Fundamentals of Conduction and Radiation

Amaresh Dalal & Dipankar N. Basu
Department of Mechanical Engineering
Indian Institute of Technology – Guwahati

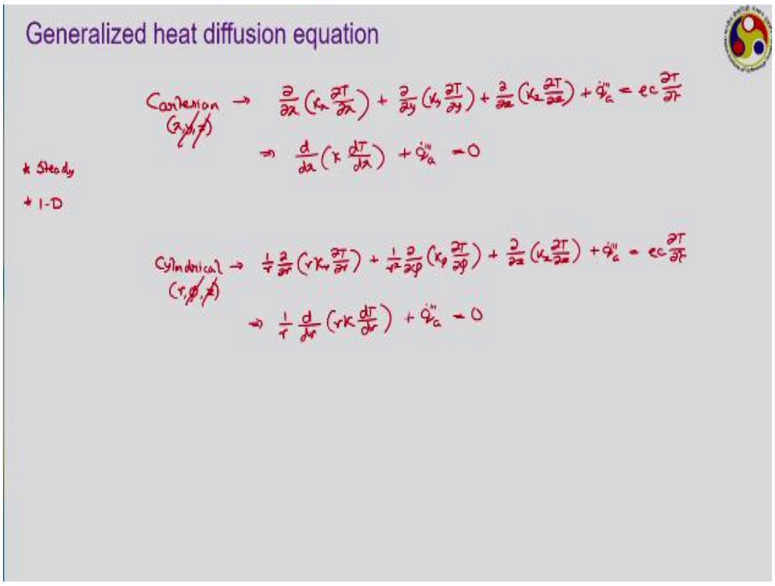
Lecture - 09
Special 1-D Heat Conduction Situations - Part 1

Morning friends, welcome to week number 4. In the previous 2 weeks, we have discussed about the development of generalized heat diffusion equation in three different coordinate systems and in the previous week, we have talked exclusively about 1-D heat conduction equation and its possible solutions primarily in all the 3 coordinate systems which led to the concept of thermal resistance.

In addition, we have seen that instead of solving the equations directly we can directly make use of the thermal resistance concept to solve several scenarios associated with 1D steady state heat conduction. However, the primary assumptions that we have taken there were not 2 rather 4 along with 1-D and steady state we have also considered the absence of any kind of heat generating source and also constant thermal conductivity.

Now in this week we shall be talking about some special 1-D heat conduction situations where the last 2 assumptions that we are going to relax. Nevertheless, it is still going to be 1-D and something not mentioned in the title of this module that is we are going to deal again with steady state systems only. Now I am just going back to the generalized heat diffusion equation.

(Refer Slide Time: 01:42)



We developed the equations in 3 different coordinate systems but here I am going to use only 2 of them. Cartesian; so in Cartesian coordinate system we have seen that the generalized heat diffusion equation can be written as

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial T}{\partial z}\right)+\left(\dot{q}'''_G\right)=(\rho c)\frac{\partial T}{\partial t}$$

The last two terms signifies the volumeteric heat generation in the system and the rate of energy storage in the system.

Similarly, in cylindrical coordinate system, we have developed the same equation again starting from an infinitesimally small cylindrical element and the equation was of the form

$$\Rightarrow\frac{1}{r}\frac{\partial}{\partial r}\left[rK_r\frac{\partial T}{\partial r}\right]+\frac{1}{r^2}\frac{\partial}{\partial\varphi}\left[K_\varphi\left(\frac{\partial T}{\partial\varphi}\right)\right]+\frac{\partial}{\partial z}\left[K_z\frac{\partial T}{\partial z}\right]+\left(\dot{q}'''_G\right)=(\rho c)\frac{\partial T}{\partial t}$$

Now in the cartesian coordinate, our coordinate directions were x y and z and in cylindrical coordinate it was the radial direction, azimuthal direction and the axial direction z. Accordingly we have got these equations. Now here in this equations we have considered like in cartesian we have written K_x, K_y and K_z are the 3 possible values of thermal conductivity.

Because we know that while for isotropic materials these 3 are equal to each other, but there are certain anisotropic materials which has some direction dependence of the thermal conductivity and that is why to make it generalized we have included the terms. Now once we are imposing the first 2 conditions that is steady and one dimensional then what we are going to have, in Cartesian coordinate if we remove the y and z dependency then the equation becomes

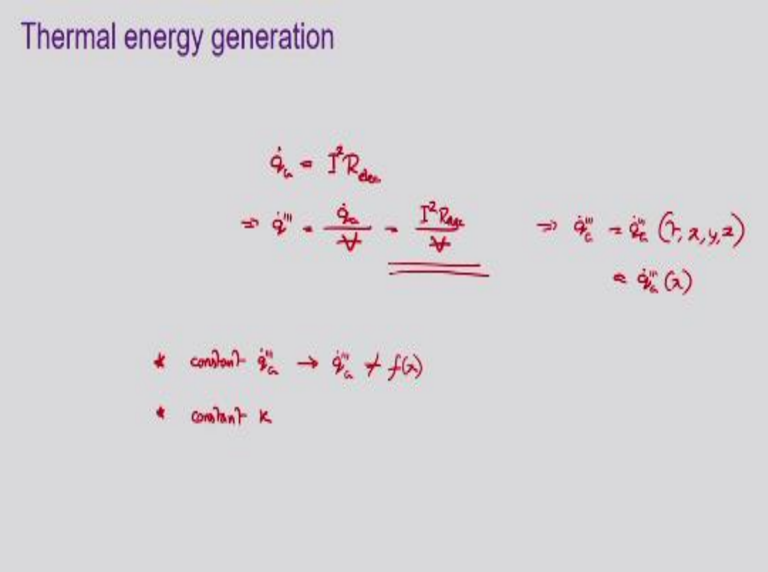
$$\frac{d}{dx}\left(K_x \frac{dT}{dx}\right) + (\dot{q}'''_G) = 0$$

Remember the last 2 assumptions which we kept on considering in the previous week that is no internal energy generation and also constant thermal conductivity that we are not considering; accordingly both K_x and this $\dot{q}'''_G$ that remains in the framework. Similarly, in the cylindrical one if we neglect any variation in the φ and z direction then the equation simplifies to

$$\frac{1}{r}\frac{d}{dr}\left[rK\frac{dT}{dr}\right] + (\dot{q}'''_G) = 0$$

In both the cases for we can drop the subscript for K, because as it is a 1-D scenario. Therefore, there is no point considering K having a spatial dependency, just one value of K is sufficient for this. Now in this way what we are going to talk about is a special scenario where either K can be variable or $\dot{q}'''_G$ is present in the system and it is also possible that both of them, that is the special dependence of K and also the heat generation needs to be considered.

