Computational Fluid Dynamics for Incompressible Flows
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Lecture 2
Iterative Methods

Hello everyone, so now we are discretizing the Laplace equation and in last class we learned how to use the Central Difference method to discretize the Laplace equation and we wrote the system of linear equations. Today, in this class we will study different iterative methods to solve the system of linear equations.

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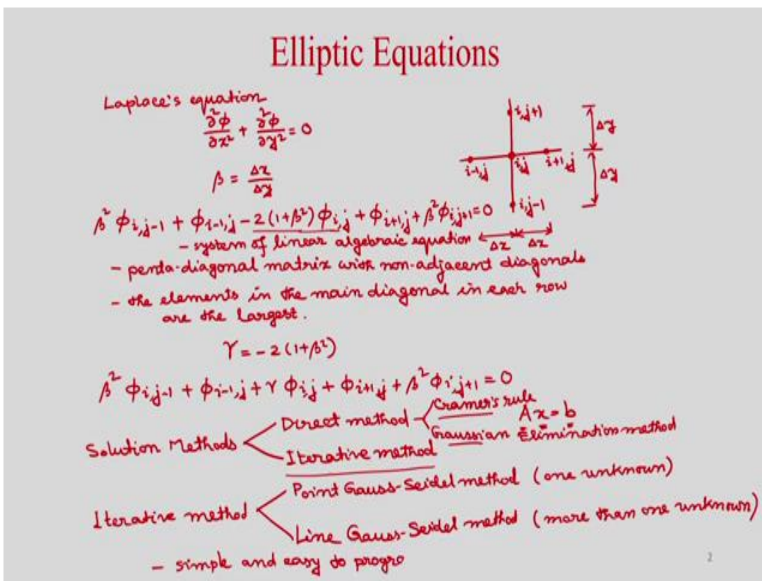

So, as you remember in last class, we tooK this Laplace equation. So, this is elliptic equation and obviously phi is any general variable. So, we can apply to fluid flow problem if we replace phi is equal to psi then we will get stream function equation for 2D steady, incompressible and irrotational flows.

So, the discrete points, so this is ij, this is your i plus 1j these are neighbor points, i minus 1j, this is your ij plus 1 and this is your ij minus 1 and we have a constant step size delta x and in x direction and in y direction also we have a constant step side delta y and we define the ratio as beta delta x by delta y and using five point rule and using central difference method discretizing this equation we wrote, the system of linear equations, that is your beta square phi ij minus 1, plus phi i minus 1j then minus 2 into 1 plus beta square phi ij, then phi i plus 1j and plus beta square phi ij plus 1 is equal to 0.

So, if you see this linear algebraic equation, you can note that it is a pentadiagonal matrix with non-adjacent diagonals. So, this is your system of linear algebraic equation and it is a penta. So, if you write for all the grid points then you will get a pentadiagonal matrix with non-adjacent diagonals.

And if you observe this equation you will see that the elements in the mid row or main diagonal, it is the largest, so you can see that this is your main diagonal, in each row, this is the largest one. So, you can write the elements in the main diagonal, in the main diagonal in each row are the largest.

So, if we write this coefficient as gamma, gamma is equal to minus 2 into 1 plus beta square, then easily you can write this equation as beta square phi ij minus 1 plus phi i minus 1j plus gamma phi ij plus phi i plus 1j and plus beta square phi ij plus 1 is equal to 0. So, here your diagonal coefficient, we are denoting as gamma, which is minus 2 into 1 plus beta square, where beta is the ratio of the step size delta x by delta y.

So, how to solve this equation so, these equations you can solve using two different methods. One is direct method and another is iterative method. So, solution method you have one is direct method and another is you have iterative method. So, in direct method actually you can write or form all these linear algebraic equation in Ax equal to b form.

So, where you can use some direct method, you can use Cramer’s rule or Gaussian elimination method and you can find x because this A known, the coefficient of all these metrics and b is known, only unknown is x and if you invert it, then you will be able to find the value of x.

So, here value of phi. So, you can use some direct method like Cramer’s rule or Gaussian elimination method. So, Cramer’s rule you can use or Gaussian elimination method. So, it takes enormous amounts of arithmetic computation per each iteration, as it takes more time. So, we can use some iterative method. So, in iterative method, we solve for each point.

So, there are two ways we can solve, one is Point Gauss Seidel method and another is Line Gauss Seidel method. Iterative method, we can use Point Gauss Seidel method, Point Gauss Seidel method another is Line Gauss Seidel method, in Point Gauss Seidel method we have only one unknown.

So, if one unknown is there then you can use Point Gauss Seidel method it is kind of explicit method in parabolic equation and if more than one unknowns are there then you can use Line Gauss Seidel method, which is kind of implicit method for parabolic equation.

So, it is one unknown and here have more than one unknown. So, these iterative methods are simple and easy to program. So, mostly we use iterative method for computation of system of linear algebraic equations. So, these are simple and easy to program, simple and easy to program.

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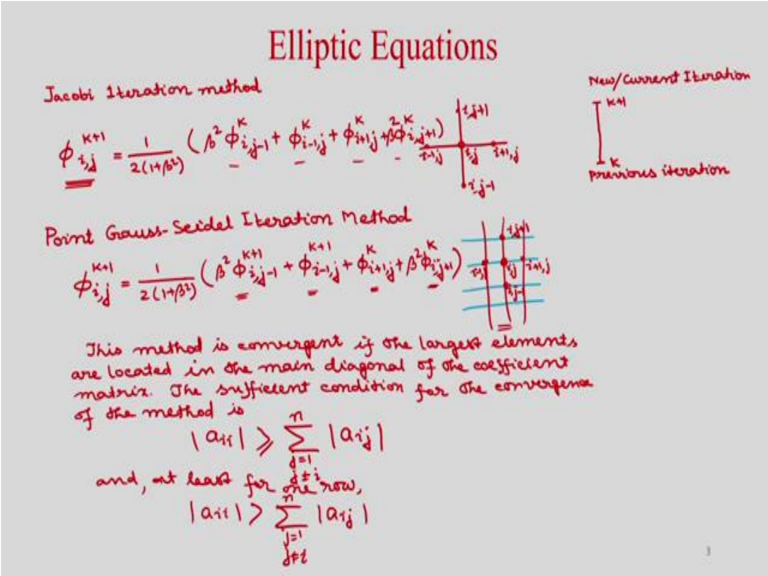

So, now we will study the Jacobi iteration method, Jacobi iteration. So, in Jacobi iteration method you have to first use some guess solution, so first you have to use the guess value and say, at level K which is your previous iteration, previous iteration, K is the previous iteration or if you are starting the program then it is the guess value and you are going to it new iteration that is K plus 1, new or current iteration. This is your K plus 1.

So, this is the iteration method, so you are going from K to K plus 1. So, if you are starting the solution then K is your guess value or earlier computed value. So, it is a previous iteration value and K plus 1 which you are actually trying to solve. So, that is your new or current iteration.

So, in this method the dependent variables at each grid point is solved using the value of the neighboring points. So, we have neighboring points, you are solving at ij and you have four neighboring points, one is i plus 1j and i minus 1j, here ij plus 1 and here ij minus 1. So, now, as you are solving for T ij, so, that is your only unknown and these T ij which is you are trying to find at iteration level K plus 1 which is your current iteration or new iteration and all the other values, neighboring points you are taking from the previous iteration value.

