**Computational Fluid Dynamics for Incompressible Flows**
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**Lecture 13**
**Finite difference formulations of Parabolic Equations**

Hello everyone, so in last module, we have discretized Elliptic Equation. So, we considered steady heat conduction equation or Laplace equation and discretize using different finite difference formulations.

Today, we will consider parabolic equations. In this lecture, we will learn the finite difference formulation of parabolic equations. The governing equations for some problems in fluid mechanics in heat transfer are parabolic. So, there are many examples like parabolized Navier-Stoke equation is parabolic in nature. Even if you consider flow over flat plate and boundary layer equation if you considered then that is also parabolic equation.

In heat transfer, also if it is unsteady conduction then that is also parabolic equation. So, you can see that different examples are there in fluid mechanics and heat transfer of parabolic equations.

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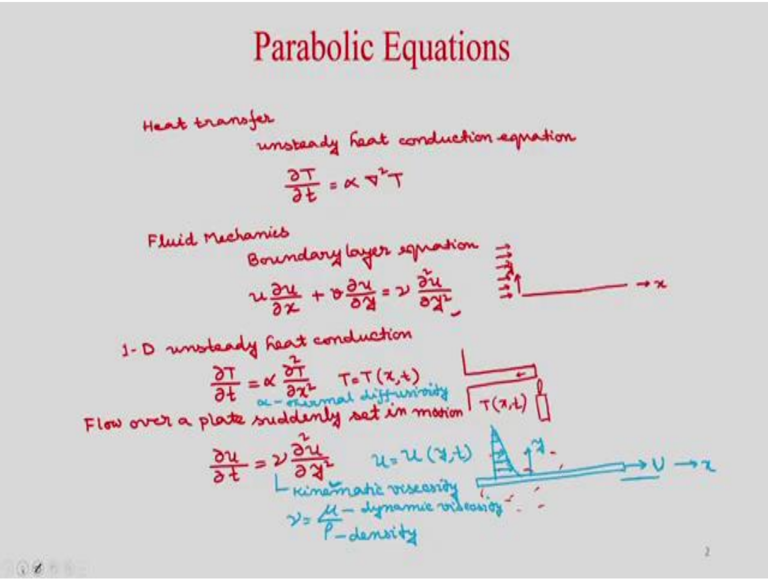


So, if you consider let us say in heat transfer, so if you consider unsteady heat conduction equation. So, if you consider del T by del t is equal to alpha del square T. Then this is parabolic in nature. Again if you consider the boundary layer equation in fluid mechanics, so boundary layer equation.

So, if you consider flow over flat plate, then you can write the governing equation as u del u del x plus v del u by del y is equal to nu del 2 u by del y square, if your flat plate is this and in this direction it is x and in this direction it is y. So, the whatever is in the F is uniform flow is coming then whatever boundary level you forms, so that you can write with some assumptions this governing equation and this is also parabolic in nature.

But you should see that there is no temporal term in this equation. So, you have to march in the x direction. So, in that way it is a parabolic equation. Now, we will consider one dimensional unsteady heat conduction then what will be the governing equations? So, if it is considered 1 D unsteady heat conduction. So, if you consider a rod, so if you consider only one dimensional heat conduction and if you are heating this side, if you are heating this side then obviously with time there will be heat transfer in this directions.

So, if is the high temperature then obviously high temperature to low temperature there will be heat transferred and this will be this T will be a function of x and t only. So, in that case you can write the governing equations as del T by del t is equal to alpha del 2 t by del x square. So, in this case T is function of, T is function of one space coordinate and time, so 1 D unsteady at connection. So, if you consider flow over a plate suddenly set in motion. So, we have a stationary fluid, one plate is there and suddenly you are moving with a velocity u.

So, in that case with some assumptions the Navier-Stoke equations you can write down in similar form. So, that will be del u by del t is equal to nu del 2 u by del y square. So, this is a problem of flow over a plate, suddenly set in motion. So, what is this, say if you have some medium, you have a fluid medium and let us say you have a plate, so infinite plate. So, that the end effects can be neglected. So, it is a infinite plate, at t is equal to 0, it is stationary both the fluid and the plate and suddenly you have set the motion in the plate with a velocity u.