(Refer Slide Time: 05:49)



Let us talk about the energy generation first. In this lecture, I am going to talk about 1 -D steady state heat conduction with energy generation. Now what are the possible scenarios, where we can have some kind of energy generation present in the system? We have talked about examples like some nuclear material, but another very common source can be an electrical resistor.

So if there is an electrical conductor through which a current is flowing in then we all know that following the Joules law there will be some kind of Joule heating loses. If the value of the resistor is R, that electrical resistor that I am talking about and I is the magnitude of current then corresponding energy generation will be equal to

$$\dot{q}_G = I^2 R_{elec}$$

This is the total amount of energy or rate of energy generation that is present in your system. If we want to know the volumetric energy generation then this energy generation rate needs to be divided by the volume of the of the electrical resistor that we are considering. That is the volumetric energy generation will be

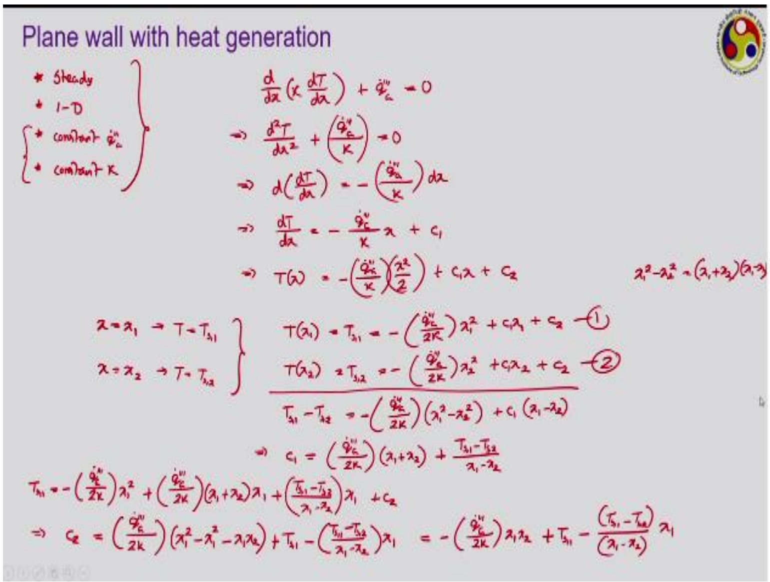
$$\dot{q}'''_G = \frac{\dot{q}_G}{V\!\!\!/} = \frac{I^2 R_{elec}}{V\!\!\!/}$$

This is the most primary kind of system that you can identified with energy generation apart from the nuclear materials which of course has very limited applicability or applicable only in certain special systems.

For the moment, we are going to deal primarily about this kind of losses. We are assuming this volumetric energy generation to be constant. Generally, the energy generation will be a function of all the three space coordinates and time. As we have reduced ourselves or restricted ourselves to a steady state, 1-dimensional scenario, so we can write this to be a function of x. Now assuming this constant value of this energy generation we are considering that this $\dot{q}'''_G$ is no more a function of x that is everywhere in your domain it is of same magnitude and another assumption that we are considering for today's discussion we are considering the thermal conductivity to be constant.

That is we just want to check the effect of this heat generation first and then the effect of thermal conductivity or variation in thermal conductivity with special location that we shall be considering in the next lecture.

(Refer Slide Time: 08:33)



So first, we are going to deal with plain wall with energy generation. Then the equation that I wrote earlier we can start exactly from here. So the generalized version of let me summarise the assumptions then,

*steady state

*1-D

*constant $\dot{q}'''_G$

*constant K

Therefore, with the assumptions, the equation that we had written in the first slide it will become

$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) + \left(\dot{q}'''_G\right) = 0$$

$$\Rightarrow \frac{d^2T}{dx^2} + \frac{\dot{q}'''_G}{K} = 0$$

So we have to solve this one now. Remember this term in the bracket is constant as per our third and fourth assumptions.

So it is not very difficult to solve this one at all. We can directly write

$$d\left(\frac{dT}{dx}\right) = -\left(\frac{\dot{q}'''_G}{K}\right)dx$$

$$\frac{dT}{dx} = -\frac{\dot{q}'''_G}{K}x + C_1$$

And if we integrate it once more with respect to x then we get

$$T(x) = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x^2}{2}\right) + C_1x + C_2$$

So this is the temperature distribution that we are trying to identify.

Once we know the volumetric energy generation rate and the thermal conductivity, we can get the temperature profile, but of course, to get the temperature profile we need to have couple of boundary conditions because there are 2 unknown constants C_1 and C_2. Let us say at some location say

$$at\ x = x_1, T = T_{s1}$$

$$at\ x = x_2, T = T_{s2}$$

So combining these 2 we can write that

$$T(x) = T_{s1} = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1^2}{2}\right) + C_1 x_1 + C_2$$

And

$$T(x) = T_{s2} = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_2^2}{2}\right) + C_1 x_2 + C_2$$

So if we write them together then

$$T_{s1} - T_{s2} = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1^2 - x_2^2}{2}\right) + C_1(x_1 - x_2)$$

This gives C1 to be equal to

$$C_1 = \left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1^2 - x_2^2}{2(x_1 - x_2)}\right) + \frac{T_{s1} - T_{s2}}{(x_1 - x_2)}$$

$$= \left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1 + x_2}{2}\right) + \frac{T_{s1} - T_{s2}}{(x_1 - x_2)}$$

Now we have to get C_2. So C_2 we can get by putting the expression for C_1 in any of this 2 equations. Let us put in 1. So putting in 1 we have

