Fundamentals of Conduction and Radiation

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Lecture - 07
1 - D Steady minus State Heat Conduction – Part 2

Good morning everyone. Welcome to the second lecture of our week number 3 where we are talking about the one minus dimensional steady state heat conduction scenario. Now in the previous lecture, we have seen the situation with a plane wall. Basically we started with the generalized heat diffusion equation in the Cartesian coordinate, and then we simplified that by removing the transient term, then removing the heat generation term and then also used the space dependence just to one coordinate. Accordingly, the corresponding partial differential equation got reduced to an ordinary differential equation and then we have seen how we can solve that to analyze simple conduction scenario and we have also seen the concept of thermal resistance or how can you use the concept of thermal resistance in such conduction analysis, so that we do not need to solve the ordinary differential equation always. Rather we can just calculate the corresponding thermal resistance and get the rate of heat transfer and that concept is even more useful and when you are dealing with a composite wall just like this.

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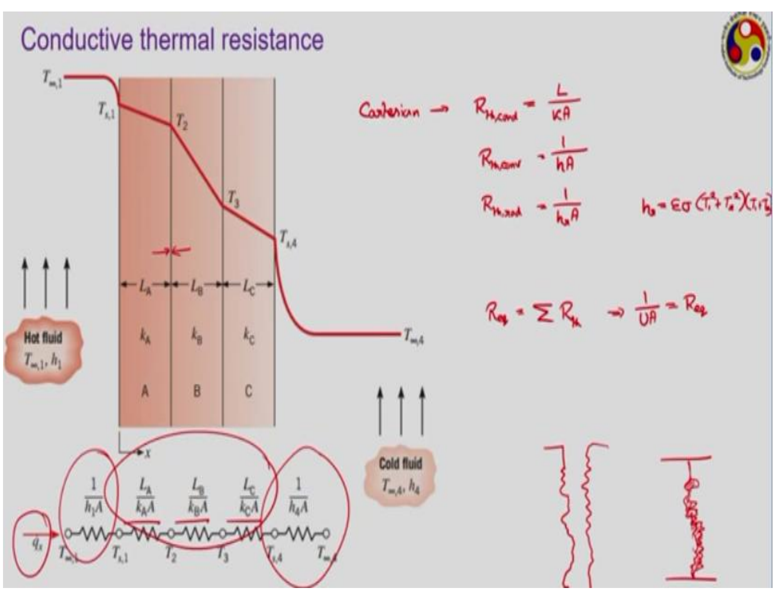

Very similar to the electrical systems, we can consider the given system to be comprising of several such kinds of thermal resistances and accordingly we can form an equivalent thermal resistance circuit and do the analysis. This is called the electrical analogy of heat transfer. Like the situation shown here, from there like yesterday as long as we are sticking to the Cartesian coordinate system, then we can write the thermal resistance for conduction to be

$$R_{th,cond} = \frac{L}{KA}$$

where L is the length scale associated with the direction on which the heat is being transferred. A is the cross section area, the area perpendicular to which the heat is being transferred and K is the corresponding thermal conductivity.

And we have seen similarly the thermal resistance corresponding to convection can be given as

$$R_{th,conv} = \frac{1}{hA}$$

Where, h is the convective heat transfer coefficient, A again is the area perpendicular to the direction of this heat transfer. When we are having multiple such kinds of layers comprising of conduction and convection or sometimes in some cases radiation, then we can easily calculate the thermal resistances for each of them to form a resistance circuit.

So just for completeness purpose, let me add the thermal resistance that we can get corresponding to radiation. I am not deriving it here because that we have done in the previous week, where we introduced the concept of thermal resistance for both convection and radiation situation and also drawing analogy with the convective thermal resistance. We can also write this one to be

$$R_{th,rad} = \frac{1}{h_r A}$$

Where h_r refers to the radiative heat transfer coefficient, which we know that can be represented as

$$h_r = \varepsilon\sigma(T_1^2 + T_2^2)(T_1 + T_2)$$

Where, ε is the emissivity, σ is the Stefan Boltzmann constant and T_1 and T_2 are the temperatures across which this heat transfer is being taking place. And then this is the corresponding expression for the resistances. And this particular diagram that is shown here yesterday we have done the analysis.

Here this situation where we can see convection is happening on either side of the wall and the wall itself comprises of three different layers. We can visualize five different resistances. This is the one corresponding to the convective heat transfer associated with the hot fluid side. This is the convective thermal resistance associated with the cold fluid side and in between we have the three conduction thermal resistance.

1, 2 and 3 corresponding to each of the layers. And as the area associated with each of the five modes of heat transfer are same, so we can assume that all of them are connected in series and also the same amount of power is passing through all of them. Therefore, we can easily connect them through a series just analogous to what we do in electrical circuit and then calculate the equivalent resistance.

