Computational Fluid Dynamics for Incompressible Flow
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Lecture 10
Finite difference formulation of Elliptic Equations with Boundary condition treatment

Hello everyone. So in last module, we have learned finite difference approximation, of partial derivatives on uniform grid and non-uniform grid. Today we will apply those finite difference approximation to discretize partial differential equations. We will find Elliptic type governing equations in particular applications for fluid flow as well as the heat transfer problems.

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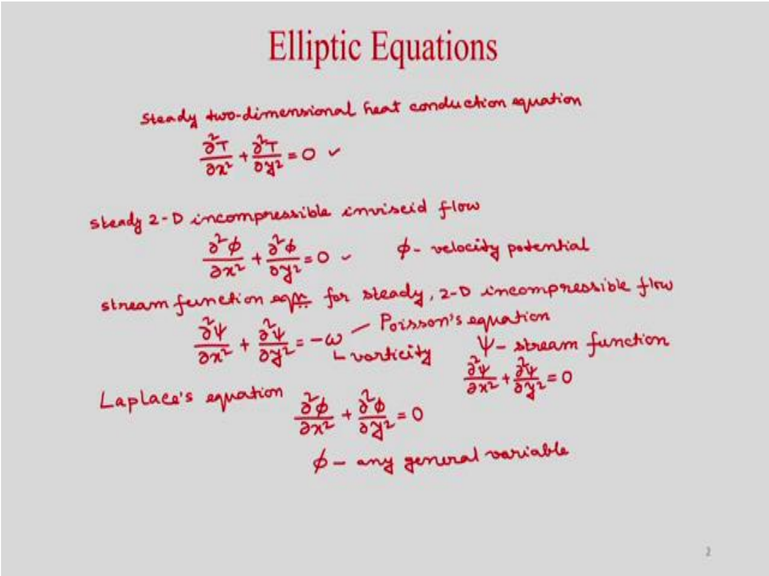

So, today we will use Elliptic equations like heat conduction equation, steady heat conduction equations. So, you know that steady conduction equations, steady 2 dimensional heat conduction equation del 2 T by del x square, plus del 2 T by del y square is equal to 0 is elliptic type equation. Similarly, if you find the velocity potential for 2 dimensional inviscid and incompressible flow, then that is also elliptical equation.

So 2 D incompressible inviscid flow and obviously steady. So, that you know that del 2 phi by del x square plus del 2 phi by del y square is equal to 0 where phi is velocity potential, phi is the velocity potential. Similarly, you know, stream function equation.

So, stream function equation is also elliptic type equation. Stream function equation for steady 2 D incompressible flow. If you write this equation, then you can write del 2 psi by del x square plus del 2 psi by del y square is equal to minus omega where this omega is vorticity and psi is your stream function.

So, if you see this equations, so of first 2 equations, this heat conduction equation and the velocity potential equations are known as Laplace equation, Laplace equation and if you write for any general variable phi, then you can write del 2 phi by del x square plus del 2 phi by del y square is equal to 0 and a third equation whatever, we have written for stream function equation. So, that is known as Poisson's equation, Poisson's equation.

And obviously if we write it for irrotational flow, then it boils down to del 2 psi by del x square plus del 2 psi by del y square is equal to 0 and which is also Laplace equation. So, we will discretize this equation for general variable phi. So, in this case we are telling that phi is any general variable, any general variable. So, phi maybe psi for stream function equation phi maybe T for the heat conduction equation. So if you, before going to discretize these equation, we need to discretize the domain into grid.