So, in the x direction, let us say it is the x direction and this is your y direction. So, suddenly this plate has started moving with a velocity u in the positive x direction, so obviously due to the CR effect, the fluid adjacent to the plate that also will start moving. So, continuously its velocity will penetrate in a y direction. So, you will get some velocity profile with time it will vary and like this. So, it is moving with velocity u.

So, this is your velocity profile will look like this and it will penetrate gradually with time. So, this is the unsteady problem and with the assumptions you can see from the Navier-Stoke location you can drop the conviction term in the left hand side. Only the temporary term will be there in the left hand side.

In the right hand side as it is infinite medium there is no pressure gradient. So, pressure gradient will be 0. So, u is only function of y, one space coordinate and time. So, you can see. So, this is your one dimensional unsteady equation which is parabolic in nature. So, today, we will use or we will learn difference finite difference formulations to discretize this simplified governing equation which is unsteady one dimensional equation, parabolic differential equation.

So, in general, we will write, so obviously you know that in this case your alpha is thermal diffusivity which is diffusion coefficient diffusivity. So, alpha is thermal diffusivity and it is known as the diffusion coefficient. Similarly, you know that this nu is kinematic viscosity. So, you can see this is also one diffusion co-efficient, so it is nothing but nu is equal to nu by row where nu is your dynamic viscosity and row is your density.

So, now we will write one model equation which is unsteady and in space coordinate, it is one dimensional. So, this parabolic differential equation just we will consider where we will write for any general variable phi.

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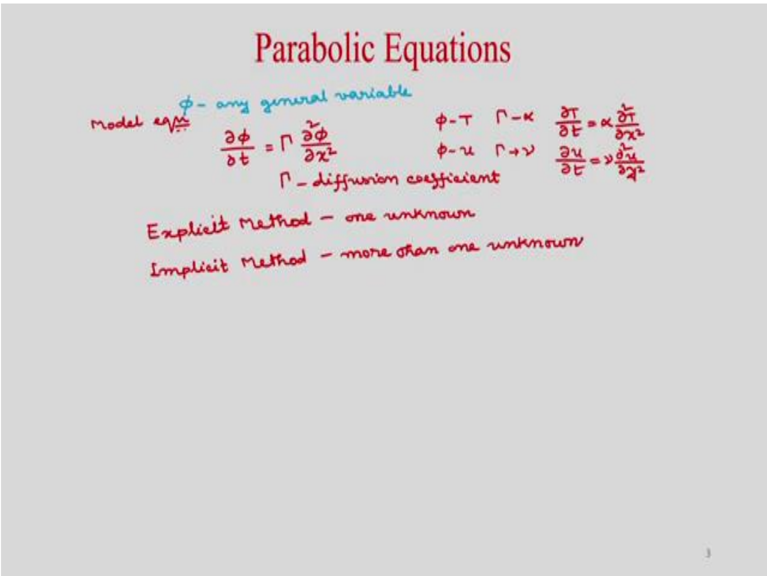


So, we will write for any general variable phi, so if you write the governing equations then we can write del phi by del t with some diffusion coefficient gamma and del 2 phi by del x squared. So, whatever we have seen that 1 D unsteady heat conduction equation, here phi is equal to say T, phi is equal to T, if you put and gamma is equal to alpha if you put then you will get this governing equation del T by del t is equal to alpha del 2 t by del x square.

So, this a governing equation for unsteady 1 D heat conduction, similarly, if you put phi is equal to u and gamma is equal to nu then you will get this equation. So, our x square you can, so x is in that direction. So, it is a model equation. So it is a model equation. So, we are considering this model equation which is your parabolic differential equation and we will digitize this equation using different finite difference formulations, where gamma is your diffusion coefficient, gamma is your diffusion coefficient.