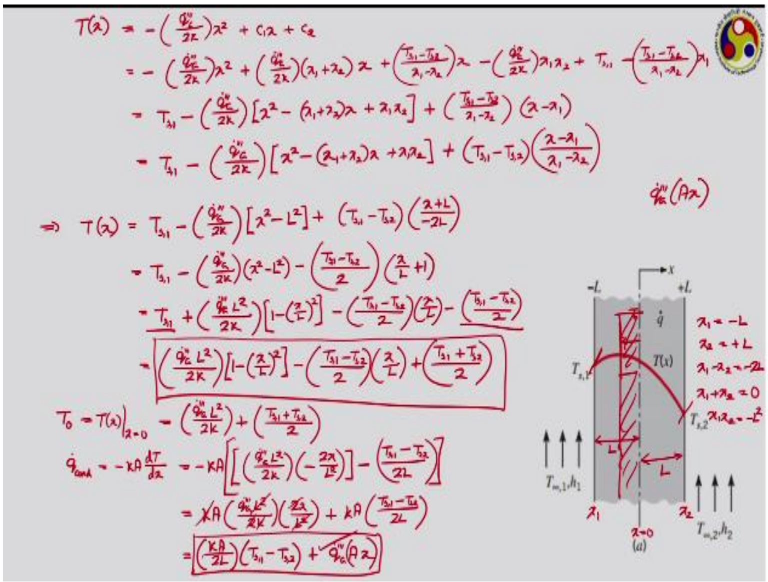
$$T_{s1} = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1^2}{2}\right) + \left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1 + x_2}{2}\right)x_1 + \frac{T_{s1} - T_{s2}}{(x_1 - x_2)}x_1 + C_2$$

After simplification, this gives you C_2 to be equal to

$$C_2 = -\left(\frac{\dot{q}'''_G}{K}\right)x_1 x_2 + T_{s1} - \frac{T_{s1} - T_{s2}}{(x_1 - x_2)}x_1$$

Hence, I have the expressions for C_1 and C_2; let us try to develop the final equation for T.

(Refer Slide Time: 15:13)



T at x originally it was

$$T(x) = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x^2}{2}\right) + C_1 x + C_2$$

Putting the value for C_1 and C_2

$$T(x) = -\left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x^2}{2}\right) + \left(\frac{\dot{q}'''_G}{K}\right)\left(\frac{x_1+x_2}{2}\right)x + \frac{T_{s1}-T_{s2}}{(x_1-x_2)}x - \left(\frac{\dot{q}'''_G}{K}\right)x_1x_2 + T_{s1} - \frac{T_{s1}-T_{s2}}{(x_1-x_2)}x_1$$

Rearranging the equation

$$T(x) = T_{s1} - \left(\frac{\dot{q}'''_G}{2K}\right)[x^2 - (x_1+x_2)x + x_1x_2] + \frac{T_{s1}-T_{s2}}{(x_1-x_2)}(x - x_1)$$

Or,

$$T(x) = T_{s1} - \left(\frac{\dot{q}'''_G}{2K}\right)[x^2 - (x_1+x_2)x + x_1x_2] + (T_{s1}-T_{s2})\frac{(x-x_1)}{(x_1-x_2)}$$

This is a generalized equation that we have developed for a simple one dimensional heat conduction scenario through a plane wall that is subjected to a uniform rate of volumetric heat generation. Quite often in an effort to avoid such complicated equation we go for simpler description of the coordinate location x_1 and x_2, so that we can reduce the equation to an even simpler form.

Just look at this, here our geometry involves again the heat generation through a plane wall this wall is kept at temperature T_{s1}, so this is our wall x_1, this is location x_2, its temperature T_{s2}. In an effort to avoid such complicated expression what has been done is that the coordinate frame or origin of the coordinate frame x = 0 has been selected as the central line of this block.

So that now both the walls are equally located. This is at a distance L and this is also at a distance of L from this and accordingly our x_1 becomes -L, x_2 becomes L and x_1 -x_2 becomes (-2L). If we are going for this kind of geometry definition then the equation becomes quite simpler. Let us try this out; also, I should add here x_1 + x2 becomes equal to 0.

You have to remember that here the x is defined with respect to that central line. So any location which is on this side of the centre line will be having a positive value of x, where as any location which is located on this side will be having a negative value of x and x =L and x = -L are the 2 extreme locations.

So in that case your expression for temperature profile is

$$T(x) = T_{s1} - \left(\frac{\dot{q}'''_G}{2K}\right)[x^2 - L^2] + (T_{s1} - T_{s2})\frac{(x + L)}{(-2L)}$$

So we can rearrange them a bit

$$T(x) = T_{s1} - \left(\frac{\dot{q}'''_G}{2K}\right)[x^2 - L^2] - \frac{(T_{s1} - T_{s2})}{2}\left(\frac{x}{L} + 1\right)$$

So this is the final form of expression probably we are looking for. Quite often instead of expressing this way we would like to express in a more compact form by doing some further substitutions.

$$T(x) = T_{s1} + \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{(T_{s1} - T_{s2})}{2L}(L + x)$$

So this is a closed form equation that we are looking for. Quite often instead of expressing this way, we can modify it even further.

$$T(x) = T_{s1} + \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{(T_{s1} - T_{s2})}{2}\left(\frac{x}{L}\right) - \frac{(T_{s1} - T_{s2})}{2}$$

$$\Rightarrow T(x) = \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{(T_{s1} - T_{s2})}{2}\left(\frac{x}{L}\right) + \frac{(T_{s1} + T_{s2})}{2}$$

So this is the final form that we are looking for regarding the steady state one dimensional heat conduction through a plane wall which is being subjected to uniform heat generation with constant thermal conductivities. You can easily see that if you put x =- L this will reduce to T_{s1}, if you put x = L this will reduce to T_{s2} and what about x = 0, the mid plane? So if we write

$$T_0 = T(x)|_{x=0} = \left(\frac{\dot{q}'''_G L^2}{2K}\right) + \frac{(T_{s1} + T_{s2})}{2}$$

This is the mid plane temperature somewhere here. If we are interested to calculate the rate of heat transfer through this then what it will be?

