diffusivity and T is function of 2 space coordinates, x and y and time t. So, now we will consider 1 model equation similar to this heat conduction equation where we will write in terms of general variable phi. So, if phi is any general variable, any general variable then, we will consider Del phi by Del t is equal to gamma which is your diffusion coefficient, Del 2 phi by Del x square plus Del 2 phi by Del y square.

So, where gamma is here diffusion coefficient. So, like earlier lectures, we will consider first explicit method and then implicit method and we will try to discretize this model equation. So, first let us consider forward time and central space which is your explicit method which is commonly known as FTCS.

So, we are considering explicit method, so in explicit method you know that there will be only one unknown and we are marching in time. So, we will go from time n to n plus 1. So, you are moving or marching from n to n plus 1 where n is your previous time and it is your present time. And this is your delta t this is the time step from tn to tn plus 1.

So, we are considering forward time, forward time and central space, so which is commonly known as FTCS method. So, we will take phi n plus 1. Now, we have 2 index because we have now space coordinates x and y, so in x direction, the discrete point will denote as index i, and in y direction j. So, if you have this grid, if you considered, so we will have this is your i and j, this is your i plus 1, j and this is your i minus 1, j and the step size is delta x and this is constant.

And this is also delta x, this is your constant and you have i, j plus 1 and this is your i j minus 1 and the step size is delta 1. So, now if you write the time derivative using forward time, so phi n plus 1 minus phi i, j n divided by delta t is equal to gamma, now you have Del 2 phi by Del x square, so you can write phi i plus 1, j minus 2 phi i, j plus phi i minus 1 j. So, this is your central difference we are taking, so it is delta x square. So, dependent variable will be at time level n because you are using explicit, so this is your n.

Similarly, if you discretize the Del 2 phi by Del y square, then you can write phi i, j plus 1 minus 2 phi i, j plus phi i, j minus 1 divided by Del y square, so all at time level n. So, this is the discretize equation. So, if you multiply delta t this side and if you have only one unknown phi i, j n plus 1 and let us define the diffusion factor gamma x as gamma Del t by Del x square and diffusion factor gamma y as gamma Del t by Del y square.

So, you can write this algebraic equation like one unknown, so phi i, j n plus 1 in left hand side. So, this is the unknown, now if you write this way then it will be, so phi i, j at n th level and you have plus it is gamma x. So, it will be phi i plus 1, j n minus 2 phi i, j n plus phi i minus 1 j n plus now gamma delta t by delta y square.

So, it will be gamma y phi i j plus 1 minus 2 phi i, j plus phi i, j minus 1 at time level n. So, now you can see that you have these phi i, j at n minus 2 phi i, j at n and minus 2 phi i, j at n. So, you can take this common and write final equation as phi i, j n plus 1 is equal to, so it will be gamma x phi i plus 1, j plus phi i minus 1, j at time level n plus 1 minus 2 gamma x minus 2 gamma y phi i, j n and gamma y phi i, j plus 1 plus phi i, j minus 1 at time level n.

So, this is the final algebraic equation and you can solve easily because you have only one unknown in the left hand side at present time level n plus 1 and in right hand side you can see that all the dependent variables are at time level n and the diffusion coefficient like gamma x and gamma y, you can calculate a prior and find the coefficients and you can solve this equation easily. Now, as it is explicit method and if you do the stability analysis, you will find that it is conditionally stable.

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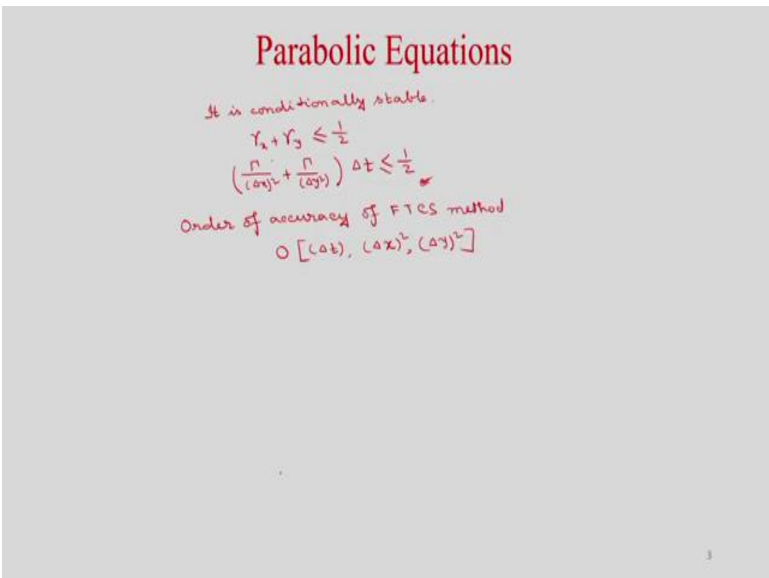