So, if you see then you will get this into 2 1 plus beta square. So, this will be phi. So, it will be beta square phi ij minus 1 plus phi i minus 1j plus phi i plus 1j plus phi ij plus 1 so here will be beta square. So, now you can see these are all neighboring points. So, all these neighboring points you take from the previous iteration value. So, previous iteration value is K.

So, you take from the K. So, their superscript in the phi, we are giving the iterative level and the suffix we are giving the index. So, ij, so, these are indices of, in x direction and the y direction. So, at any two dimension their grid point is denoted by the indices ij and the superscript in phi we are denoting with the iteration level. So, K plus 1 is the current iteration level and K the previous iteration level.

So, previously iteration level we know all the values and K plus 1 is the unknown. So, we are trying to find the value of phi at ij at time level K plus 1 with the known previous values of phi of the neighboring points.

So, you can see here one unknown. So, it is phi ij K plus 1 is unknown and all are known from the previous iterative values. So, it is having some disadvantage because it is very slow because although we have some current available values are there but we are not using. So, now it is these Jacobi iterative methods are not used. So, to accelerate these

Jacobi method we can use the based available values or current available values of the neighboring points so, that is your Gauss Seidel method.

So, Gauss Seidel iteration methods, so that is your Point Gauss Seidel iteration method. So, you can see that when you are iterating or you are going in this direction. So, each row you are solving, so row by row if you are solving. So you are solving, say these are different rows, so this is your ij, this is your i plus 1j, this is your i minus 1j and this is your ij minus 1 and this is your ij plus 1.

So, if you see that when you are going line by line, then when you are solving for the ij already in the earlier row you have solved for ij minus 1 point and also this i minus 1j, because this is when you are solving it already you have solved, when you are going or you are solving row by row.

So, the current values are already available at point ij minus 1 and i minus 1j. So, those values if you use then it will be faster, the solution will be faster than the Jacobi method. So, for that now whatever we are using, so you can write phi ij K plus 1, one unknown is there, but whatever already you have the available values that you use.

So, that is your phi i j minus 1 plus phi i minus 1j plus phi i plus 1 j plus beta square phi ij plus 1. So, you can see from this figure that ij minus 1 already you have computed from the earlier row so this is known and also i minus 1j you have already solved, so this is also known.

So, these two dependent variables are known at the current iteration level K plus 1. So, that you use ij minus 1 at K plus 1 level because already you have solved so, the current values are available and also phi i minus 1j so, that is also K plus 1, it is available but i plus 1j and ij plus 1 you have not solved yet because you are going row by row. So, these are unknown. So, this you take from the previous iteration.

So, now you see all the values are available, either at kth iteration that means the previous iteration or the current iteration K plus 1. So, the best available values we have used for i minus 1j point and ij minus 1 point and i plus 1j and ij plus 1 we have used the previous iteration level.

So, here this method is convergent if the largest element are located at the main diagonal. So, that means, just let me write first then I will explain, this method is convergent. So, if the largest elements are located in the main diagonal, in the main diagonal of the coefficient matrix, coefficient matrix and the sufficient condition, sufficient condition for the convergence, for the convergence of the method is say the diagonal element.

Its value should be greater than equal to summation of j is equal to 1, j naught is equal to i to the value n, aij and at least for one row, one row, this aii should be greater than j is equal to 1, j naught is equal to ina ij. So, you can see that this method is convergent if the largest element are located in the main diagonal of the coefficient matrix and that, for this discretization method, this Laplace equation we have seen that largest diagonal, largest element lies at the main diagonal.

So, there is no problem but for the convergence of the Point Gauss Seidel method, the diagonal coefficient should be at least greater than equal to summation of all the neighboring coefficients, summation of neighboring coefficients and for at least one row, your main diagonal cooperation should be greater than summation of all the neighboring coefficients. So, it is the sufficient condition, so if it is not satisfied still your solution may converge.

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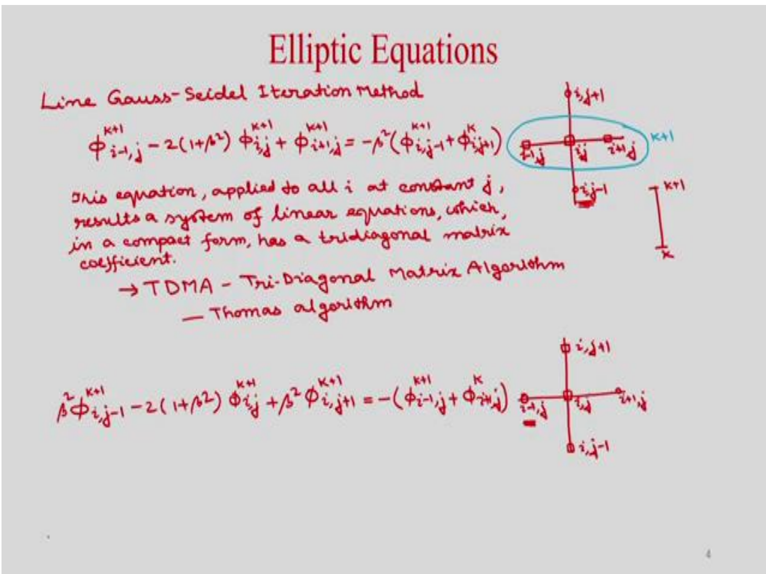

Now we will use Line Gauss Seidel method, Line Gauss Seidel iteration method. So, in line Gauss Seidel method we will solve row by row or column by column. So, when you are going row by row in each row, whatever dependent variables are there will find at K plus 1 time level.

So, that means if you see, so this point is your ij, this is your i plus 1j and this is your i minus 1j and this is your ij plus 1 and this is your ij minus 1. So, now when we use this line Gauss Seidel method, so all these three points values will use at K plus 1 iteration, so, all these three are unknown.

So, if you write the algebraic equation, then you will get it like this phi i minus 1j is unknown, then minus 2 into 1 plus beta square phi ij and phi i plus 1j. So, these three are in one row. So, these are unknown and which are known that will take in the right hand side. So, you can write minus beta square phi ij minus 1 plus phi ij plus 1.

Now, you see, K is your previous iteration and this K plus 1 is current iteration. So, in single row whatever points are there, so that all are unknown. So, we want to solve for those dependent variables. So, those will keep at that K plus 1 iteration level. So, K plus 1 and whatever is best available that we will use or we will use from the previous iteration.

So, you see phi ij minus 1 so this is all ready to go row by row then already this point you have computed. So, you can put at K plus 1 because the based level values at K plus 1 level and phi ij plus 1 you have not solved yet. So, you use it from the previous iteration level.

So, you can see that there are more than one unknowns and for each row if you write for all the points then you will get a tridiagonal matrix coefficient which you can solve using tridiagonal matrix algorithm which is known as Thomas algorithm. So, you can see this equation, this equation applied to all i at constant j results a system of linear equations, linear equations which in compact, in a compact form has a tridiagonal, tridiagonal matrix coefficient.

And this matrix is easy to solve and you can solve using TDMA, tridiagonal matrix algorithm which is known as Tomas algorithm, TDMA, tridiagonal matrix algorithm and also known as Thomas algorithm. So, this we will learn later, but it is easy to solve as there are three unknowns.