$$\dot{q}_{cond} = -KA\frac{dT}{dx} = -KA\left[\left[\left(\frac{\dot{q}'''_G L^2}{2K}\right)\left(-\frac{2x}{L^2}\right)\right] - \frac{(T_{s1} - T_{s2})}{2L}\right]$$

$$= KA\left(\frac{\dot{q}'''_G L^2}{2K}\right)\left(\frac{2x}{L^2}\right) + KA\frac{(T_{s1} - T_{s2})}{2L}$$

$$= \left(\frac{KA}{2L}\right)(T_{s1} - T_{s2}) + \dot{q}'''_G(Ax)$$

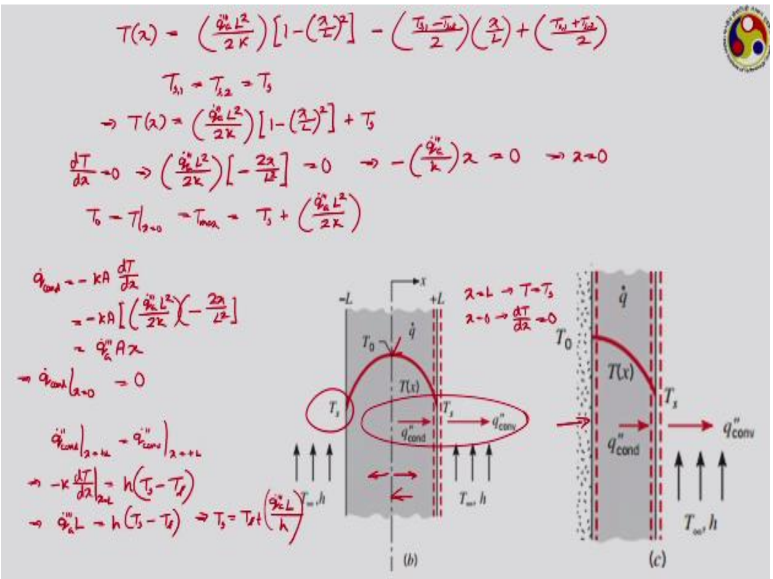
So the first term that you are getting here is nothing but the heat transfer that we could have derived from thermal resistance or using the resistance concept, by using a constant thermal conductivity of K and length scale of 2L. However, here we have an additional term in the form of this $\dot{q}'''_G(Ax)$, which is coming because of this volumetric energy generation.

And also it is important to note that there is an x present in the last term, that is because of the presence of volumetric energy generation the heat transfer rate is not constant rather it is a function of x. At x = 0, it will be equal to 0 that is at the central line heat generation will be equal to 0, but as we move away from this you will be having an increased contribution coming from the second term.

And also this term $\dot{q}'''_G(Ax)$, what is the physical significance of this? A is the cross section area and x refers to any distance. So if we have. Say at this particular distance we want to calculate the contribution to conduction coming from this term that basically will lead to the volume corresponding to this portion, I should have drawn the entire volume.

This Ax refers to this entire volume because A is the cross section area and x is the distance and whatever volumetric energy generation is taking place inside this, this multiplied by this corresponding volume contributes to this second term in energy production or energy transmission I should say.

(Refer Slide Time: 30:43)



Now one special case we could have had, I should just repeat the term, the entire expression for Tx that I have got.

$$\Rightarrow T(x) = \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] - \frac{(T_{s1} - T_{s2})}{2}\left(\frac{x}{L}\right) + \frac{(T_{s1} + T_{s2})}{2}$$

Now if we consider a special scenario, something like this, where at both x = L and -L, we are having the same temperature T_s.

The temperature is same at those locations. Then what we should have? Here we are talking about your T_{s1} and T_{s2} both are equal to T_s, here the system is subjected to symmetrical boundary condition. In the previous case it was asymmetric, and look at the temperature profile that we have got here.

It is an arbitrary drawn temperature profile of course, but the temperature profile that we have got here that shows that assuming T_{s1} to be higher, we are having a parabolic kind of nature because you are having an x square term present, but this parabola is not symmetric with respect to central line, as T_{s1} is higher than T_{s2}.

Now we are having a geometry where T_{s1} and T_{s2} both are equal to T_s, then in that case what will be the temperature distribution T_x. Your T_x in this case is going to be

$$\Rightarrow T(x) = \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[1 - \left(\frac{x}{L}\right)^2\right] + T_s$$

The second term T_{s1} - T_{s2} that cancels out and the third term reduces to T_s, which is a perfectly parabolic profile. That is what we are getting the base value is T_s and then we are having a parabola on top of this.

Now, at which point this temperature profile is going to have its maxima? to get that let us calculate dT/dx and assign that to 0. So we are going to have

$$\frac{dT}{dx} = \left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[-\frac{2x}{L^2}\right] = 0$$

$$\Rightarrow -\frac{\dot{q}'''_G}{K} x = 0$$

Now $\dot{q}'''_G$ cannot be 0 because there is a volumetric rate of energy generation, if there is no energy generation then there are 2 things we can identify. K cannot be 0 because K is the thermal conductivity, a property of the system, it cannot be negative as well, it is always a positive number, like if there is no volumetric energy generation then the entire temperature profile in that care $T_x = T_s$.

Only if $\dot{q}'''_G$ is equal to 0; that means throughout the system you have the same temperature. Secondly if $\dot{q}'''_G$ is having a nonzero value in that care the maxima will appear at x = 0, that is only at the centre line. So this is the location where you are going to have the maximum of this temperature. And this maxima

$$T_0 = T(x)|_{x=0} = T_{max} = T_s + \left(\frac{\dot{q}'''_G L^2}{2K}\right)$$

So this is the maximum temperature you can identify. In the previous case, we developed the rate of heat conduction like this. Let us try to develop the expression for rate of heat conduction in this situation as well. So here,

$$\dot{q}_{cond} = -KA\frac{dT}{dx} = -KA\left(\frac{\dot{q}'''_G L^2}{2K}\right)\left[-\frac{2x}{L^2}\right]$$

$$= \dot{q}'''_G Ax$$

So in the previous expression, look at this here; the expression for $\dot{q}'''_G$ that we got, there was a constant term, and then we had that $\dot{q}'''_G Ax$ and that constant term was appearing because of the difference between these 2 end temperatures. Now as the end temperatures are not different, in that case, we are not having that first term and therefore what will be your conduction at x = 0, that is going to be 0 only. So that means this mid plane, this particular plane is not having any heat transmission across.

This it is acting like a plane of symmetry. Whatever heat conduction taking place on this side, the same amount of heat conduction is taking place on the other side as well. So that net heat conduction is equal to 0, and the temperature profile across this centre line is going to be mirror image like this. Temperature profile on either side will be mirror image of each other. So the centre line can be considered a plane of symmetry.