So, this method is conditionally stable. And the condition (())(10:19) stability is gamma x plus gamma y is should be less than equal to half that means gamma by delta x square plus gamma by delta y square delta t should be less than equal to half. So, there is restriction in choosing the time step if we were grid is fixed then from this relationship you have to find that what is the maximum delta t you can choose for the stable solution using this scheme.

So, obviously, we have discretized the time derivative using forward time and the special derivative using central difference, so order of accuracy is, in time it is first order and in space it is second order. So, order of accuracy of FTCS method, so first order in time because forward time you have used and central dependency you have used in space, so delta x square and delta y square and obviously it is conditionally stable and condition is given by this relation.

So, now let us take backward time and central space which is your implicit method and obviously, you know that implicit measure, we will get unconditionally stable. So, you have you can choose higher time step value.

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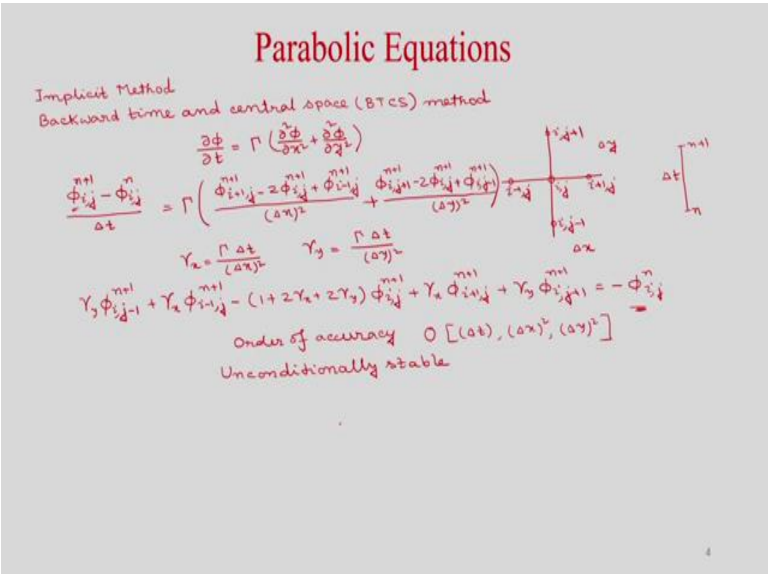

So, now we will use implicit method, so we will use backward time, Backward Time and Central Space so it is BTCS method. So, our model equation is Del phi by Del t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square. So, now if we consider the grid, so you have this is your i, j this is your i plus 1 j this is your i minus 1 j and this is your i j plus 1 and i j minus 1.

Consider uniform grid and these delta x and delta y are constant and you have a constants time step size delta t. So, when you are moving from n to n plus 1. So, you have delta t. So, now if we discretize this equation, so similarly we will use phi i, j minus phi i, j. So, it is n plus 1, this is your n divided by delta t is equal to gamma.

So, you have phi i plus 1, j minus 2 phi i, j plus phi i minus 1, j divided by delta x square. And what will be the time level? As you were using backward time, it will be n plus 1. And similarly, the other term plus phi i, j plus 1 minus 2 phi i, j plus phi i, j minus 1 divided by delta y square, we have used central difference and time level will be at n plus 1.

So, you can see that we have more than one unknowns because the dependent variable at discrete point i, j, i plus 1 j, i minus 1 j i, j plus 1 and i, j minus 1 at phi discrete points, the value of phi is unknown. So, obviously you will get one penta diagonal matrix if you write these algebraic equation for each grid point and if you write in a matrix format then you will get pentane diagonal matrix.