So, this method obviously is faster than the point Gauss Seidel method, because you are involving more points, unknown points while solving the system of linear algebraic equations, although it is faster, but part iteration it takes more time than the Gauss Seidel method.

So, as you have written for the row now, if you go by column by column, then for a constant i you can write the equation as. So, if you have let us say, column by column if you are solving then this is your ij plus 1, this is your ij and this is your ij minus 1. So, these three points are unknown and these are known. So, this is i minus 1j and this is your i plus 1j.

So, for a constant i if you write the equations, then you will be able to write this equation as beta square ij minus 1 minus 2 into 1 plus beta square phi ij and plus beta square phi ij plus 1. So, these are all in one column and these are unknown, means we will evaluate at K plus 1 iteration level and whatever known will keep in the right hand side.

So, it will be phi i minus 1j plus phi i plus 1j. So, you see, so this ij minus 1, ij and ij plus 1 obviously, these are, we are trying to evaluate at iteration level K plus 1. So, this superscript will give K plus 1, K plus 1, K plus 1. Here you see phi i minus 1j and phi plus 1j. So, you can see that when you are going column by column from left to right, then phi i minus 1j already you have evaluated at the current iteration level K plus 1.

So, you can write these superscript as K plus 1 and i plus 1j you have not evaluated yet, so it will be at k. So, that you can write, so for the each column. Now, whatever we discussed for this Point Gauss Seidel method and Line Gauss Seidel method that can be further accelerated using some relaxation factor. So, that will study.

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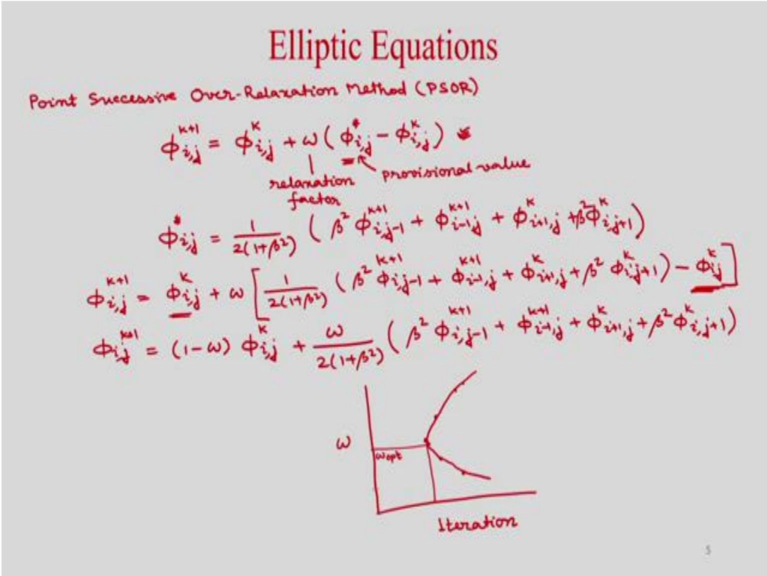

So, first we will study Point Successive over Relaxation, point successive over relaxation method, so which is known as PSOR. So now, whatever we have studied in point Gauss Seidel method now, we will use some relaxation factor and this solution can be accelerated. So, whatever now we want to find phi ij at iteration level K plus 1.

So, that will evaluate using phi ij, whatever we have in the previous iteration level plus the realization factor omega into phi ij star minus phi ijK. So, this is some provisional value, star quantities is some provisional value. So, omega is your relaxation factored, relaxation factor and this is some provisional value. So, this provisional value will calculate using the, whatever way we have calculated for Point Gauss Seidel method.

So, in Point Gauss Seidel method if you remember we are calculated phi ijK plus 1. So, instead of K plus 1 we will use the star which is your provisional value equal to 2 into 1 plus beta square and we have now beta square phi ij minus 1 plus phi i minus 1j plus phi i plus 1j and plus phi ij plus 1 and we have beta square here and in which iteration level?

So, phi ij minus 1, so already it is calculated, so K plus 1, i minus 1j so, that is already available at K plus 1 iteration level, so it is K plus 1, i plus 1j and ij plus 1 you have not solved, so it is available at the previous iteration level.

So, the provisional value we have calculated using the Point Gauss Seidel method and we have used the over relaxation method here, where we are calculating phi ijK plus 1 level equal to phi ijK which is your previous iteration level plus omega into the difference between the provisional value and the previous interactive value.

So, now if you substitute phi ij here, so what you will get? So, if you have substituted there you can see, you will get phi ijK plus 1, phi ijK plus omega. Now, you are substituting by star idea there. So, it is 1 by 2 into 1 plus beta square, beta square phi ij minus 1 K plus 1 plus phi i minus 1j K plus 1 plus phi i plus 1jK plus beta square phi ij plus 1 K and you have minus phi ijK.

So, now you rearrange it, so, whatever phi ijK is there. So, that you take common so, you will get. So, now you see here you have phi ijK and here plus pki ijK and omega is there so, you can write 1 minus omega phi ijK and plus omega by 2 into 1 plus beta square into beta square phi ij minus 1 at K plus iteration level plus phi i minus 1 jK plus 1 plus phi i plus 1j at K plus beta square phi ij plus 1K.

So, you can see that if you use these relaxation factor then you will get a faster solution than the Point Gauss Seidel method but there is no proper way to find the optimum value of the omega. So, for that you have to use trial and error method and you have to do some numerical experiment and find for a particular problem what will be the omega optimum,

So, you can see that if you, if you plot say total number of iteration to converge, if you are plotting in the x direction or x axis and in the y axis if you are plotting the value of omega, then you input the value of omega and find total iteration it took for the convergence.

So, if you do that then say let us say for a particular omega, say particular omega you got this iteration, another omega you got this, another omega you got this, another omega you got this, then you can see if you join these points, then you can see that for these omega you are getting minimum number of iterations. So, this is your omega optimum, omega optimum, because this is your omega optimum for which actually it is taking minimum number of iteration for the convergence.

So, these you can use some trial and error method doing the numerical experimentation and find the value of omega optimum for a particular problem, but if you have a rectangular domain with constant step size and you have only Dirichlet boundary conditions, then there is a formula which you can use to find the optimum value of the omega.