Therefore, this centre line we can almost assume to be like an insulated surface. Just like this, we can just do the analysis for half of the domain and whatever we are getting the solution we can easily extrapolate that to the other side. Therefore, if we are having a plane of symmetry like this, you could easily solve the problem by putting your boundary condition at x = L to equal to T_s and at x = 0, the rate of heat conduction is equal to 0. So that dT/dx = 0, which is the symmetry boundary condition that we can very conveniently use in several scenarios as long as you are having a symmetrical temperature profile like this.

Now one final thought on this, quite often we may not be knowing the end temperatures T_{s1} T_{s2} or the T_s in this case, we may not know them and to identify that we may have to use the information available outside the block, like shown here the block may be subjected to some kind of convective heat transfer.

And therefore we can easily make use of the balance between conduction and convection at the surface under steady state condition so that the conduction heat flux at x = L should be equal to the convective heat flux again at x = L. So what is your conduction heat flux that is equal to

$$\dot{q}''_{cond}\Big|_{x=L} = \dot{q}''_{conv}\Big|_{x=L}$$

$$-K\frac{dT}{dx}\Big|_{x=L} = h(T_s - T_\infty)$$

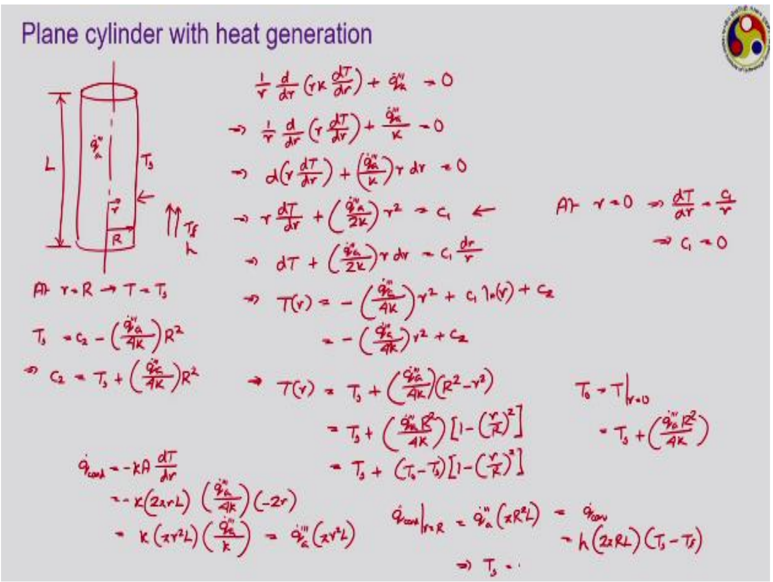
Expression for dT/dx we have developed, so if we have knowledge about T_∞ and h you can easily identify T_s from here. Subsequently you can incorporate that into your analysis, because we want to get a close form expression. Putting the values

$$\dot{q}'''_G L = h(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}'''_G L}{h}$$

In the same way, we can calculate the value on the other surface if the temperatures are different. We have considered heat generation is uniform, if that is non-uniform then we may have to depend on some other technique. Calculation will become much more complicated. Let us quickly check the scenario with a plane cylinder.

(Refer Slide Time: 40:10)



So for plane cylinder, this is the geometry that we have, we are having cylinder like this, this is the centre line of the cylinder and let us say the radius of the cylinder is R. This is your radial direction and the length of the cylinder maybe L. We have to get an expression for the temperature profile when the cylinder is experiencing uniform heat generation inside this.

We have already seen the basic equation that can be written taking $\dot{q}'''_G$ and K both to be constant as

$$\frac{1}{r}\frac{d}{dr}\left[rK\frac{dT}{dr}\right] + (\dot{q}'''_G) = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] + \frac{(\dot{q}'''_G)}{K} = 0$$

$$\Rightarrow d\left[r\frac{dT}{dr}\right] + \frac{(\dot{q}'''_G)}{K} r\,dr = 0$$

Integrating

$$\Rightarrow r\frac{dT}{dr} + \frac{(\dot{q}'''_G)}{2K} r^2 = C_1$$

$$\Rightarrow dT + \frac{(\dot{q}'''_G)}{2K} r\,dr = \frac{C_1 dr}{r}$$

$$T(x) = -\left(\frac{\dot{q}'''_G}{4K}\right) r^2 + C_1 \ln(r) + C_2$$

We have to identify the constants. Let's check at this particular situation at r =0, then

$$\left.\frac{dT}{dr}\right|_{r=0} = \frac{C_1}{r}$$

Here if C_1= 0 then dT/dr will be 0. As dT/dr is C_1 / r, so to have any finite value of this only possible scenario is $C_1 = 0$. So then C_1 we can directly eliminate from this. Hence

$$T(x) = -\left(\frac{\dot{q}'''_G}{4K}\right) r^2 + C_2$$

Second boundary condition is quite straightforward, let us say the surface temperature is T_s, so at r = R, T is equal to T_s, accordingly

$$T_s = C_2 - \left(\frac{\dot{q}'''_G}{4K}\right) R^2$$

Giving you C_2 to be equal to

$$C_2 = T_s + \left(\frac{\dot{q}'''_G}{4K}\right) R^2$$

Taking it back there,

$$T(r) = T_s + \left(\frac{\dot{q}'''_G}{4K}\right)(R^2 - r^2)$$

$$= T_s + \left(\frac{\dot{q}'''_G R^2}{4K}\right)\left(1 - \left(\frac{r}{R}\right)^2\right)$$

Which is the closed form solution that we could have got about temperature profile within this plane cylinder subjected heat generation. If we want to calculate the central line temperature

$$T_0 = T(r)|_{r=0} = T_s + \left(\frac{\dot{q}'''_G R^2}{4K}\right)$$

Therefore, sometimes we also write this expression in an alternate form as

$$T(r) = T_s + (T_0 - T_s)\left(1 - \left(\frac{r}{R}\right)^2\right)$$

Now what will be your rate of heat conduction? Then $\dot{q}"_{cond}$ will be equal to

$$\dot{q}_{cond} = -KA\frac{dT}{dr}$$

$$= -K(2\pi rL)\left(\frac{\dot{q}'''_G}{4K}\right)(-2r)$$

L is the length of this particular cylinder.

$$= \dot{q}'''_G(\pi r^2 L)$$

Again, $\pi r^2 L$ refers to the volume of an incremental cylinder. If we take an incremental volume inside $\pi r^2 L$ will be the total volume of this. Then how much is the total heat conduction at the surface

$$\dot{q}_{cond}|_{r=R} = \dot{q}'''_G(\pi r^2 L)$$

at r = R, heat energy conducted at the surface at this location or surface of this cylinder is the entire volume of the cylinder multiplied by the energy generation that is the amount of heat that will be coming out.