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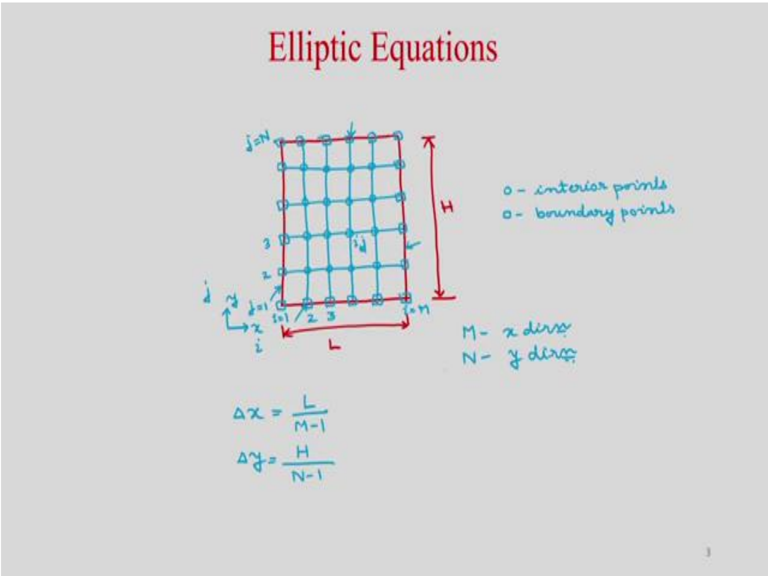

So, let us take a square domain. So, let us take this domain, whose length is L and height is H. So, the governing equations, whatever we have, we, if we use the finite difference approximation, we need to find the dependent variables at any discrete points. So, to find those values at discrete points, we need to divide this domain into grid. So, let us divide it into grids on uniform grid.

So, for this particular domain you can see there is 4 boundaries. This is 1 boundary. This is another boundary. This is another boundary and this is you another boundary. And the discrete points, you can find that these are the discrete points at interior points and these are boundary points. So, these are all interior points where we need to find the value of that dependent variables, the governing equation of whatever we have that is Laplace equation.

After discretising, we need to find all those interior points. So, these are interior points and we will apply the boundary conditions, at boundary points. These are boundary points. So, all these are boundary points. So, if we have M number of points in x direction and N number of points in y direction. So if you use uniform grid, then you will be able to find what is the delta x and delta y. So what will be your delta x in this case, so delta x is your, if L is the length, then in x direction, then the length.

So, this is your x direction and this is your y direction and the index will use in x direction is i and in y direction is j. So, at any discrete point in 2 dimensions, we will use i,j notation. So, delta x will be L and M minus 1, if your i starting from 1. So, 2, 3, so on i equal to M. Similarly, if you start j is equal to 1, 2, 3 and j is equal to N, then delta y will be your H by.

So, H is the height in the y direction. So H by in N minus 1. If you are starting j is equal to from j is equal to 1. So, we are assuming the uniform grid in the x direction. So delta x is constant and similarly uniform grid in y direction. So, delta y is also constant. So, with this now we will discretize this Laplace equation at discrete point i,j.

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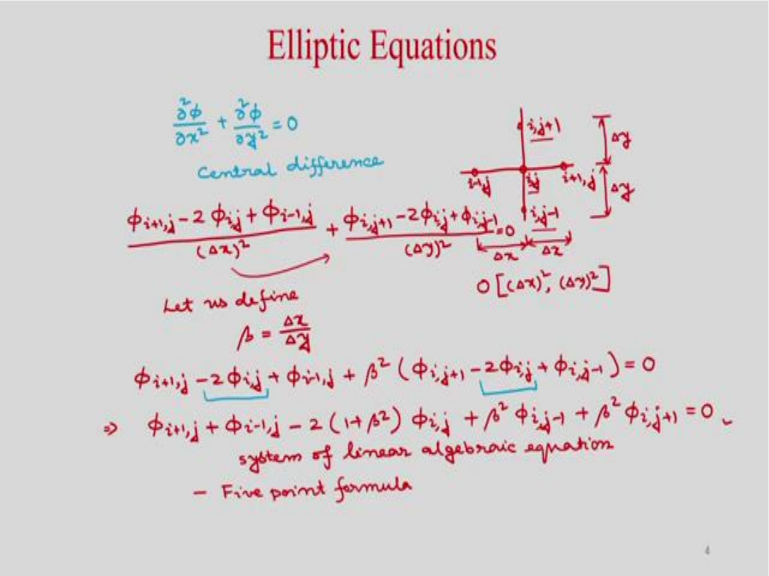

So our governing equation is del 2 phi by del x square plus del 2 phi by del y square is equal to 0. So, this is your governing equation. Now we will discretise this partial derivative using second order accurate scheme, which is known as central difference. So we will use central different scheme. So we already, you have learned in last class that central different scheme is your second ordered accurate.