Now, in last model also we have learned that when you are going from iterable k to k plus 1 and when you are discretizing if there is one unknown then you have the explicit scheme or explicit method and if you have more than one unknown then you get implicit scheme or implicit method. So, here also we will discretize this equation using these 2 methods, one is explicit, explicit method where you have one unknown. So, the governing equation only one unknown will be there.

So, all known terms you can take in the right hand side and you can solve this equation easily and if you have more than one unknown then it will be implicit method, implicit method, so more than one unknown. So, in each grid point you will get one linear algebraic equation and for each grid point, if you write those equations and you will get a set of linear algebraic equations and that if you form in a matrix then you will get some matrix form and you need to solve that setup governing equations in implicit method.

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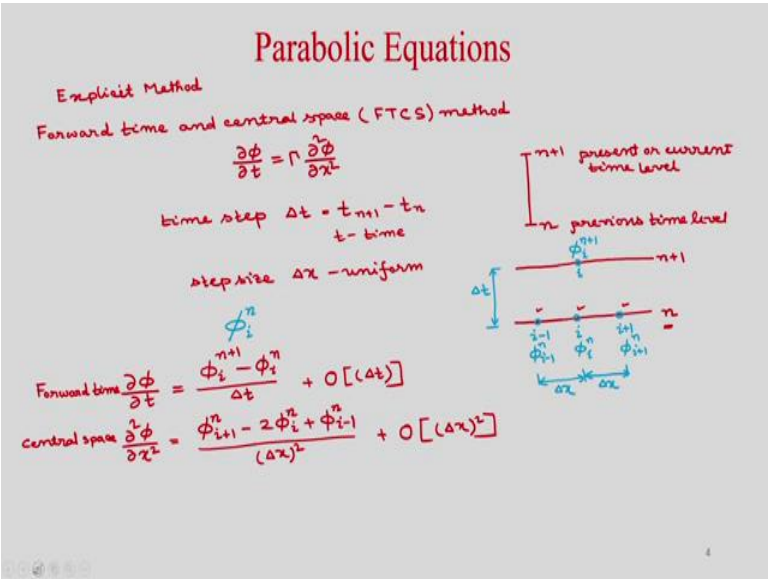

So, today first we will study the simplest one and which is explicit method, explicit method. We will use and this scheme is known as forward time, forward time and central space. This is commonly known as FTCS, F is for forward, T Time, central, space. So, the model equation is your del phi by del t is equal to gamma del 2 phi by del x square.

So, now as we have to march in time, so we will take in the nth level as previous time level, previous time level and in plus 1 as present or current time level, present or current level. So, now we are marching in time from n to n plus 1, n is your previous time level and n plus 1 is the present or current time level where we are actually interested to find the value of dependent variable phi.

So, if you remember in elliptic equation we actually went from previous iteration level to present iteration level from L to L plus 1 or K to K plus 1. So, similar way here, we have to march in time, so we have to go from n to n plus 1 and the time difference between n and n plus 1 will take as time step delta t. So, time step delta t, so this will be delta t is nothing but the time at n plus 1 minus time at n.

So, this is your t is just you are representing as time. So, now we are, we have to take the delta t as timestep, like in grid when you discretize the domain into grid you get the step size as delta x. So, this is also time step size delta t. So, you are moving from time level n to n plus 1 and the difference between these 2 times step to time is known as timestep delta t.

And obviously here also we will use the uniform step size in space, so we will consider step size as delta x which is uniform. So, if you consider the grid level at level n, so this is the nth time level. We have the one dimensional grid because only we have considered x and the index in x direction will take as i, for each discrete points, we will just write with the index i.

So, if you can see, so if you have, if you take 3 points, this is your i, this is your i plus 1 and this is your i minus 1. So, we are considering at the nth time level which is previous time level and the values of dependent variable at that time step are known. So, this is your phi n i, this is your pi i plus 1 n and this is your phi i minus 1 n.

Now, you consider the present time level, so that is your n plus 1. So, this is in time we are marching, so we are going to n plus 1 and similar this grid points will be there and where you have a uniform step size, you have uniform step size. So, this is your delta x and this is also your delta x.