And if the cylinder is subjected to some kind of convective boundary conditions a fluidic temperature T_∞ and heat transfer coefficient of h then this should be equal to

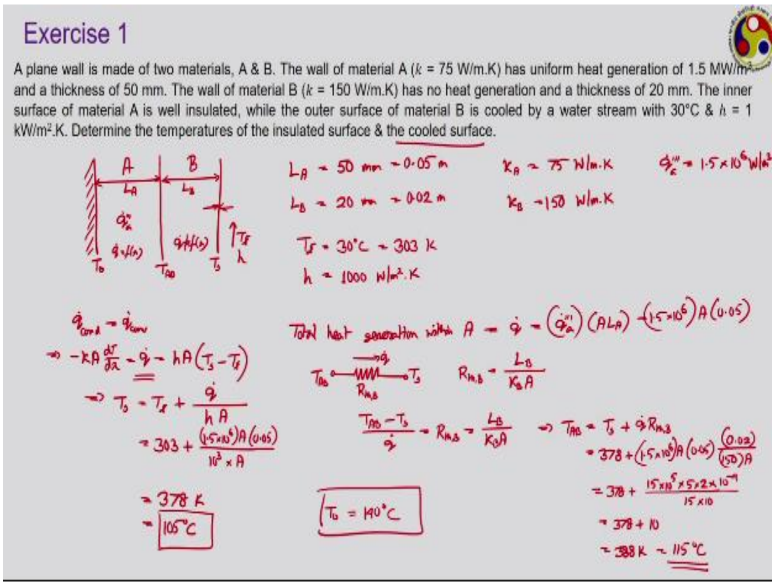
$$\dot{q}_{cond}|_{r=R} = \dot{q}'''_G(\pi r^2 L) = \dot{q}_{conv} = (2\pi r L)(T_s - T_\infty)$$

Remember conduction convection both in this cases happening on the same surface area, the peripheral area, which is $2\pi r L$. So using this we can also get an expression for $T_{s.}$ if T_s is not known.

So this way we can take care of energy generation inside both plane wall and plane cylinders. Now let us quickly solve a couple of numerical problems to get a taste of what we have done. Before solving any problem, you have to be careful about one thing. The resistance concept that we have used in the previous week that is not applicable when energy generation is present. Because when energy generation is present we have seen in case of both plane wall and cylinder that the rate of energy conduction is a function of location, and as it keeps on varying from one point to another in the same system therefore we cannot go for the resistance concept.

We have to stick to the basic formulation. However, if your system can be broken into several components where some component is expressing energy generation and some components are not then for the components where there is no energy generation you can still stick to that resistance concept. That is precisely what we are going to check in this first numerical problem.

(Refer Slide Time: 49:12)



Here it is given that we are having a plane wall, which is made of 2 materials A and B. The wall of material A has uniform heat generation and the thickness of 50 mm and wall of material B is having no heat generation and thickness of 20 mm, so we can draw the walls. So we have 2 blocks, this is your A, this is your B, block A is having thickness of L_A, block B is having thickness of L_B.

Here as per our given information L_A, the thickness of the first block is 50 mm that is 0.05 m L_B is equal to 20 mm, 0.02 meter. Their thermal conductivities are given also, K_A is equal to 75 W/mK, K_B is equal to 150 W/mK. The wall A is having energy generation, so $\dot{q}'''_G$ for A is given as 1.5 MW that is $1.5X10^6$ W/m^3.

No heat generation in the second one, that is layer B. Now the inner surface of material A is insulated well. So this side is well insulated, no heat transfer across this; while the out of surface of material will be cooled by water stream with 30 ºC and h is equal to 1 kW/m^2K. On this side you are having a fluid flowing with temperature T_∞ and heat transfer coefficient h.

It is given that T_∞ = 30 ºC that is 303 K and this convective heat transfer coefficient is 1000 W/m^2K. You have to determine the temperature of the insulated surface and the cold surface. So let us say temperature at this point is T_0, temperature at this surface is T_s. This we have to calculate. So there are 2 layers and therefore and as we are having heat generation only in one of the layers.

So we can treat the layer separately. For layer B we can use the resistance concept, but for layer A we have to follow the mathematical approach that we have just done. Now for layer A as there is heat generation, so here $\dot{q}$ will be a function of x, but layer B as there is no heat generation there is no heat $\dot{q}$ will be a constant in this case.

Then let us try to tackle q_B and write an energy balance at this point.

$$\dot{q}_{cond} = \dot{q}_{conv}$$

$$\Rightarrow -KA\frac{dT}{dx} = hA(T_s - T_\infty)$$

So this is the energy balance at the cold surface. From where the $\dot{q}$ is coming? The $\dot{q}$ is coming because of whatever energy that is getting generated in A, that is passing through B without any change.

So total energy generation or heat generation I should say within A that should get transmitted through B as it is; so this is the $\dot{q}$ that I am looking for and that should be written as

$$\dot{q} = \dot{q}'''_G(AL_A) = 1.5 \times 10^6(0.05)A$$

So you can easily calculate the total value of $\dot{q}$, and once we take it back here then we have an estimate of T_s

$$T_s = T_\infty + \frac{\dot{q}}{hA}$$

$$= 303 + \frac{(1.5 \times 10^6)A(0.05)}{10^3 \times A}$$

$$= 378\ K = 105\ ℃$$

So this is one of the solution for the cold surface, which we can get very easily. Now we have to do for this surface number B. What about the temperature point T_{AB}, how can we calculate T_{AB}? If we talk about energy transmission via resistance following electrical resistance concept through the layer B, then we can express it something like this. Here your temperature is T_{AB}, the temperature is T_s, which you have just calculated, and how much will be this resistance? It should be equal to