So, we will apply the centre difference at this discrete point. So, if your i,j. So, your neighbour points is i plus 1j. This is your i minus 1j. This is your ij plus 1 and this is your ij minus 1 and obviously the distance between these 2 discrete points, is delta x. This is also delta x and this is your delta y and this is also delta y. So, now use central difference if you use the finite difference approximation, then you will use phi i plus 1j, because in x direction you are discretising del 2 phi by del x square.

So, we will use the neighbour points, 3 points i plus 1, ij and i minus1j. So, it will be 2 phi ij plus phi i minus 1j by delta x square and you know the order of accuracy of this finite different scheme is second order. That means del x square. Similarly, if we discretise del 2 phi del y square. Then you can write it, so in y direction. Now we are doing, so you will use 3 points ij plus 1, ij and ij minus 1. So, we can write phi ij plus 1 minus 2 phi ij and phi ij minus 1 divided by delta y square is equal to 0.

And obviously this is also having the order of accuracy delta y square. So, over all accuracy of this finite difference approximation is, delta x square and delta y square. So, now let us define the ratio of step size. Delta x by delta y as beta. So, let us define beta is equal to the ratio of step size, delta x by delta y. So, what you can right now, so if you take this multiply both into delta X square, then you can write it. phi i plus ij minus 2 phi ij plus phi i minus 1j. Now you can write delta x square by delta y square, which is your beta square. So, because delta x is going here.

So delta x square by delta y square is beta square. So you can write phi ij plus 1 minus 2 phi ij plus phi ij minus 1 is equal to 0 and if you rearrange then we can write it as phi i plus 1j plus phi i minus 1j. So, you can see here you have phi ij here and here we have phi ij. So, if you take together then you can write minus 2 into 1 plus beta square because this minus, here minus, we have taken the outside.

So, it will be phi ij then beta square phi ij minus 1 and beta square phi ij plus 1 is equal to 0. So after discretising, we have got this equation which is known as system of linear algebraic equation. So, this equation we have got for the particular discrete point ij. So, there are 5 points are involved. Phi plus 1j, phi minus 1j, phi ij, phi ij plus 1 and phi ij minus 1 and that is why it is know as 5 point formula. 5 point formula we have used.

If you want to use higher order scheme then you will get other more number of points. Neighbour points so total number, maybe 9 or so. So depending on the, what is the order of accuracy you are using. But it is that simplest and it just way to use 5 point formula and commonly used this 5 point formula and you can see that we have got some linear algebraic equation which you can solve using some computer.

So, before going to how to solve this equation, let us find how will apply different types of boundary conditions? Earlier you have learned there are 3 different types of boundary conditions 1 is Dirichlet type boundary conditions, 1 is Neumann type boundary condition and anther is mixed or robin type of boundary conditions. So let us consider a heat conduction in a square plate.

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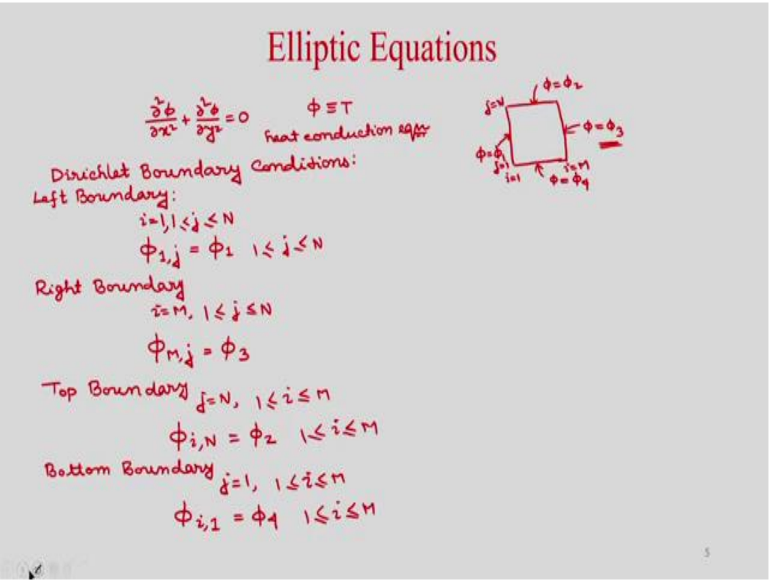