So, your delta x, this is your delta x, this is also your delta x, this is your uniform step size in grid. So, that is your delta x and this is your delta t, so you are moving from here. So, this is your time step. So, this is your delta t. And obviously, this is at i and this value will be denoted as phi at point i and you can see the time level whatever we are writing as a superscript.

So, this is your n plus 1. So, you see, so when we are writing phi, so grid point we are writing as a subscript i or i plus 1 or i minus 1 and the superscript, we are writing the time level whether it is n, n means previous time level and n plus 1 is current or present time level.

So, we are marching in time and the unknown dependent variable is at time level n plus 1. So, now with this discretized grid, you can see that we can write these governing equations. So, if you have a time derivative which is your first derivative of phi with respect to time.

So, in this case to discretize this first derivative we will use forward time, finite difference approximation. So, if you use it that, so del phi by del t. So, if you discretize this first derivative of phi with respect to time with the forward time difference approximation, then you can write as at point level i you are doing, so phi i n plus 1 minus phi i n divided by delta t.

So, you can see this is the forward time phi i n plus 1 minus phi i n divided by delta t. So, what is the order of accuracy of this method finite difference formulation? Obviously, it is a first order accurate. So, it is first order accurate, order of delta t. Now, the special discretization if you do, so we have del 2 to phi by del x square. So, to discretize this equation you just use second order central difference, so central difference method you know that is your second order accurate method.

So, we will use central difference for this, so it is forward time we are using for time derivative and this is using’s in central difference method. So, that means central in space. So, what will be your discretization? So, phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square. So, this is the step size in space, so delta x square. So, obviously order of accuracy is delta x squared.

So, now when we use this explicit scheme which is forward time and central space method, so in the special discretization all the values we will take from the previous time level, so that is at time level n. So, here you can see, here we will take the superscript as n, so super script denotes the time level. So, here n and all the values of phi, so you can see here this one and this one we are taking from the time level n to discretize the spatial derivative.

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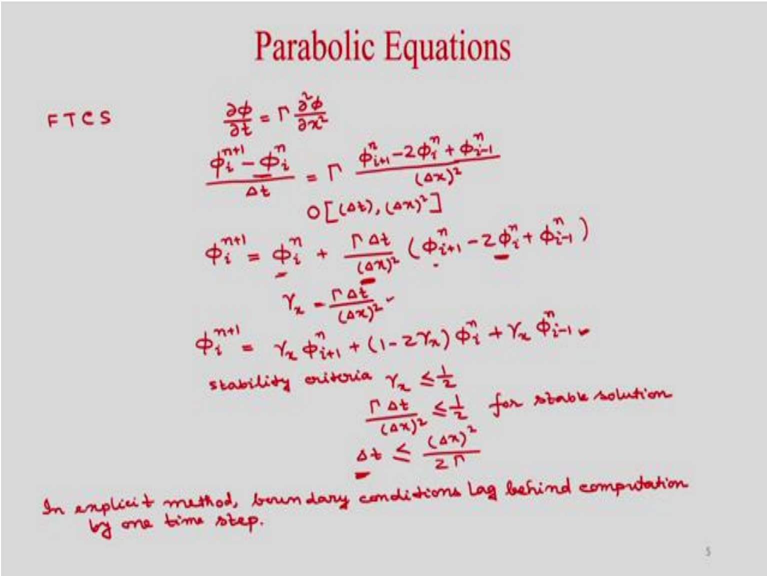


So, now if you write this equation then what you will get, you can write. So, your governing equation is del phi by del t is equal to gamma is the diffusion coefficient del 2 phi by del x square. So, you are using forward time, so superscript is the time level of n plus 1, subscript is your grid index i minus phi i n divided by delta t.

So, it is first order accurate and in the right hand side, the special discretization we have used central difference method and the superscript we will take all n time level means previous time level and this is your subscript which you are using the for the grid notation.