The equivalent resistance for such as a circuit can be just summation of all the individual thermal resistances. And also we have introduced the concept of overall heat transfer coefficient, where the product of overall heat transfer coefficient into the corresponding area is defined as this equivalent resistance.

$$R_{eq} = \sum R_{th} = \frac{1}{UA}$$

So we can also calculate the overall heat transfer coefficient at any specified area once we know the value of the equivalent resistance for the circuit.

And when the areas are different, then you have to visualize them to be connected in parallel and accordingly you can perform the analysis as well. Now, just look at this particular surface, which is connecting this material or layer A and layer B. Here while writing this expression in this particular format; we are assuming that the contact surface connecting layer A and layer B is a perfect one.

There is no irregularity on the surface and on each point of the surface both A and B are tightly connected with each other, but that may not be true in practice, because in several scenarios practical surfaces can be rough. They may have certain irregularities. Suppose your surface A may be of shape like this, whereas surface B may have a shape somewhat like this. Now if you connect them, then what will happen?

If this is the way you are connecting them, ideally they should have been connected just along this line, but practically the contact surface may have lots of variations or pores, something like this. Now this intermediate portion, which I am showing by this hash line; these portions are voids. They are generally filled by air or if there is some liquid or other gases that can enter into this, they are filled with that.

And accordingly the concept of contact resistance comes into picture. Contact resistance refers to exactly what I am talking about the layer A and B are connected with each other and once we assume this connection using a microscope or some similar device, you will find lots of discontinuities, lots of pores and these voids once they are filled up with certain material like air, then they also will lead to some kind of resistance or they are also going to offer certain additional resistance to the heat transfer. Because the thermal conductivity of air generally is expected to be much lower than corresponding solid materials, the heat when it is passing through these voids passing through these pores, they will face additional resistances and that resistance is referred as this contact resistance.

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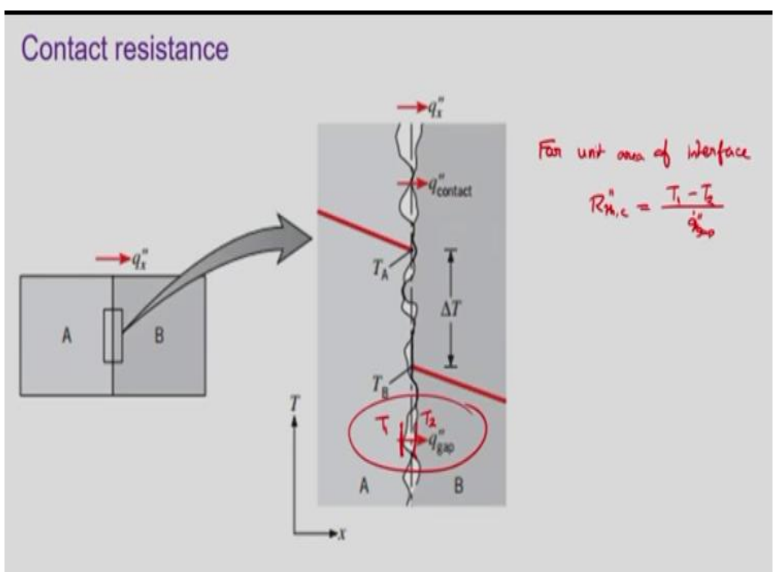

Suppose for unit area of the interface, you are talking about at one point where the temperature is T_A and another point the temperature is T_B, then the corresponding thermal resistance at this contact surface will be

$$R"_{th,c} = \frac{T_A - T_B}{\dot{q}"}$$

We are putting the double prime symbol to indicate that we are going to use heat flux. Here we are assuming that the heat transfer is taking place from T_A to T_B. For better convenience let us talk about this point, let us put the temperatures T_1 and at this point the temperature is T_2, then as per this diagram if you talk about this heat transfer, then corresponding thermal resistance will be

$$R"_{th,c} = \frac{T_1 - T_2}{\dot{q}"_{gap}}$$

So this additional resistance now also needs to be taken into consideration, while forming the circuits for thermal resistances. Of course the magnitude of the thermal resistance depends upon the surface, the nature of the contact depends upon both layer A and B and also by modifying the nature of this gap, we can modulate the value of this resistance. Like suppose if these gaps are vacuum, then what will happen?

Then there is nothing which can be used to transfer the heat and you know that both conduction and convection requires certain kind of medium for transport of heat. So if the pores are vacuum and then there will be no heat transfer through them at all, and accordingly the heat will only get transmitted through this connected portion; through these portions there will be no heat transfer. So the total contact resistance will be much more significant in that case.

On the contrary, if we can somehow fill up these gaps with some conducting liquid, then there will be an increase in the thermal conduction property through this gap, so the contact resistance will decrease.

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Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)

(a) Vacuum Interface			(b) Interfacial Fluid	
Contact pressure	100 kN/m²	10,000 kN/m²	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m²)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m²)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m²)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m²)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m²)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

These are the certain values for contact resistances for certain common situation, when you are talking about vacuum interface. For stainless steel the value of this thermal resistance will be in the range of$6-25\times10^4$. Please note this notation. It indicates that whatever values they are given there are $R''_{th,c}$ after multiplying $R''_{th,c}$with 10^4.