So if you considered a heat conduction in a square plate, so if you have the Dirichlet boundary conditions in all the boundaries, then let us have here phi is equal to phi 1, here phi is equal to 2 phi 2 and phi is equal to phi 3 and this is your phi is equal to phi 4. So, some constant temperatures are given at each boundaries. So, for these governing equations del 2 phi by del x square plus del 2 phi by del y square is equal to 0, where phi whatever these general variable we have used.

If it is equivalent to T, then you can write it as a heat conduction equation. Steady state heat conduction equation and for this we are trying to find how to implement the boundary conditions. So first, we are considering Dirichlet boundary conditions. So, Dirichlet boundary condition, we know the value of that variable is specified at the boundary. So if you consider, say this is your i is equal to 1 to i is equal to m and j is equal to 1 to j is equal to n.

So, you can say that your phi. Which is your left boundary if you considered first. So, if you consider the left boundary, then your j is varying 1 to n and i is equal to 1. So, i is equal to 1 and j is varying 1 to n. So, for that you can write phi. So, this is your 1j is equal to phi 1. Where j is varying between 1 and N.

So, similarly you can write for right boundary. So right boundary, you can write, so right boundary, your i is equal to M and j similarly it is varying from 1 to N. So, you can write phi Mj is equal to phi 3. Right boundary, we have seen that the boundary condition is phi 3 Dirichlet boundary condition.

Similarly, your top boundary. So, top boundary j is equal to N and i is varying between 1 and M. So, you can write the boundary condition for the top boundary phi is your i N, is equal to phi 2, because phi 2 is specified and i is varying between M and 1. Similarly for bottom boundary, your j is equal to 1 and i varying between 1 and M. So, you can write phi i1 is equal ti phi 4. Where i varies between 1 and M. So it is very easy to apply these Dirichlet type boundary condition, because there are no gradients are involved. Only the value of the variable is known. So, easily you can apply these boundary conditions.

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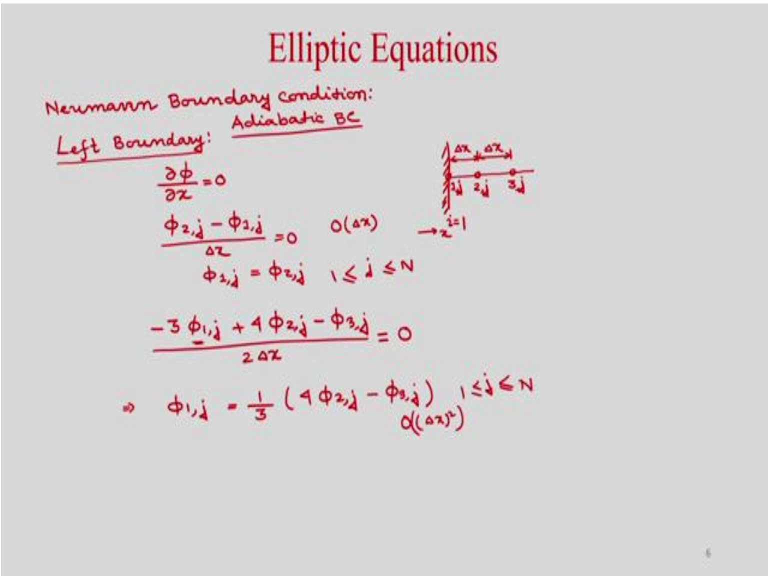

Now let us consider some Neumann type boundary conditions. Neumann. So, in heat transfer you can have two types of Neumann type boundary condition. One is your adiabatic boundary conditions where the temperature gradient is 0. Temperature getting normal to that surface is 0 and another is given heat flux boundary conditions. So, let us consider left boundary. So, in the left boundary, one to find what is the boundary condition for adiabatic condition.