$$\frac{T_{AB} - T_s}{\dot{q}} = R_{th,B} = \frac{L_B}{K_B A}$$

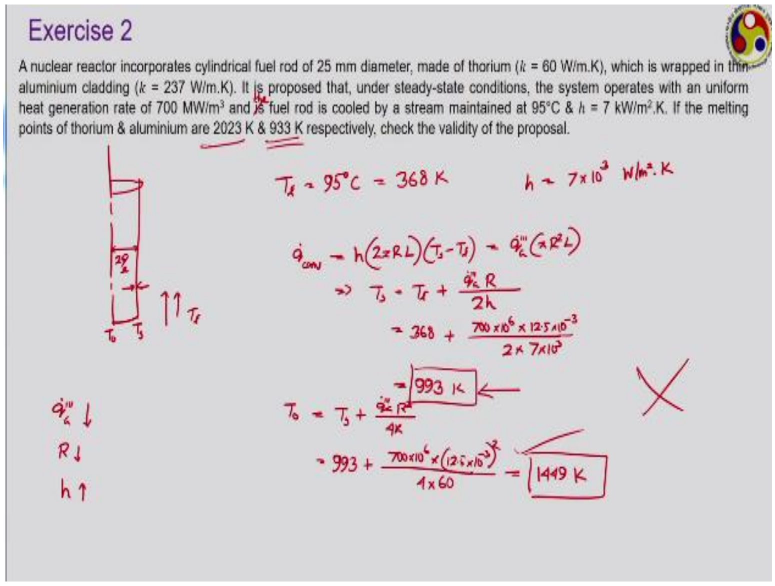
$$\Rightarrow T_{AB} = T_s + \dot{q}R_{th,B} = 378 + \frac{(1.5 \times 10^6)A(0.05)(0.02)}{(150)A}$$

$$= 388\,K = 115\,℃$$

That is T_{AB}. Once you know T_{AB} then you can easily go back to the earlier approach that we have done to get the values of T_0.

If I just go back to the surface plane wall with heat generation, this was the temperature profile that we had in this. You can easily follow this approach to get the temperature at the other end. Here in this case we are looking to identify the T_{s2} itself and T_{s1} is the T_{AB} that we have just got and accordingly you can do the calculations to get the final result. So, final solution for T_0 is going to be equal 140 degree Celsius. You please do the calculation to see whether you are going to get the results or not.

(Refer Slide Time: 58:51)



Now second problem I would like to quickly go through this one. Here the problem involves a nuclear reactor, which incorporates cylindrical fuel rod of 25 mm diameter; it is made of Thorium, which is having thermal conductivity of 60 W/mK. It is wrapped in a thin aluminium cladding. It is proposed that under steady state condition, the system operates with uniform heat generation rate of 700 MW/m^3 and the fuel rod is cooled by stream maintained at 95 ºC and corresponding h is given. So you are having a cylindrical fuel rod. Let us draw only half of the domain. So this is your fuel rod made of thorium and around this rod, we are having a thin cladding. Cladding is nothing but a jacket kind of thing. So this is the fuel rod with diameter is 25 mm, so radius is 25/2. And aluminium cladding is very thin, so the cladding thickness is not given which can be neglected for the situation shown here. We can almost assume that the surface is made of this aluminium. So we are not going to draw the cladding surface separately, rather this is the half of the domain that I am considering, half of the cylinder. There is heat generation that is happening inside the cylinder and outside we are having a fluid flowing with temperature T_∞, in this case T_∞ is set at 95 ºC that is equal to 368 K. Corresponding convective heat transfer coefficient is 7 X 10^3 W/m^2K. So you have to solve this problem. Again, there is a heat generation part.

So using heat generation let us try to calculate the temperature T_0 here and the surface temperature T_s here. You already know the way you have to derive the expressions for this. For the moment you just neglect the cladding, just consider this is a cylinder and which is having uniform volumetric heat generation, you have to calculate this temperature T_0 and T_s.

Performing a balance between conduction and convection here, energy convected away = total energy generation

$$\dot{q}_{conv} = h(2\pi RL)(T_s - T_\infty) = \dot{q}'''_G(\pi R^2 L)$$

$$\Rightarrow T_s = T_\infty + \frac{\dot{q}'''_G R}{2h}$$

$$= 368 + \frac{700 \times 10^6 \times 12.5 \times 10^{-3}}{2 \times 7 \times 10^3} = 993\ K$$

Now you have to calculate the value of T_0

$$T_0 = T_s + \frac{\dot{q}'''_G R^2}{4K}$$

$$= 993 + \frac{700 \times 10^6 (12.5 \times 10^{-3})^2}{4 \times 60} = 1449\ K$$

So the thermal conductivity of aluminium actually never coming into picture because aluminium layer we are not considering anywhere during the calculation. So the temperature we have identified, this is the maximum temperature the system is having, this is temperature the surface of that aluminium cladding.

We have to judge whether this proposal is valid or not. The proposal is valid or not that depends upon the melting point, see the melting point of thorium is 2023 K and the maximum system temperature is well below there. So the thorium will remain solid, but the melting point of aluminium is 933 K and we are having the surface temperature to be 993 K, which is going to be the aluminium temperature.

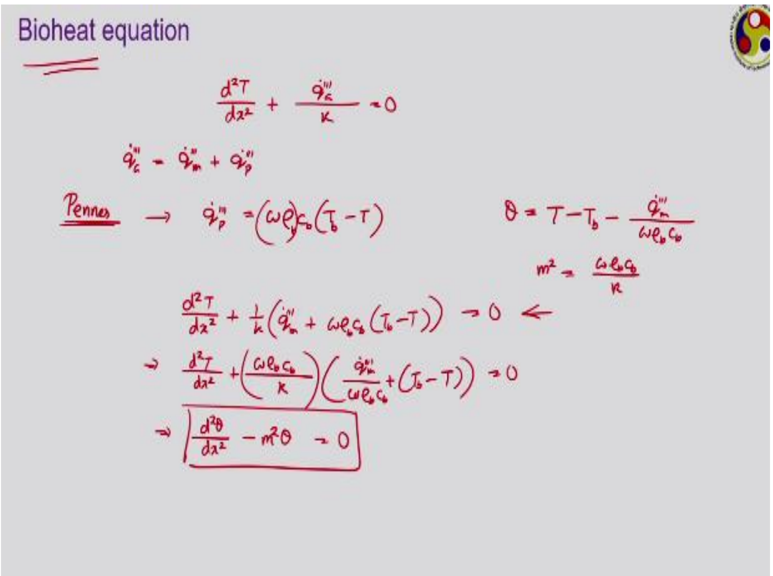
So aluminium is going to melt if we run the system. So this cladding will not be able to stay and hence this proposal is not valid. To make the design valid then there are several ways we can make modifications. One way is if we reduce the volumetric rate of heat generation but that depends upon the rate of fusion reaction and also we always want higher volumetric energy generation in such kind of systems.