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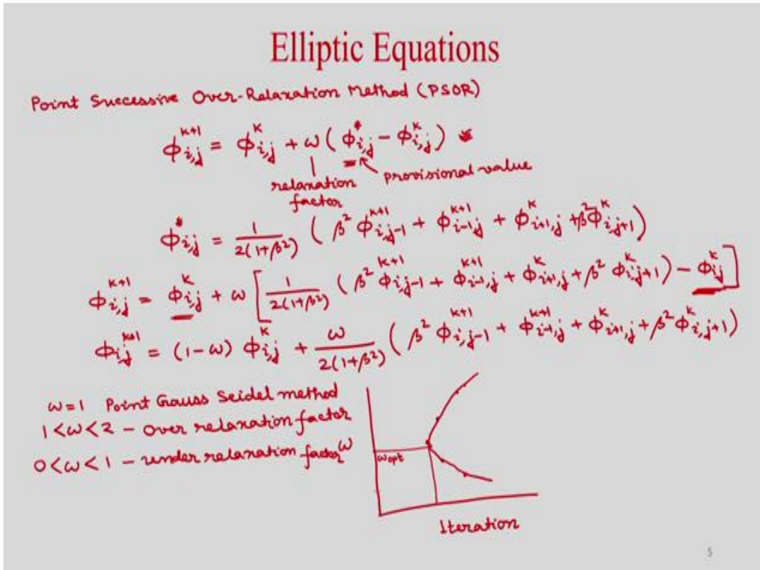

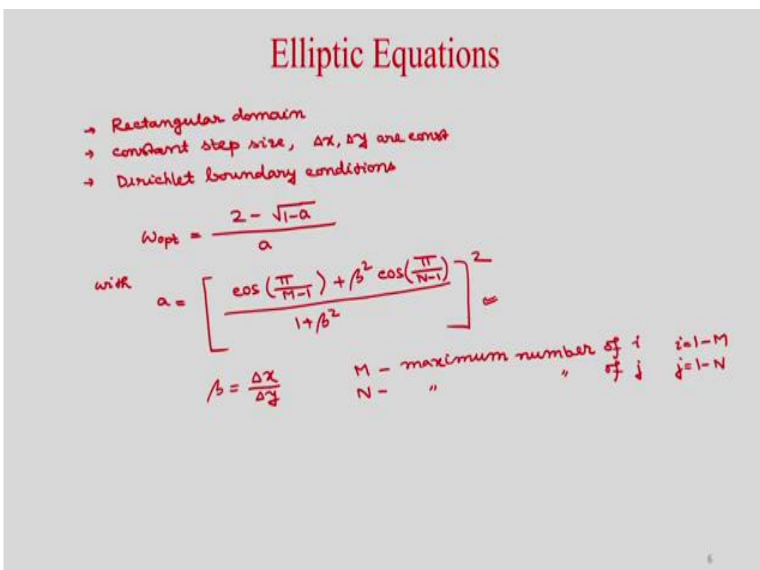

So, if you have a rectangular domain, rectangular domain in Cartesian coordinate then you have a constant step size, constant step size that means delta x and delta y are constant step side that means delta x and delta y are constant, you have a uniform grid and if you have a Dirichlet boundary conditions, Dirichlet boundary conditions.

So, for this you can have a formula to find the optimum value of omega. So, that is your omega optimum is 2 minus root 1 minus a divided by a, where with a is equal to 1 plus beta square cos Pi by M minus 1 plus beta square cos Pi N minus 1 its square. So, omega optimum can find with 2 minus root 1 minus a, in denominator it is a.

So, a you can find it this formula, where beta obviously is the ratio of the step size delta x by delta y and M is the maximum number of points in x direction, so maximum numbers i and i is varying 1 to M and N is your maximum number of j, where j is varying 1 to N and if you do not have Dirichlet boundary conditions or if you have a non uniform grid, then obviously, you have to use trial and error method to find the omega optimum.

So, here this omega is the relaxation factor. So, if omega is 1, so you can see from the previous slide, if omega is 1 then you can see that it will boils down to the Point Gauss Seidel method. So, if omega is 1 then it is Point Gauss Seidel method and if omega is in between 2 and 1.

So, omega if it is in between 2 and 1 then it is known as over relaxation factor, over relaxation factor. So, whatever problem we have taken say if it is a heat conduction equation and if you are solving this algebraic equation, then you can use over relaxation factor to get the faster solution than the Point Gauss Seidel method and for some equations you may need to use the omega value less than 1 and greater than 0. So, that time it is known as under relaxation factor.

So, if omega is in, the value of omega is in between 0 and 1 then it is known as under relaxation factor. So, but for heat conduction equation for steady two dimensional heat conduction equation you can use over election factor but you need to find the value of omega optimum so that it will take minimum number of iterations for convergence.

Now, whatever way we have used the points successive over relaxation method similar way we can introduce these relaxation factor in line Gauss Seidel method to accelerate the solution.

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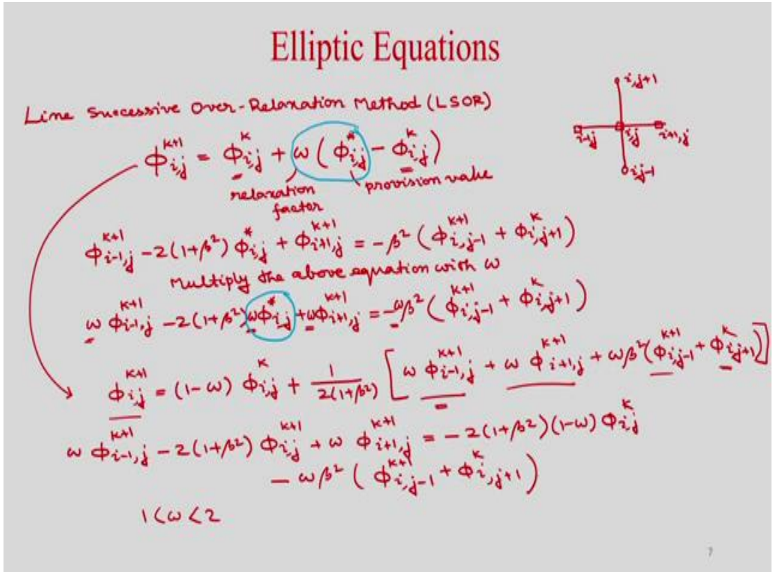

So, now line successive over relaxation method. So, this is known as LSOR and for easily, so this is your ij, this is your i plus 1j, this is your i minus 1 j. So, if you are solving row by row then these three points are unknown and this is known ij minus 1 and this is your ij plus 1.

So, in line successive over relaxation methods similar to way we will find the value of phi at ijK plus 1 level as phi ij at previous iteration level K plus the relaxation factor omega into phi ij provisional value minus phi ijK. So, here omega is your relaxation factor and here it is over relaxation we use that is why it is known as over relaxation factor and this is your provisional value, provisional value of phi ij to be calculated similar way we calculated in Line Gauss Seidel method.

So, if you remember in Line Gauss Seidel method which way we have calculated? We have used phi i minus 1j K plus 1 minus 2 into 1 plus beta square into phi ij and this is your provisional value plus phi i plus 1 jK plus 1 and is equal to minus beta square and whatever value, based value is available.

So, it is phi ij minus 1 plus phi ij plus 1 and obviously this is your K plus 1 because it is available and this is your K. So, now you multiply this equation with omega so, what you will get? Multiply the above equation with omega. So, what you will get?

Omega phi i minus 1j K plus 1 minus 2 into 1 plus beta square phi ij star plus phi i plus 1j K plus 1 is equal to minus beta square phi ij minus 1 K plus 1 plus phi ij plus 1K. So, here omega you have to multiply, so here omega, here omega and omega. So, we have multiplied omega in both side, so here omega we have multiplied, omega, omega and omega.

So now, you substitute this phi ij here. So, here you can write phi ij K plus 1 is equal to, so this K and this K you can take together say 1 minus omega phi ij. Now, you substitute this phi ij star here, omega phi ij. So, this is your omega phi ij, omega phi ij star. So, that you substituted here. So, you will get, so this if you take in right hand side, so it will be plus 1 by 2 1 plus beta square. So, it will be omega phi i minus 1j K plus 1 plus omega phi i plus 1j K plus 1 and this will be in the right hand side it will be omega into beta square phi ij minus 1 K plus 1 plus phi i j plus 1 K.