So, we are considering adiabatic boundary condition for the heat transfer. So, what is adiabatic boundary condition. So, that means there will be no heat flux across that boundary. That means heat flux is 0, if it flux is 0, then the temperature gradient will be 0, for this left boundary. So as this is your x direction. So it will be del phi by del x is 0. Because it is your heat flux is 0 at that boundary. So, here i is equal to 1 and if you see the points interior points, so it will be 1 at any point j, this will be 2,j and this will be 3j because i is going 1, 2, 3.

So, boundary points is i is equal to 1. Then the next neighbour point is 2, i is equal to 2 and i is equal to 3 for a constant j. So 1,j, 2,j and 3,j. So, now we need to find what is the value of this dependent variable at the boundary 1j. So, now you can discretize this equation using either first order or second order approximation. So first, we will use the first order accurate scheme. So, which is your you can write phi 2,j minus phi 1j divided by delta x is equal to 0.

So, this is your delta x and this is also delta x. We have considered uniform grid. So, this is your delta x. So, what approximations we have used this is forward difference approximation we have used because we have used forward point phi 2,j minus phi 1,j divided by delta x equal to 0. And our concern is to find the value of phi at 1,j.

So, now you can find phi 1, j is equal to phi 2,j was phi 2,j you will get from the solution of the system of linear algebraic equation. So, once you solve the interior points then you will be able to find the value of phi at the boundary. So, phi 1j is equal to phi 2,j and j will vary your between 1 and N.

So, for the left boundary just we have considered and we have shown. How to discretize this adiabatic boundary conditions. Now here also you can use 3 points, so that you will get the second order accurate accuracy. So in this case, if you use, one sided approximation then will use minus 3 phi 1,j plus 4 phi 2,j minus phi 3,j. So you are using 3 points, phi 1,j, phi 2,j, phi 3,j. So the distance between phi 1,j and phi 3,j is 2 delta x, so divided by 2 delta x and you know that this final difference approximation is second order accurate. So equal to 0.

This is adiabatic boundary condition, so it is equal to 0. So, now our concern is to find phi 1,j. So you write phi 1,j is equal to. So, you take the other 2 terms in the right hand side and divide by 3, so you can find 1 by 3, 4 phi 2,j minus phi 3,j and which is valid for j between 1 to N. So, as we have discuss that this approximation is order of delta x first order accurate. And this is the delta x square.

So, this is the order of accuracy delta x square, second order accurate and that we have done for the adiabatic boundary condition for particular left boundary. If you considered some other boundaries, let us say top or right accordingly you will be able to discretize this boundary condition. Now and under type Neumann boundary condition as we discussed that given heat flux. So, let us discretise the boundary condition for a given heat flux.

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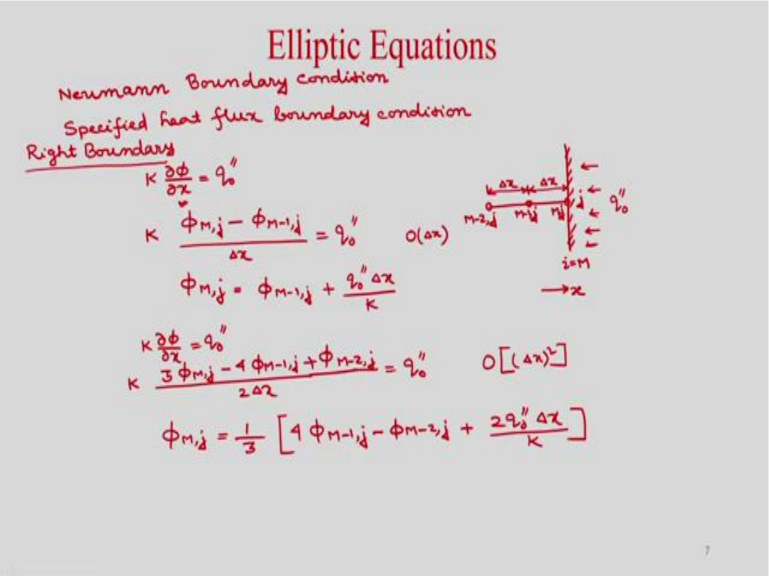

This is also Neumann boundary condition. So, we are considering specified heat flux boundary condition. So, let us consider now right boundary and let us assume that specified heat flux is given at the right boundary. So, if you consider a right boundary, so this is your boundary. Where i is equal to M and let us consider the points at a particular j. So, this is your j.