So, this is your phi i n, this is will be, so minus twice phi i n plus phi i minus 1 n. So, you can see that all these superscript, we have taken from the previous time level which is n divided by delta x square. So, the overall accuracy of this finite difference formulation is order of first order delta t and second order delta x square. So, you can see it is first order accurate in time and second order accurate in space, so that is known as FTCS, forward time and central space.

So, now if you write the final algebraic equation, so you can write as phi i n plus 1 is equal to, so you take delta t here, so gamma delta t divided by delta x squared, so this minus phi i you take in the right hand side. So, it will be phi n plus gamma delta t by delta x square and you have phi i plus 1 n minus twice phi i n plus phi i minus 1 n.

So, you can see in the right hand side, all the dependent variables are at time level n which are known. So, only one unknown is phi i n plus 1 which is at the current time level and this value you have to find from the known values of phi at time level n.

This term, we will write as gamma x, so gamma delta t by delta x square. So, if you denote gamma delta t by delta x square as gamma x, then you can ride these discretize equation as phi i n plus 1 which is unknown and you can see that here 2 phi i n is there and here phi i n is there.

So, if you combine these two with the gamma x then you can write gamma x, this you write first phi i plus 1 n plus 1 minus 2 gamma x phi i n plus gamma x phi i minus 1 n. So, this is the discretize equation and obviously, we have used expressive method. So, 1 unknown is there in the left hand side and right hand side, all are known from the previous time level n.

So, if gamma x you can see, so gamma x is gamma delta t by delta x square, so gamma is the diffusion co-efficient and delta t is the time step and delta x is your special step size. So, you can see that if you do the stability analysis, for this scheme which we will learn later. So, we can show that the solution is stable when gamma x is less than equal to half.

So, that is known as stability criteria we will learn in later. So, the stability criteria for this finite difference formulation stability criteria, so that means solution will be stable if gamma x is less than equal to half. We will show it later, so what does it mean? That gamma delta t by delta x square should be less than equal to half for the stable solution for stable solution.

So, you can see here gamma is the diffusion coefficient. So, that is obviously constant, delta x if you fix the grid then delta x is constant. Then delta t you have to find from, so delta t will be less than equal to delta x square divided by 2 gamma. So, delta is it is restricted.

If your grid is fixed and fluid obviously fluid if you see, so gamma will be fixed for a particular fluid, gamma will be fixed and for a period grid delta x will be fixed, so delta t up to choose such a way that it should be less than equal to delta x square by 2 gamma for getting the stable solution.

So, now here you can see that in explicit scheme, in explicit as we are using explicit scheme here, so you can see that boundary conditions will lack by computation by one time step. Because you are solving for n plus 1 then obviously the boundary condition will be at nth time level.

So, in explicit method, explicit method boundary conditions, boundary conditions, lag behind computations by 1 time step. So, when you are solving at the interior points this discretize equation, you have to get the boundary values and that boundaries values you have Neumann type boundary conditions then obviously, that is not constant. Then that will lag behind computations by 1 time step. So, the values you have to take from the previous time level.

So, obviously, so you can see that FTSC is very simple to solve this equation because 1 unknown is there and in right hand side all are known from the previous time level. So, computations or writing the program is very easy, but you have the time step restriction. So, from the stability criteria you can show that solution of this finite difference formulation is stable if gamma x is less than equal to half. So, delta t is restricted. So, we have to choose the delta t from your fixed step size delta x and the diffusion coefficient gamma.