So, let us simplify this algebraic equation with introducing the diffusion coefficient gamma x and gamma y. So, if you write gamma x as gamma delta t by delta x square and gamma y as gamma delta t by delta y square. So, then what you will get? So, now, you see there are more than one unknowns only known term is phi i, j.

So, phi i j let us keep in this side and phi i, j n plus 1, you take in the right hand side and if you write it then you will get gamma y T i, j minus 1 n plus 1 plus gamma x T i minus 1, j n plus 1 minus 1 plus, so this is 1 and this will be 2 gamma x and 2 gamma y.

So, 2 gamma x plus 2 gamma y sorry this is not T this is your phi and phi i, j n plus 1 then plus gamma x phi i plus 1, j n plus 1 plus gamma y phi i, j plus 1 n plus 1 is equal to minus phi i, j n. So, you can see, in the right hand side whatever we have that is known from the previous time level and in left hand side, all are unknowns.

So, obviously you can see the order of accuracy, order of accuracy is first order accurate in time and second order in space because you have used central difference. So, obviously, in a space CL direction it is second order and backward time you have used, so it is first ordered, so it is delta t.

Then if you do the stability analysis later we will show that it is obviously unconditionally stable, unconditionally stable, so obviously you have to use some method to solve these linear algebraic equations. Now, we will move to Crank Nicolson and we have shown in last lecture that Crank Nicolson method is second order accurate at least for the unsteady 1 dimensional equation and it is true for unsteady 2 dimensional equation as well.

So, we will consider the same equation and we will use the Crank Nicolson method and we will write the final algebraic equation. And obviously it will be unconditionally stable and order of accuracy in time it is second order and in space any way it is second order.

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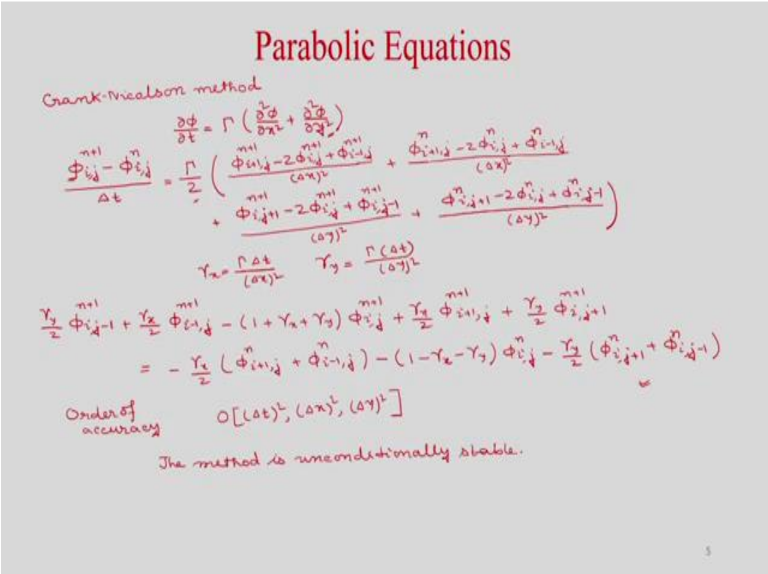

So, it is now we will use Crank Nicholson method, so in Crank Nicholson method now what we will do? Obviously, we have seen that whatever way we have used the BTCS, now in the special derivative, we will take the dependent variable at each discrete point average of n and n plus 1.

So, you can consider in a way that it is combination of BTCS plus FTCS method. So, whatever way we have done for the 1 dimensional equation, parabolic equation similar way also we will discretize this 2 dimensional unsteady equation. So, our equation is Del phi by Del t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square.

So, now in spatial derivative now, we will take both these BTCS type discretization and FTCS, so average of these two. So, it will be, we will write as gamma by 2. So, in the left hand side now, we will write phi i j at n plus 1 minus phi i, j at time level n divided by delta t. And it is now, gamma by 2 we writing and now you have this delta 2 phi by Del x square.

So, this you can write at phi i plus 1, j minus 2 phi i, j plus phi i minus 1, j divided by delta x square and at time level n plus 1 and also we will take at time level n because already you have taken the average. So, gamma by 2, you have written, so it will be phi i plus 1, j minus 2 phi i, j plus phi i minus 1, j divided by Del y square and it will be at time level n.