So, if you rearrange it, so you can see that K plus 1. So, K plus 1, you have K plus 1, this K plus 1, this K plus 1, this K plus 1. So, if you multiply into 2 into 1 plus beta square and rearrange. So, you will get omega phi i minus 1j K plus 1. So, this is we have written and now, 2 into 1 plus beta square we have multiplied and we have taken in this side.

So, it will be minus 2 into 1 plus beta square phi ij K plus 1 then you will get this. So, it will be plus omega phi i plus 1j K plus 1. So, now you have these 2 into 1 plus beta square you will be multiplied with these. So, it will be in the left hand side, so minus 2 into 1 plus beta square 1 minus omega phi ijK and these two also will be there in the right hand side, which are known. So, it will be plus, this side you have taken plus. So, this will be going minus, minus omega beta square phi ij minus 1 K plus 1 plus phi ij plus 1 K.

So, you see this system of linear algebraic equations where you have i minus 1j, ij and i plus 1j are unknown and you have used the relaxation factor omega and we got successive, lines successive over relaxation method where omega your between 2 to 1 used for this equation, for this Laplace equation.

That is why it is known as over relaxation method and obviously, if you find the optimum value of the omega using some trial and error method, then you will get much, much faster solution than the Line Gauss Seidel method. So, these we have used for a single row now, if you write for a single column, then you will be able to write similar way.

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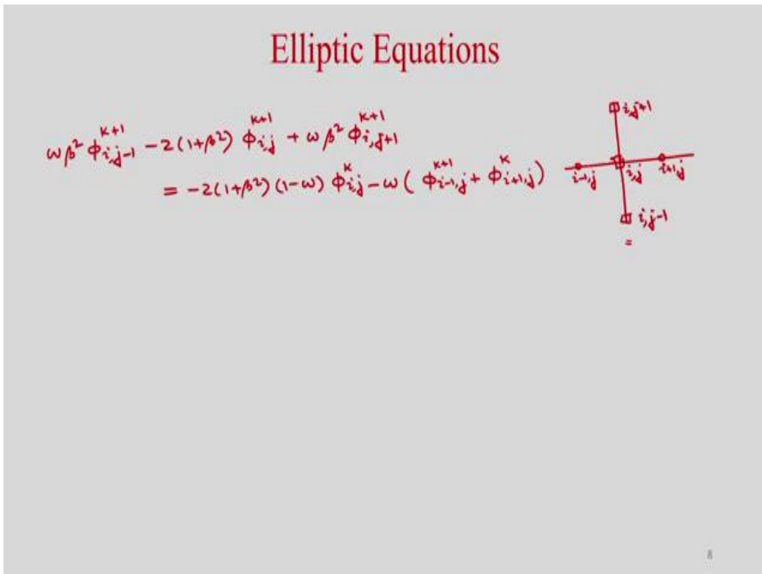

So, now, if you write for the single column, then these three are unknowns ij, ij plus 1 and ij minus 1. So, these three points values will be at K plus 1 iteration level and these i minus 1j, i minus 1j and i plus 1j. So, this whatever based values are available that you will use. So, for this if you use for a constant i, for a constant i which is your column.

So, you can use similarly omega beta square phi ij minus 1 K plus 1 minus 2 into 1 plus beta square phi ij K plus 1 plus omega beta square phi ij plus 1 at K plus 1. So, these three values are unknown and in the right hand side you have minus 2 into 1 plus beta square 1 minus omega phi ij K.

So, K in the previous iteration level minus omega into phi minus 1j and this value is known because i minus 1j already in this column you have solved. So, K plus 1 is known value and ki plus 1j you have not solved yet. So, you take it from the previous iteration level K. So, for a single column or a constant j you can write the algebraic equation like this and which you can solve using some tridiagonal matrix algorithm or Thomas algorithm.

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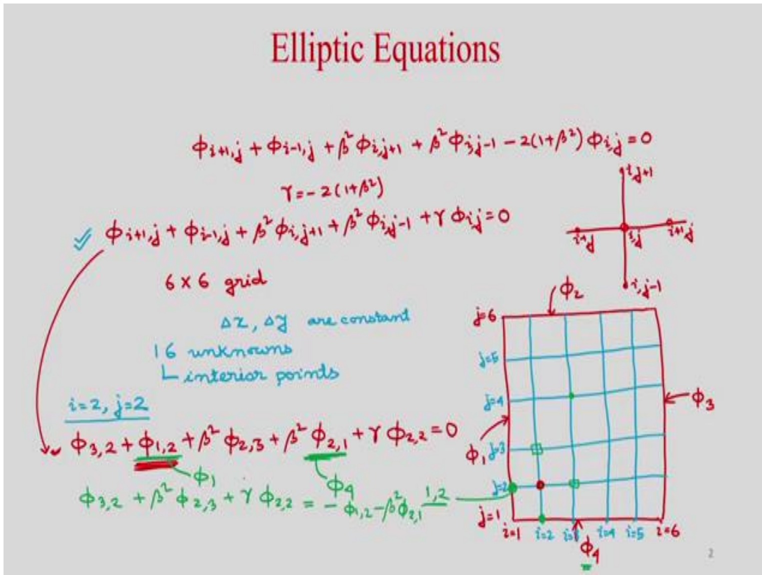

So, let us consider the discretize equation for that Laplace equation whatever we have derived phi i plus 1j plus phi i minus 1j plus beta square phi ij plus 1 plus beta square phi ij minus 1 minus 2 into 1 plus beta square phi ij is equal to 0. So, that equation we have written for a discrete point ij.

So, you have seen that there are phi points involved. So, this is your ij, this is your i plus 1j, this is your i minus 1j, this is ij plus 1, and this is your ij minus 1. So, you can write that this gamma is equal to minus 2 into 1 plus beta square and hence we can rewrite this equation as phi i plus 1j plus phi i minus 1j plus beta square phi ij plus 1 plus beta square phi ij minus 1 plus gamma phi ij is equal to 0. So, gamma is obviously the diagonal coefficient.

Now let us take a domain and considered the Dirichlet boundary conditions. So, let us consider the domain with six by six grid. So we are considering a squared domain with six by six grid. So, if you considered that so you can see six by six squared domain we are considering. So, this is your i is equal to 1, 2i is equal to 6 and j is equal to 1 to j I equal to 6.

So, these are the four boundaries of the square domain and let us consider that we have only Dirichlet boundary conditions. So, the value of the variable phi is known in all the boundaries. Let us say this is your phi 1, this is your phi 2, this is your phi 3 and this is phi 4. So, these are the constant values Dirichlet boundary conditions. Now, we have total number of grid points varying from i is equal to 1 to 6.

So, we will have in between the grid. So, let us draw the grid first. So, we have uniform grid. So, constant step side we have delta x is constant and delta y is constant and similarly, we have so, we have delta x and delta y are constant, constant steps size. So, obviously you can see this is i is equal to 2, i is equal to 2, i is equal to 3, i is equal to 4, i is equal to 5 and j is equal to 2, j is equal to 3, j is equal to 4 and j is equal to 5.