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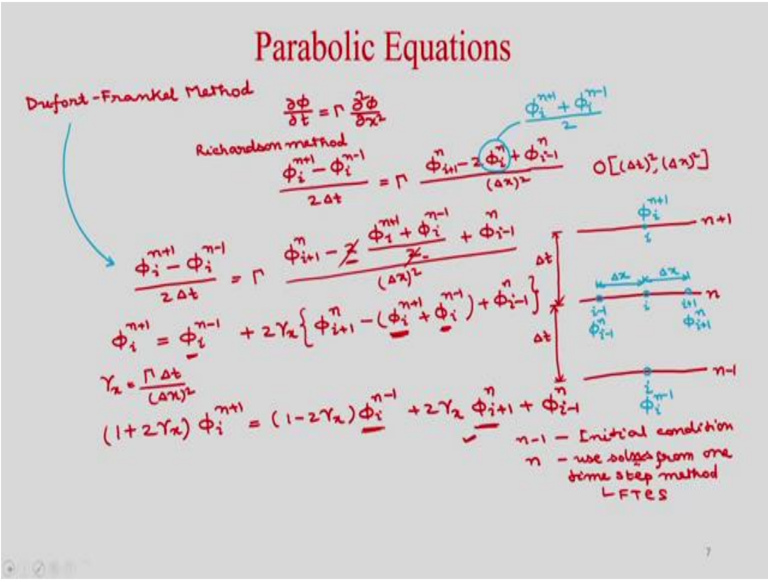


Now, next we will learn Richardson method, Richardson method. In Richardson method, what we will do? We will use central difference in time derivative as well. So, that overall accuracy will be delta t square and delta x square that means second order accurate in time and second order accurate in space.

So, the governing equation you have seen that this is your del phi by del t is equal to gamma del 2 phi by del x square. So, now you use central difference for this first derivative del phi by del t. So, what you can use then, phi, so if you use it, so phi i n plus 1 minus pi i n minus 1 divided by delta t divided by 2 delta t. So, this is your second order in time and second order in space. So, that means phi i plus 1 n minus 2 phi i n plus phi i minus 1 n divided by delta x square.

So, what is the order of accuracy of this finite difference formulation that is order of delta t square and delta x square. But if you do the stability analysis of this finite difference formulation then you will see that it is unconditionally unstable. So, the method is unconditionally unstable. So, we will show it later using the stability analysis, we will show later that thus method this finite difference method is unconditionally unstable. So, the solution of this method is unconditionally unstable.

So, obviously it has no practical use because it is unconditionally unstable. So, it has no practical value. Because whatever steps size, time steps size you take it will be the solution will be unstable, so it has no practical value. So, now we will learn another explicit method that is known as Dufort-Frankal method, so Dufort-Frankal method.

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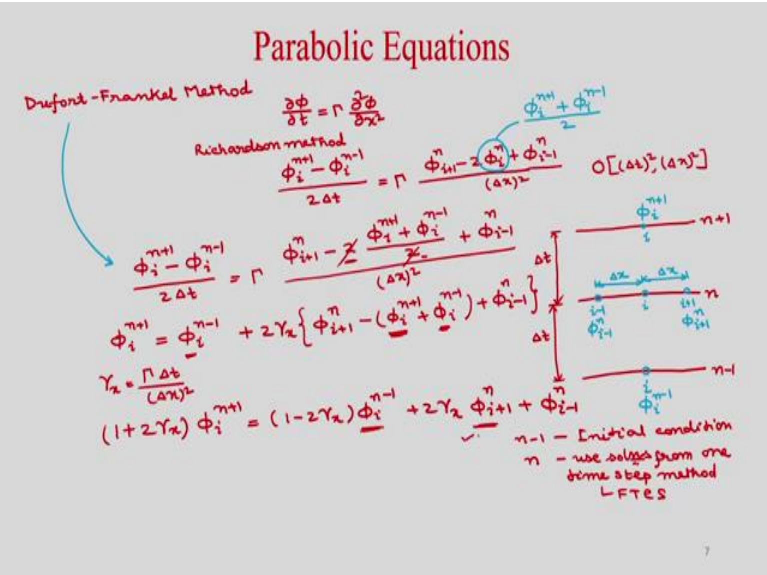


So, Dufort-Frankal method is the modification of this Richardson Method. Richardson method was unconditional and stable. So, you cannot use it because it has no practical value, but with some modification, you can use it which is second order accurate in time and space.

So, in this method what we will do? So, we will use the Richardson method, but in the temporal discretization you have phi i n. So, in temporal discretization, if you see the phi n that will take the average value of phi i at n plus 1 and phi i at n minus 1, so let us write first Richardson method.