Similarly, Del 2 phi by Del y square, this also you discretized plus phi i, j plus 1 minus 2 phi i, j plus phi i, j minus 1 divided by Del y square at time level n plus 1, n plus 1 and plus phi i, j plus 1 minus 2 phi i, j plus phi i, j minus 1 divided by Del y square.

Sorry this is your delta x square at time level n. So, now you define gamma x as gamma delta t by delta x square and gamma y as gamma Del t by Del y square. And you rearrange this equation, now you have more than one unknown. All the unknowns you take in left hand side and all the known value which are at time level n, you take in the right hand side.

So, you can see from this equation, so you can see these whatever way we have write, written BTCS, you can write gamma x by 2 phi i, j minus 1 n plus 1 plus gamma x gamma sorry this is your gamma y gamma y by 2 gamma plus gamma x by 2 phi i minus 1, j at n plus 1 and this you can see, so it is 2, 2 will get cancel out.

So, this you are taking this side, so it will be it will minus because you are taking this in the right hand side. So, it will be minus 1 plus gamma x plus gamma y phi i, j at n plus 1 plus gamma x by 2 phi i plus 1, j n plus 1 plus gamma x by sorry gamma y by 2 phi i, j plus 1 n plus 1.

So, all are unknown terms and now, in the right hand side you write all the known terms which are at time level n. So, if you see now, you take all this in the left hand side so you will see you will get minus gamma x by 2 phi i plus 1, j at n plus phi i minus 1, j at n time level and now i j so i j if you take this side. So, you will get to the minus 1 so minus gamma x minus gamma y phi i, j n minus gamma y by 2 phi i, j plus 1 plus phi i, j minus 1 at time level n.

So, we know that this scheme is second order accurate both in time and space so, it is delta t square, delta x square and delta y square. So, this is the order of accuracy and the method is unconditionally stable. So, now we learn another method which is known as alternating direction implicit method, you have already learned this method for elliptic equation.

So, when we are iterating from K to K plus 1 for 2 dimensional equation, we have sown that in 2 steps method, we generally go from K to K plus half and K plus half to K plus 1. But we solve it line by line, so the unknowns will be 3.

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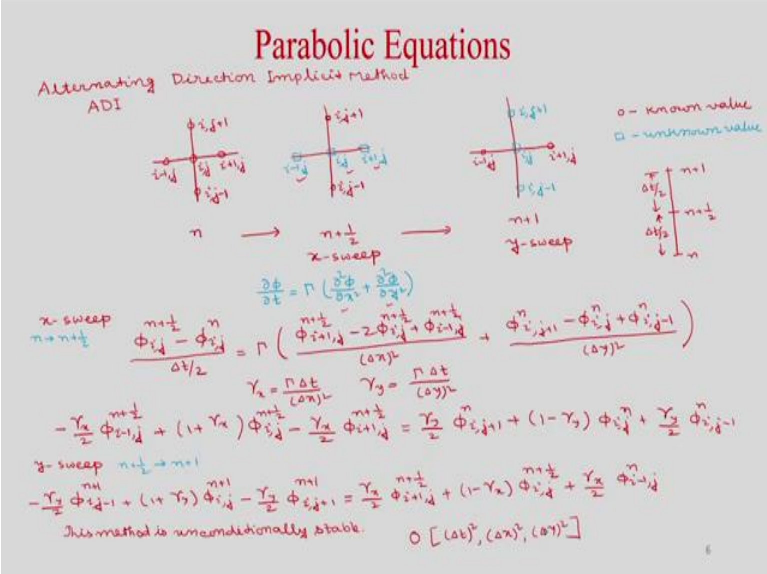

So, in this equation also, we will use this alternating direction implicit method, Alternating Direction Implicit method which is commonly known as ADI. So, in this case when we are going from n to n plus 1, we will split into 2 steps. When we will go to n to n plus half, we will do x sweep and from n plus half to n plus 1, we will do the y sweep or vice versa.

So, if you see, is that if you have the grid in 2 dimensions, so these are the grid. So, this is your i j, this is your i plus 1 j this is your i minus 1 j i j plus 1 and i, j minus 1. So, this is at time level n. Now, when we are going from n from n plus half, let us say we are using x sweep.