So, how many interior points are there where you need to solve these algebraic equation? So, there are, you can count, so total 16 points, these 1, 2, 3, 4 and this direction 1, 2, 3, 4 so 4 into 4. so 16 are unknown. So, you have 16 unknowns so, these are interior points, you need to solve for the dependent variable, interior points.

So, the discretized equation whatever we have written here, so this is for a particular point ij we have written. So, now for these 16 points you can write these algebraic equation. So, let us say for i equal to 2 and j is equal to 2 you can write this equation. So, if you put, let us say i is equal to 2 and j is equal to 2, then you will get one this algebraic equation.

Similarly, for i is equal to 3, j is equal to 2 so, accordingly if you substitute and write this equation, so, you will get 16 equations, for 16 points you will get 16 equations. So, 16 unknowns are there and 16 equations will be there. So, for i is equal to 2 and j is equal to 2 if you write the equation, this algebraic equation then what will you get?

So, you can see from here, so i is equal to 2. So, it will be phi 3 to plus phi i minus 1. So, it will be phi 1 2 plus beta square. So, it is phi ij plus 1. So, it will be phi 2, 3 plus beta square phi i is 2 and j minus 1 is 1 plus gamma phi 2 2 is equal to 0. So, for these grid point, for these grid points, we have written this discretize equation.

Similar way you can write for each grid point and 16 grid points are there so, for each grid point you can write this equation. So, you can see from this equation that your i is equal to 2 and j is equal to 2. So, for i is equal to 1 and j is equal to 2 so what is this point? So, you can see phi 1 2, so, this is your 1 and this is 2. So, this point is your boundary point.

So, this is your boundary point and this is nothing but you can have this is the index is 1 2 and this is known value because you have a Dirichlet boundary condition on the left wall. So, phi 1 2 is known and this is nothing but your phi 1, because the Dirichlet boundary condition is given on this left wall as phi 1. So, ph 1 2 is known and you can see again you have 2 1. So, this phi 2 1, so phi 2 1 is this point, because this is your boundary point. So, at this boundary the value is known which is your phi 4 so, you can these value is your phi 4.

So, you can see in this equation for i is equal to 2 and j is equal to 2 you have only three unknown points because for which point you are solving ij that is 2 2 and this point is unknown and this point is unknown and these two are the 2 phi 1 2 and phi 2 1. So, these are falling on the boundary, so from boundary values you will be able to know. So, this you can take in the right hand side, so, you can write phi 3 2 plus beta square phi 2 3 plus gamma phi 2 2 which are known you take it in the right hand side.

So, you will get minus phi 1 2 and minus beta square phi 2 1 because these two values are known. So, we have taken the right hand side and in left hand side for this particular discrete points you have three unknowns, but obviously you have a interior points then if you consider a point let us say here.

So, you can see these phi points are unknown. So, all phi will be phi coefficient and the phi discrete values will be in the left hand side. So, for all the grid points if you write then you will get the equations you can see.

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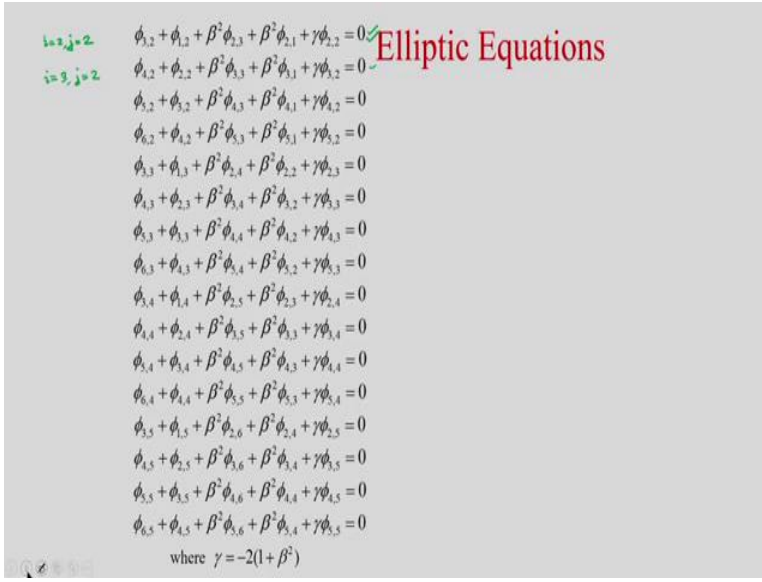

So, this is already we have written for i is equal to 2 and j is equal to 2 already we have written, then you can go i is equal to 3 and j is equal to 2. So, if you write similarly, i is equal to 3 and j is equal to 2 you can see. So, it is phi i plus 1j so, 3 plus 1 is 4, 2, then phi i minus 1j. So, i minus 1 is 3 minus 1 is 2 so, 2 2. So, similarly, you can see that you can write 16 equations. So, there are 16 equations and 16 unknowns. So, here all these equations you can put in a matrix form as like this.

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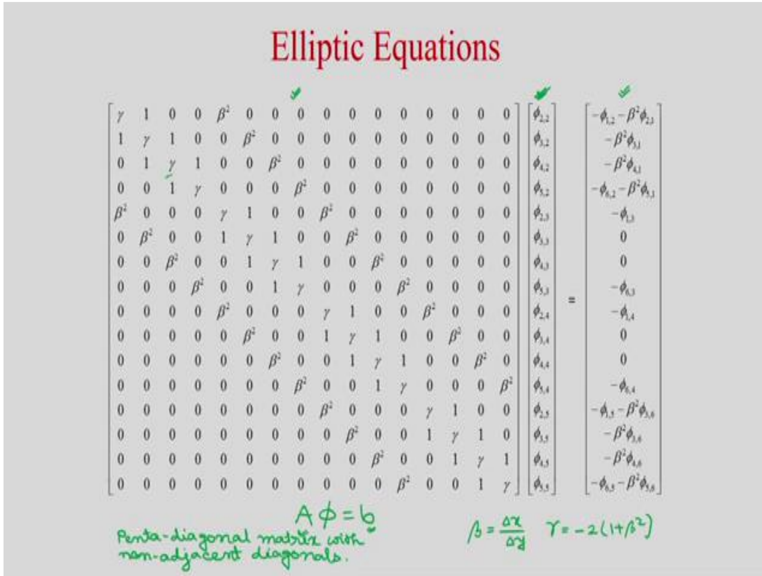

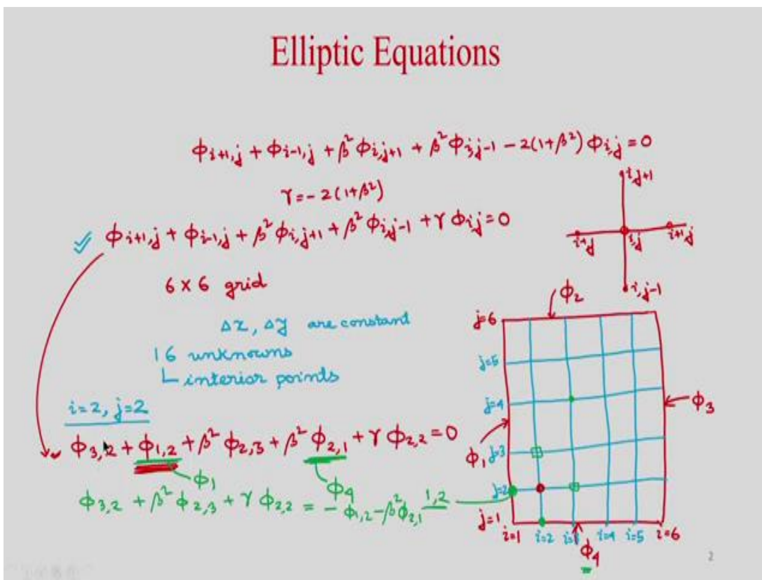

So, all the coefficients you can see. So, this is your coefficient matrix, these are the unknown. So, these vector is unknown. So, for all the interior 16 points we have phi 2 2, phi 3 2, phi 4 2, phi 5 2 then again j is equal to 3, so ph 2 3, phi 3 3, phi 4 3, phi 5 3 similarly j you increase to 4 and j to phi.