So, you will get. So these, are distance between these 2 points is delta x. So, this point is Mj. This is your M minus 1j and this is your M minus 2j. So, 3 points we are considering along the x direction. So, the boundary points, it is the right boundary. So, this is your grid point is M. So Mj at a particular j. So, the previous point is M minus 1j and another previous point is M minus 2j.

So, now we will consider again first order accurate scheme as well as second order accurate scheme. So, what is the heat flux boundary conditions? So, heat flux boundary conditions is given that at this boundary you have q naught, q naught double prime which is the heat flux.

But we are solving for the temperature or phi. So, we need to write the boundary condition in terms of the variable phi. So for that, now let us. So what is heat flux? So at this right boundary, so we can write, so K del phi by del x is equal to q naught double prime.

So you see, so x direction, we have considered this way. So, obviously q naught is in negative x direction. So, whatever q naught double prime is coming. It is going in this direction. So we have written the Fourier law as K del phi by del x. In general, we know that in positive x direction if your heat flow is taking place, then your Fourier heat conduction equation is minus K del T by del x. But in this case as in negative x direction. We are considering the heat flux q naught double prime. So, it will be k del phi by del x is equal to q naught double prime.

So, now they are first derivative is involved. This first derivative you can discretize using first order accurate scheme or second order accurate scheme. So first, let us consider first order accurate scheme. So, if it is first order accurate scheme, so we will take a backward finite difference approximation. So, if you use backward difference approximation, then you will use phi mj minus phi M minus 1j diverted by delta x is equal to q naught double prime. So why we used a backward point M minus 1j.

So, it is backward finite difference approximation. So, if we rearrange it and we are solving for phi Mj. So you can write phi Mj is equal to phi M minus 1j plus q naught double prime del x by K and what is the order of accuracy? This accuracy is order of delta x. Similarly, now let us use second order accurate scheme. So, we will involve three points, Mj, M minus 1j and M minus 2j. So you can write, similarly K del phi by del x is equal to q naught double prime and we are doing for the right boundary.

So, now your discretize using a secondary accurate scheme using 3 backward points, Mj,M minus 1j and M minus 2j. So you can write del phi by del x using 3 phi Mj minus 4 phi M minus 1j and plus phi M minus 2j. What is the distance between Mj and M minus 2j? It is 2 delta x is equal to q naught double prime and order of accuracy is Delta x square.

So, you can rearrange it. So after rearranging you will get phi Mj because that we are interested to find the value of phi Mj at the right boundary. So, you will get, so it will be 1 by 3, 4 phi M minus 1j minus phi M minus 2j and you will get plus 2 q double prime delta x by K. So, similarly if you have that specified heat flux boundary condition at other boundary similar way you will be able to derive.

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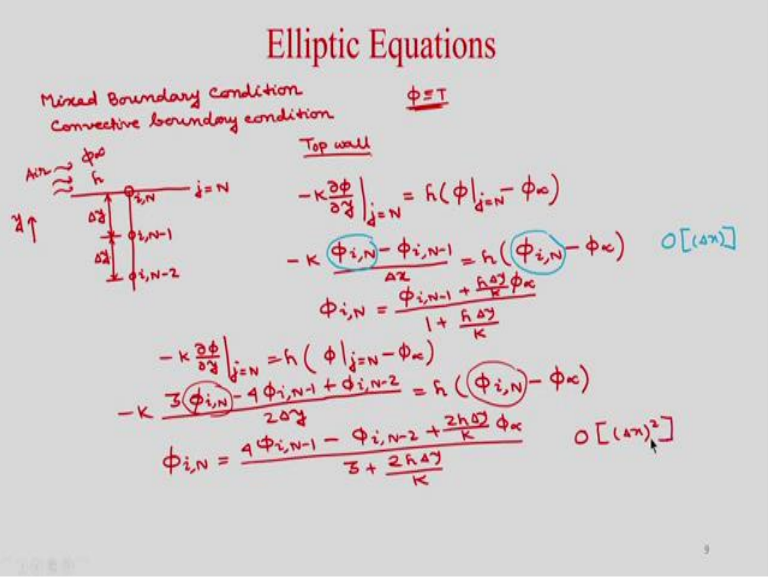