So, if you will see the Richardson method, whatever we have seen. So, your governing equation is del phi by del t is equal to gamma del 2 phi del x square. So, this is the second time accurate, so it will be phi i at n plus 1 minus phi i at n minus 1 divided by 2 delta t. So, it is second order accurate is equal to gamma. Now, this is also second order accurate in space.

So, it will be phi i n minus 2 phi, this is i plus 1 this is your 2 phi i n plus phi i minus 1n divided by delta x square. So, you can see that this is order of delta t square and delta x square, so this is the order of accuracy, second order accurate in space and time. So, here now what we will do the modification? In the spatial discretization you can see this term, this term in the special discretization we will take the average value of phi at n plus 1 plus phi i at n minus 1 divided by 2. So, the average value we will take.

Then if you write it then for this Dufort-Frankel method you can see that the discretization will be phi i n plus 1 minus phi i n minus 1 divided by 2 delta t is equal to gamma phi i plus 1 n minus 2. So, now we are replacing the average value at n plus 1 and n minus 1 phi i n minus 1 divided by 2.

It is average value plus phi i minus 1 n divided by delta x square. So, now if you see the time level so, you have 3 time levels are involved, one is n, n minus 1 and n plus 1. So, n plus 1 obviously your current time level, n is your previous time level and previous to previous time level is n minus 1.

So, if you see the time level, so these are the time levels and we will also see the grid points. So, this is your n, this is your n plus 1 and this is your n minus 1. So, this is your delta t time step you are going from n minus 1 to n and similarly from n to n plus 1, you are going to with the timestep delta t.

So, now you can see the grid points, so grid points obviously, will be say n minus 1. So, n minus 1 will be your, only the phi i is required. So, it is i point, where phi i n minus 1 is required. At nth level you need 2 points. So, this is your i, this is your i plus 1 and this is your i minus 1. So, this will be phi i plus 1 at n and this is your phi i minus 1 at n and at n plus 1, you actually need to find the value of phi at i point, so this your phi i n plus 1. So, obviously we have a uniform step size delta x.

So, you can see now, so to find the unknown at time level n plus 1 which is your current time level, you need the values at previous time level n and previous to previous time level n minus 1. So, at the starting you need to have the values known at time level n and n minus 1. So, it will be very difficult sometime same values will be used at the starting point.

But it will have some error because if it is time accurate then if you take the both the values at same as 2 time levels then obviously there will be some error. So, it is better that you use single time step method at the starting point.

When you are starting the solution, the first iteration or fist time iteration you use 1 time step method like explicit method, we have learned FTCS, forward time and center space. So, this is a single time step method where only you need the values at previous time level n. So, if you use this 1 time step method then you will get some values at n plus 1.

So, that you can use as n and n minus 1 as the initial condition to find the value at n plus 1 in this Dufort Frankel scheme. So, what I am telling, you can see, first let me write the algebraic equation then I will discuss again. So, now to write it, so you can see that it will be phi i n plus ones only 1 unknown.

So, phi i n plus 1 which is only one unknown, so that will keep in the left hand side and we will take gamma x as gamma delta t by delta x square. So, you can write it now as, so this will go in this side, so phi i n minus 1 plus 2 into gamma x. Now, you write in bracket phi i plus 1 n. So, this, you can see these 2, these 2 will cancel out. So, you can write minus phi i n plus 1 plus phi i n minus 1 plus phi i minus 1 n.

So, in the right hand side also phi i n plus 1 is there, so that you take in the left hand side, so this phi i n plus 1, if you take in the left hand side, then you can write, so this is 1 plus 2 gamma x is there. So, 2 gamma x, so this is your phi i n plus 1. So, this is your unknown. So, in the right hand side, now you can see 1, so this is i n minus, you can see here, phi i n minus 1 and here phi i n minus 1.

So, you can write 1 minus 2 gamma x phi i n minus 1 and here you have 2 gamma x phi i plus 1 n plus phi i minus 1 n. So, you can see in this discretize equation, so left hand side you have only one unknown term that is your phi i n plus 1 which is to be evaluated at time level n plus 1.