So, that way you will get total 16 unknowns for the interior points. So, this is your unknown vector and whatever is known or whatever are known that we have kept in the right hand side. So, these are known values and this is your coefficient matrix. So, essentially you are writing in this form a phi is A equal to b, where b is the vector which is known and a is the coefficient matrix that is also known, because beta is nothing but the ratio of the step size delta x by delta y and gamma already we have defined as minus 2 into 1 plus beta square.

So, all these coefficients are known, so unknown is only this vector phi. So, all the interior points, so if you somehow, if you solve these matrix then you will be able to get the values at all interior points and as we are discussing that you can see the diagonal matrix. So, you can see that there are pentadiagonal matrix. So, this is a how many say here you can see, how many unknowns are there? 1, 2, 3 so you can see.

So, there are phi unknowns. So, obviously, you can see it is a pentadiagonal matrix. So, it is a pentadiagonal matrix with non adjacent diagonals. So, it is a pentadiagonal matrix with non adjacent diagonals and obviously, you can see that the elements in the main diagonal, in each row are the largest. So, you can see gamma, so in each row if you considered. So, in the main diagonal, this is the maximum because gamma is minus 2 into 1 plus beta square. So, obviously, this is the maximum this element.

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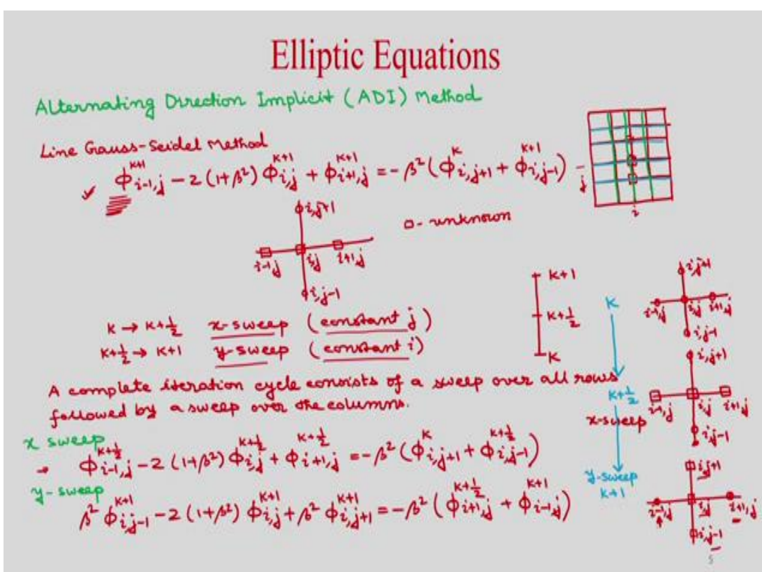

Now, let us learn another iterative method that is known as Alternating Direction Implicit method which is commonly known as ADI, alternating direction implicit. So, this is your

ADI method. So, in Line Gauss Seidel method already we have learned and line Gauss Seidel method proceeds by taking each line in a same direction, in a repetitive order.

That means, for a constant i you are solving or constant j you are solving. So, if you see that in a line Gauss Seidel method, what we have done so, these are all constant i or these are constant j. So in line Gauss Seidel method what we are solving, tridiagonal coefficient matrix we are solving or tridiagonal system we are solving and you can see that we are just we have three unknowns in a particular line so if you are moving in constant j.

So, there are three unknowns, so, this line by line we are moving or if a constant i you are solving then there will be three unknowns so, there will be three okay. So, for a constant i there are three unknowns and this column by column we are solving, but this further can be the converging rate, further can be improved if we solve this tridiagonal system by each row first then followed by the each column or vice versa.

So, let me explain it. So, if you have in alternating reaction method if a constant j you write then you are writing phi i minus 1j minus 2 into 1 plus beta square phi by ij then phi i plus 1j equal to minus beta square phi ij plus 1 and plus phi ij minus 1 and obviously, these are all unknown So, K plus 1 level we have written and this is ij plus 1 obviously is unknown so, it is K and ij minus 1 already at K plus 1 level it is available.

So, in this form we have written Line Gauss Seidel method. So, you can see, so for a particular case, there are three unknowns in a particular line. So, this is your ij, this is your i plus 1j and this your i minus 1j and this is your ij plus 1 and this your ij minus 1. So, these are all unknowns, these are unknown points.

So, if you are solving for a constant j, then there are three unknowns and you get tridiagonal system which you can solve using Thomas algorithm. So, this way we solve in Line Gauss Seidel method. So, how to improve it further? So, actually, the iterative cycle you can go from K to K plus half and from K plus half to K plus 1.

So, K is your previous iteration and K plus 1 is the current iteration, but while sweeping we are going from K to K plus half maybe for a constant j we will solve which is known as x sweep. So, constant j and from K plus half to K plus 1 we will go to the columns that means constant i and that is known as y sweep.

So we will take K plus half to K plus 1 iteration level will go this y sweep for or constant i. So, you can see from K to K plus half, we are going row by row and we will complete in the whole domain, then after that K plus half to K plus 1 iteration level, it will go column by column that means constant i will just sweep.

So, K to K plus half is known as x sweep as it is constant j and y sweep where you have constant i or you can go vice versa. So, K to K plus half you can go y sweep first, then K plus half to K plus 1 you can go to x sweep. So, in that way the convergence rate can further be improved. So, now to write the equations, the algebraic equations. So, you can see. So, what is actually a, so a complete iteration cycle consists of x sweep over all rows followed by a sweep over the columns. So what we will do so, when you are doing let us say x sweep then x sweep you have constants j.

So, you solved this line, then this line, this line, this line then x sweep is completed. So, after that from K plus half to K plus 1 you go to y sweep so, through y sweep, you just solved this column, this column and this column. So, if you write the algebraic equation now so you can see for x sweep.

So, x sweep we can write similar equation whatever whoever even for Line Gauss Seidel methods similar way we will write, but first we will do the x sweep from K to K plus half. So, first you write phi i minus 1j minus 2 into 1 plus beta square phi ij plus phi i plus 1j is equal to minus beta square phi ij plus 1 plus phi ij minus 1. So, what we are doing? So, you have at Kth level, Kth level you have all these known values.