Then another possibility is if we can reduce the radius of this one, radius of this rod; so that the surface temperature becomes smaller. Another definite possibility if we can increase the heat transfer coefficient h but again that depends upon the coolant side conditions and of course if somehow we can use a cladding material which is having a higher melting point that is also very much possible.

Finally, I would like to give you another example of a scenario where we have heat generation or rather conduction with heat generation that is inside our body only. You know that because of the metabolic activities our body is always converting chemical energy to thermal energy and the thermal energy is going to get conducted through our tissues, come to the skin and then that get distributed to the surrounding via convection.

But before it reaches the skin, it is only conduction; and conduction with energy generation and that is where the bio heat equation comes in to picture, it is nothing but the thermal conduction equation modified to incorporate the metabolic heat generation inside our bodies.

(Refer Slide Time: 01:06:02)



The earlier equation that we wrote was

$$\frac{d^2T}{dx^2} + \frac{\dot{q}'''_G}{K} = 0$$

Now here this $\dot{q}'''_G$ generally can have 2 contributions, one contribution coming from the metabolic activities and other contribution is coming from the perfusion.

$$\dot{q}'''_G = \dot{q}_m + \dot{q}_p$$

Perfusion refers to as the blood flows through the tissues then it exchanges heat with that, depending on the temperature of the blood is higher or temperature of tissue is higher this $\dot{q}_p$ can be positive or negative.

But the $\dot{q}_m$ is always positive. We have standard charts available for $\dot{q}_m$, when a person is sleeping, generally$\dot{q}_m$is in the range of 75 W or rather total body generates heat in the range of 75 W. Here we are writing in per unit volume, so that 75 W needs to be divided by total volume of our body to get the value of this $\dot{q}_m'''$.

When someone is swimming it is in the range of 200 to 250 W. When someone is playing some high activity sports like tennis etc., it can be in the range of 400 W as well. But $\dot{q}_p$ is the one that need some attention. Without going into detail I shall be giving you the expression that was developed by Pennes,

$$\dot{q}_p''' = \omega\rho_b c_b(T_b - T_t)$$

Here T_b is the temperature of the blood, T_t is the temperature of the tissue, ρ_b and c_b refers to the density and specific heat of the blood and ω is the rate of this perfusion. Generally, $\omega\rho_b$ can also be considered as the mass of the tissue that is concerned in this heat transfer. So if we put this form then the heat conduction equation can be written as

$$\frac{d^2T}{dx^2} + \frac{1}{K}\left(\dot{q}_m''' + \omega\rho_b c_b(T_b - T_t)\right) = 0$$

This is the bio heat equation. Generally, we used to go for some kind of non-dimensionalization by introducing a term θ. Before defining this θ let us just take this term out.

$$\frac{d^2T}{dx^2} + \frac{\omega\rho_b c_b}{K}\left(\frac{\dot{q}_m'''}{\omega\rho_b c_b} + (T_b - T_t)\right) = 0$$

This T_t is actually the temperature that you are trying to identify because tissue temperature is the one that is our objective. So let us drop the subscript.

$$\frac{d^2T}{dx^2}+\frac{\omega\rho_b c_b}{K}\left(\frac{\dot{q}_m'''}{\omega\rho_b c_b}+(T_b-T)\right)=0$$

Θ we can define as

$$\theta = T - T_b - \frac{\dot{q}_m'''}{\omega\rho_b c_b}$$

So the equation becomes

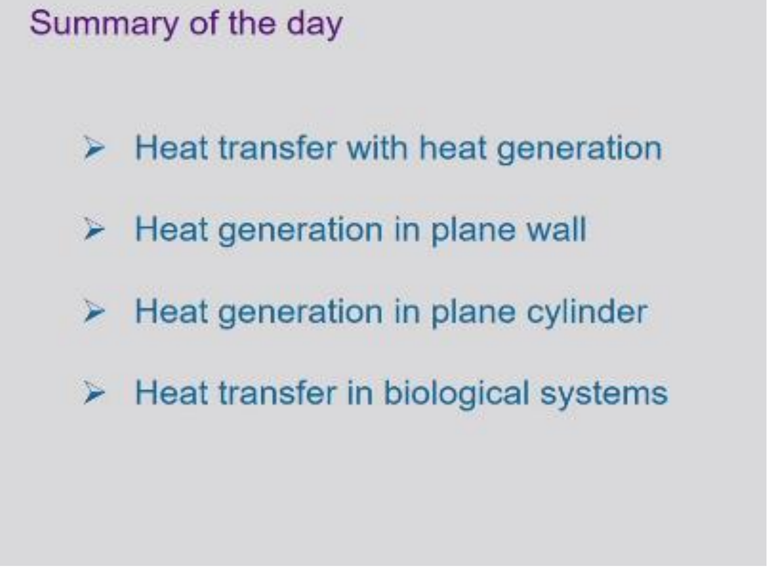
$$\frac{d^2T}{dx^2} - m^2\theta = 0$$

Where,

$$m^2 = \frac{\omega\rho_b c_b}{K}$$

So this is the bio heat equation or the Pennes equation. This can easily be solved using the standard boundary condition and we can get an idea about the temperature profile inside our body from the core to the skin using suitable boundary conditions. Nevertheless, I am not going to the deep of this, if any of you are interested you can just refer to the book of Incropera and DeWitt, the recent versions where some more information about this is available. So this is where I shall be stopping today.

(Refer Slide Time: 01:10:51)



Today we have talked about the heat transfer with heat generation in both the plane wall and the plane cylinder. Several scenarios of that has been discussed like symmetric temperature profile, asymmetric temperature profile, the condition of a symmetrical surface or symmetrical boundary condition and then we have briefly discussed about heat transfer in biological system.

So that takes us to the end of our discussion for the day. There is one more lecture this week where I shall be talking about some other special scenario like variable space thermal conductivity during heat conduction. Thank you very much.