In the right hand side, you can see that you have phi at time level n and n minus 1. You can see, it is phi i n minus 1, this is your phi i n, so phi at time level n. So, now at the starting point, when you are starting the solutions, so that time only initial conditions are available, only initial condition is available.

So, if only initial condition is available, then you will not get the values in both n and n minus 1. So, you start with some 1th time step method like FTCS, forward time and center space, then you phi evaluate the n plus 1 from the known value of the from the initial condition.

So, one it is known then n minus 1, you take as initial guess sorry initial condition. So, n level, so now, you get the solutions using one time step method. So, use solutions from one stand step method, one time step method. So, like such as FTCS, forward time in central space.

So, now, you have the values available at n, n minus 1 after getting the solution from one time step method. Now you can use this Dufort Frankel method which is second order accurate both in space and time, so that you will be able to use and solve this algebraic equation.

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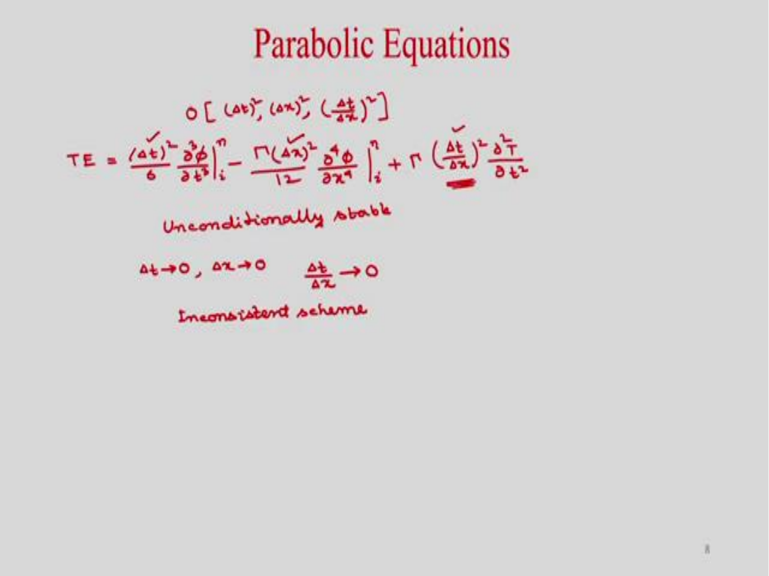


But, if you see the order of accuracy of this method, so the truncation error if you find. So, the order of accuracy will be delta t square delta x square and del t by del x square. So, this is the transition error. This is the order of accuracy. It is not only delta t square and delta x square, it is also the order of accuracy is del t by del x square.

So, if you write the truncation error for this Dufort Frankel method then it will be truncation error as delta t square by 6, del cube T by del t cube at grid point i, time level n, it is not T, it is del cube phi minus gamma del x square by 12 del 4 phi by del x 4 at time level n and grid point i plus gamma del t by del x square del 2 T by del t square.

So, you can see the order of accuracies, you have delta t square delta, x square and del t del x square. So, and if you do the stability analysis you will find that it is unconditionally stable. So, the solutions are unconditionally stable, unconditionally stable.

So, although the solutions are unconditionally stable, but the problem is from this truncation error. So, it is inconsistent scheme, why inconsistent scheme? Because as you refine the grid delta x tends to 0 and if you reduce the timestep delta t tends to 0 then you will see that delta t by del x not tends to 0. So, your truncation error will not go tends to 0.

So, it is an inconsistent scheme. So, what I am telling that if it delta t tends to 0, if you reduce the timestep size and delta x refine the grid, so delta x tends to 0. But del t by del x, it may not tend to 0. So, in the truncation error this term will remain and your truncation error will not tend to 0.

So, obviously it is inconsistent scheme. So, but the advantage is that it is unconditionally stable, because you do not have any restriction to choose the delta t. But at the same time, you have that it is a inconsistent method. So, carefully you have to use delta t and delta x to get a good solution. Today, we will stop here, thank you.