This is your K iteration level, now when you are going to K plus 1, you are going for x sweep. So, three unknowns in a constant i. So, these three are unknown and these you take from the previous iteration or whatever based available values you can take. So, you can see this is your ij, this is your i plus 1j, this is i minus 1j, ij plus 1, ij minus 1 when you are going, this is your K plus half, because we are moving from K to K plus half.

So, you can see this is your ij, this is your i plus 1j, this is your i minus 1j, this is your ij plus 1 and this is your ij minus 1. So, this is the x sweep we are doing so, this is your x sweep. So, three unknown so, that we have written if you see the equation here phi i minus 1j minus 2, 1 plus beta square phi ij plus phi i plus 1j.

So, these are all unknown so, we will write we are solving at K plus half level. So, this is a iteration level K plus half and in the right hand side now, you can see phi ij minus 1. So, this already you have solved at K plus half level because you are going row by row, so it is K plus half and what about phi ij plus 1? So, you see here phi ij plus 1, so this row you have not solved yet.

So, it will be at previous iteration level that means K. So, this is your x sweep. So, you solved for all the rows and find the value of phi at K plus half iteration level. Now once you get the all the values at K plus half level, then you move to K plus 1 with y sweep. So, you just move with through the columns.

So, now if you see here. So, we are moving to K plus 1 which is we are now moving here and this is your y sweep we are doing. So, if you see what are the unknowns at K plus 1. So, now we are doing y sweep, so this is unknown, this is unknown and this is unknown three unknowns are there. So, this is your ij, this is your ij plus 1, ij minus 1 and this is your known value either from the previous iteration or from the current iteration, this is your i plus 1j and this is your i minus 1j. So, when we are doing the y sweep.

Now, let us write the equation, y sweep. So, now we can write beta square phi ij minus 1 minus 2 into 1 plus beta square phi ij plus beta square phi ij plus 1 equal to minus beta square phi i plus 1j plus phi i minus 1j. So, now you see what will be iteration level, so you can see ij plus 1, ij and ij minus 1.

So, these three points you have unknown, so that will be at K plus 1 iteration level and in the right hand side you can see phi i plus 1j. So, phi i plus 1j you have not solved because you are going column by column. So, if you are moving in the right direction, column by column then i plus 1j point you have not solved because you are solving column by column and if you are moving in the right direction. So, this column is not solved yet.

So, these you have to take from the previous iteration levels and previous iteration level is K plus half, because we are moving from K plus half to K plus 1 and if you see phi i minus 1j. So, phi i minus 1j already you have solved, so this is available at iteration level K plus 1. So, now what we are doing so, we are moving the iteration level from K to K plus half and you do either x sweep or y sweep, then you go K plus half to K plus 1 iteration level and you do the y sweep or x sweep.

So, this way you can see that alternate way you are sweeping through the columns or through the rows and the convergence rate is faster than the Line Gauss Seidel method, as we have discussed this ADI method here now, here also it can be further improved using your relaxation factor, omega.

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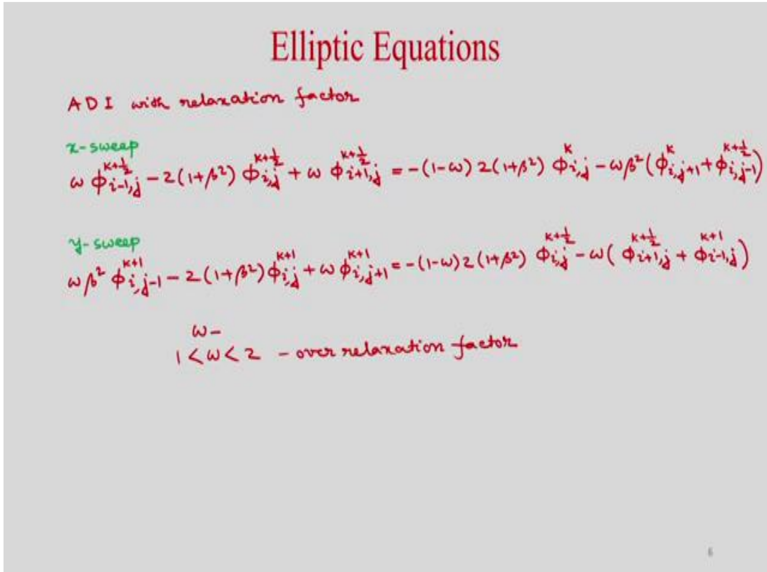

So, if you write the similar equation like line successive over relaxation method, then they will get this ADI with relaxation factor. So, this I am not going to derive because this already we have derived for the line successive over relaxation method, just I am going to write the final algebraic equation.

So, if you write x sweep. So, if you write x sweep with the relaxation factor omega, then you will write omega phi i minus 1j minus 2 into 1 plus beta square phi ij plus omega phi i plus 1j is equal to minus 1 minus omega 2 into 1 plus beta square phi ij minus omega beta square phi ij plus 1 plus phi ij minus 1.

So, now you see the iteration level obviously, we are going x sweep that means K to K plus half, so K plus half iteration level we have unknowns. So, in the left hand side all will put at K plus half and in the right hand side phi ij. So, that obviously will take from the previous iteration level K and phi ij plus 1.

So, which you have j plus 1 you have not solved. So, it is K and phi ij minus 1 already that row you have solved so, it will be K plus half and now if you do the y sweep. So, you can write omega beta square phi ij minus 1 minus 2 1 plus beta square phi ij plus omega phi ij plus 1 is equal to minus 1 minus omega 2 1 plus beta square phi ij minus omega phi i plus 1j plus phi i minus 1j.

So, for the y sweep now, we are moving K plus half iteration level to K plus 1. So, K plus 1 is your unknown value. So, you will get phi K plus 1, K plus 1, K plus 1. Now, in the right hand side obviously we are moving from K plus half to K plus 1. So, phi ij will be at K plus half and phi i plus 1j. So, phi i plus 1j you have not solved yet, so you take it from the K plus half iteration level and phi i minus 1j. So, when we are moving column by column, so you have already solved, so this will get at K plus 1 level.

So, obviously, as we discuss that omega now in this particular case whatever we have solving the Laplace equation, in this case, it is over relaxation factor and you will use the value of omega between 1 and 2 and this is your over relaxation factor and as we discussed before, that by trial and error method you have to find the optimum value of omega.

So, that you will get the, it will take minimum time for the convergence. So, if you see that in this lecture, we have discretize the Laplace equation which is your Elliptic type equation and iterative method we have used to solve because it is very simple and easy to program.

Different types of iterative methods we have discussed first we have discussed the Jacobi method, which is nowadays almost obsolete, we do not use because it is very slow. Then we have used Point Gauss Seidel method which is faster than the Jacobi method. Then we have learned Line Gauss Seidel method where you solved in one row three unknowns and you can get tridiagonal system with which you can solve using Thomas algorithm and it is much faster than the Point Gauss Seidel method.

Then we have used successive over relaxation method. So, first we have learned Point Successive Over Relaxation method and then Line Successive Over Relaxation method. And after that we have further improved the convergence rate by using Alternating Direction Implicit method, where we go the iteration level from K to K plus half with x sweep and K plus half to K plus 1 with y sweep or vice versa and it improves the convergence rate considerably than the Line Gauss Seidel method and further it can be accelerated using Over Relaxation method, thank you.