Now let us consider the third type of boundary condition that is your mixed boundary condition and when do we apply this in the heat conduction equation then this is known as convective type boundary condition. So, let us consider a top wall of a solid and that is cooled by a convection. So, if you are considering a top wall, so it is your top wall we are considering.

On the top wall so obviously we have j is equal to N and we have the points, grid points. So this is your iN and this is your iN minus 1 and this is your iN minus 2. So, the distance between these points is delta y and y we are considering in this direction positive and on the top wall let us say some airflow is there and due to this whatever heat is conducted that is convicted through this air and it is temperature is infinity and heat transfer coefficient is h.

So, now for this heat conduction equation, now you can write the boundary condition as whatever heat is convicted. So, del y at j is equal to N, so now whatever heat is conducted equal to whatever it is convicted.

So, in heat is convicted is h is the heat transfer coefficient of this air, you have phi at j is equal to N minus phi infinity. So, let us consider here phi infinity here. So, this is the phi is actually as we are considering heat transfer equation, it is phi equivalent to t. So, now if you discretize this equation, now we have first derivative. So first derivative we can use either first order accurate scheme or second order accurate scheme.

So, first let us use the first order accurate scheme. So, we will use the backward difference because it is the top wall. So, only the points are available below the top wall. So it will be a backward differencing we will use. So, minus K phi I N minus phi I N minus 1. The distance between these two point is delta x equal to h phi i N minus phi infinity. So, now you can rearrange it. You have phi iN here and phi iN here and what is the order of accuracy, order of accuracy is delta x.

So, now you rearrange and write the value of phi i N which is the boundary point and we are interested to find the value at the boundary. So, on the numerator you will get phi i N minus 1 plus h delta y by K into T infinity divided by 1 plus h delta y by k, not T infinity it is phi infinity. So similarly, now let us use second order accurate scheme. So to use the second order accurate scheme. We will use three points in the backward direction. So we will use phi i N, phi i N minus 1 and phi i N minus 2.

So, if you use K del phi by del y at j is equal to N is equal to h phi j is equal to N minus phi infinity. Then if you use second order accurate scheme, then will be using three points. So you will use 3 phi i N minus 4 phi i N minus 1 and plus phi i N minus 2 divided by the distance between these 2 points phi i N and i N minus 2 is 2 delta y. So, you will write 2 delta y is equal to h phi i N minus phi infinity. So, now in the left hand side you have phi i N. And in the right hand side also you have phi i N. So you rearrange it and find the value of phi i N.

So phi i N is equal to. So in the numerator you will get 4 phi i N minus 1 minus phi i N minus 2 plus twice h delta y divided by K phi infinity divided by 3 plus twice h delta y by k and it is order of accuracy is delta x square. So, we can see that we have considered three different types of boundary conditions. Dirichlet, Neumann and mixed type of boundary conditions. And in Neumann and Robin, boundary conditions or mixed boundary condition. We have the derivatives at the boundary.

So, that we discretize using first order accurate scheme, as well as second order accurate scheme. Although for the interior points we have used centre difference where the order of accuracy is second order. So, in the boundary it is better to use the second order accurate scheme so that overall you will get a second order accuracy in the solution.

But if you use first order accurate scheme at the boundary, then obviously in the interior points, although you are using second order accurate scheme, but overall solution will not be secondary order accurate. So, using these techniques you can apply to other type of boundary condition, for stream function equation or velocity potential equation where you might get similar type of boundary conditions, but mostly this mixed type of boundary condition. It will get in the heat conduction equation that we have shown here.