So, if you use x sweep, so in the x direction all will be unknowns. So, if you again draw it, so it will be i j plus 1, this will be i, j minus 1 and these 3 values at i, j, i plus 1, j and i minus 1, j will be unknown. And next, when we will move from n plus half to n plus 1, we will do the y sweep.

So, now you are moving from n plus half to n plus 1. So, now we are doing y sweep. So, now you are, (())(28:04) this will be known. So, it is i minus 1, j. This is your i plus 1, j and the circle unknown, known value and unknown will be at point i, j plus 1, i j and i, j minus 1. So, this square box wherever we are showing this is unknown.

So, now let us write the x sweep and y sweep for the governing equation Del phi by Del t is equal to gamma Del 2 phi by Del x square plus Del 2 phi by Del y square. So, if you write the discretize equation, so if you write x sweep first. So, in x sweep only the value of phi at these 3 discrete points i, j, i plus 1, j and i minus 1, j will be unknown.

So, we are moving, so now you can see. So, when you are moving from n to n plus half, it is intermediate time step 2 n plus one. So, obviously this is your Del t by 2 and this is your Del t by 2. So, you are moving from n to n plus half with the time step size delta t by 2. So, for this equation, if you write, it will be phi i, j at n plus half minus phi i, j n divided by Del t by 2.

And in the right hand side, it will be only in the as we are doing in the x sweep. So, in the Del 2 phi Del x square, we will take all at n plus 1 time level and when we are discretising Del 2 phi by Del y square, we will take at the time level n and Del 2 phi by Del x is square, we will take at the time level n plus half.

So, it will be gamma phi i plus 1 j minus 2 phi i j plus phi I minus 1 j divided by delta x square. So, that will take at n plus half and plus phi i j plus 1 minus phi i j plus phi i j minus 1 divided by delta y square. So, it will be at previous time level n.

So, now if you write the equation, the final algebraic equation you will write in this way. So, gamma x let us define as gamma Del t by Del x square and gamma y as gamma Del t by Del y square. So, in the x sweep equation, you will write minus gamma x by 2 phi i minus 1 j at n plus half plus 1 plus gamma x by 2 phi i j at n plus half minus gamma x by 2 phi i plus 1 j n plus half. So, these are unknown. So, in the right hand side, you write all the known term.

So, it will be gamma y by 2 phi i j plus 1 at n plus 1 minus sorry this will be gamma x only because 2 at there, so it will be cancel out. So, 1 minus gamma y phi i j n and phi gamma y by 2 phi i j minus 1 at n, so how many unknowns are there in the left hand side? You can see 3 unknowns at 3 discrete points i j, i plus 1 j and i minus 1 j because we have written discretise equation for x sweep.

So, obviously, you can see that if you write for each point these linear algebraic equation and from the matrix, you will a tridiagonal matrix and you can easily solve using a using Thomas algorithm or which is known as tridiagonal matrix algorithm. Similarly, now if you do the y sweep then there will be 3 unknowns, phi i j, phi i j plus 1 and phi i j minus 1.

So, if you write the discretize equation, so it will be now we are doing y sweep, I am writing the final equation. So, it will be minus gamma y by 2 phi i j minus 1. So, in y sweep what we are going, so in y sweep we are moving from n plus half to n plus 1. In y sweep, we have moved n to n plus half, so time step size delta t by 2. So, we can write it as 1 plus gamma y phi i j.

So, it will be n plus 1, this will be n plus 1 and minus gamma y by 2, phi i j plus 1 n plus 1. So, all unknown terms, we have written in the left hand side. Now, you write the right hand side, so gamma x by 2 phi i plus 1 j. So, it will be at n plus half because we are moving from n plus 1 to n plus half. So, n plus half time level, all the values are known from the x sweep equation.

So, it will be 1 minus gamma x phi i j n plus half and gamma x by 2 phi i minus 1 j. So, in this method, this ADI method obviously we are moving in 2 step method, one is from n to n plus half either you do x sweep or y sweep then when you are moving from n plus half to n plus 1, you do y sweep or x sweep.

So, now the advantage of this ADI method is that it is second order accurate in time. So, this method is second order accurate in both, space and time and also it is unconditionally stable. So, this method is unconditionally stable. And order of accuracy is delta t square, delta x square and delta y square.