So here with stainless steel, we are actually talking about values of contact resistance to be of the order of $6-25\times10^4$. Similarly, for aluminum, it is slightly lower because aluminum has a higher thermal conductivity compared to stainless steel. If the interfacial fluid is air, then you are having 2.75, whereas you are using something like glycerol, resistance is lesser. These are certain interfaces or metal combinations or material combinations.

You can see when we are having aluminum coupled with aluminum with indium foil filter, then it is only of the order of 0.07. Similarly, when silicon chip has been coupled with aluminum with 0.02 mm epoxy, it is in the range of $0.2 - 0.9 \times 10^4$. Also note that unit for this one; here we are also having a m^2 in the numerator, because we are expressing this one in terms of heat flux and not in terms of power.

So contact resistance is another important parameter while dealing with the real surfaces and to mathematically calculate the rate of heat transfer through such contact resistance, we need to know precise value of this contact resistance, otherwise we cannot do it and for that we generally have to depend on certain kind of experiments and maybe certain analogies with real life situations. Let us now solve one numerical problem to use this concept of thermal resistance.

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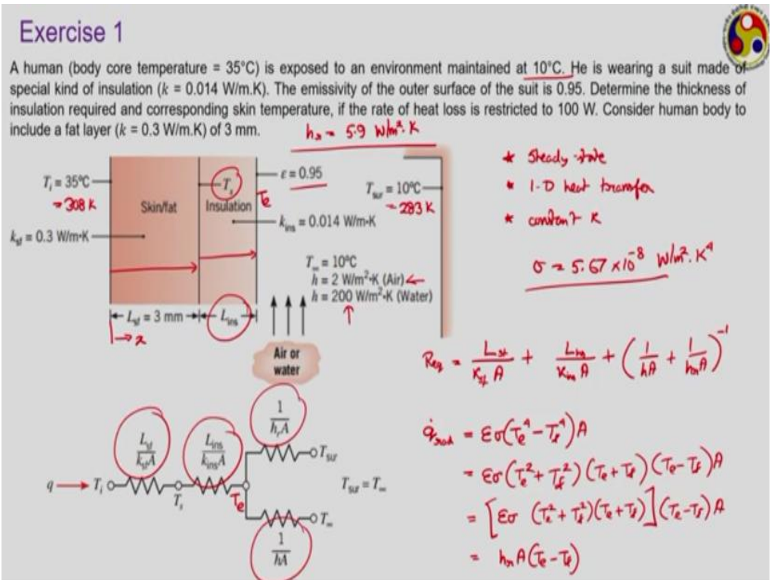

I am not putting the contact resistance in here. I am just talking about thermal resistances. Now, here is my situation. I am talking about a human which is having a body core temperature of 35 ºC. He is exposed to an environment maintained at 10 ºC. He is wearing a suit made of a special kind of insulation; the corresponding thermal conductivity of insulation for the suit is mentioned.

Emissivity of the outer surface of the suit is 0.95. So radiation heat transfer is also relevant. And we have to determine the thickness of the insulation required and corresponding skin temperature, if the rate of heat loss is restricted to 100 W. We have to consider the human body to include a fat layer of 3 mm; corresponding thermal conductivity of fat layer is also given. This is somewhat kind of a schematic representation.

This surface represents core body, then we have the fat layer of 3 mm thickness and then we have the insulation. So this is our body core and this is the skin, in between you have the layer of fat. So this T_s here refers to the skin temperature which we have to find at the end and the insulation corresponds to the suit that he is wearing. So this is the thickness of the insulation or thickness of the suit which also you have to identify.

Emissivity of the outer surface and thermal conductivity of the insulation material is given and on outer surface we can say that the temperature is 10 ºC. Now, we can consider two kinds of scenarios. We can have air flowing over the surface at 10 ºC. This is the corresponding heat transfer coefficient, whereas when water is flowing the transfer coefficient will be much larger. Let us say it is 100 times more than air.

We shall be solving for both the scenarios, but quite a few assumptions that we have to put, the standard assumptions. One is steady state, we have to consider. Then 1D heat transfer, let us say this direction is the x direction, then the entire heat transfer is taking place only in this direction. Then we are talking about constant value of the thermal conductivities for both the materials. Similarly the emissivity is also constant.

And so is the heat transfer coefficient for air or water whatever is the working medium. And another consideration that is related to the radiation application here is that we are assuming the body or this outer surface of this insulation layer, it is total area to be too small compared to the area of the room, so that we can assume it is having radiative heat transfer only with the wall maintained at 10 ºC. So we have to form the circuit of thermal resistances.

Find the modes of heat transfer, how many modes of heat transfer involved here, all the three modes. Firstly, from here up to this, it is conduction from the core to the skin by conduction. Then again conduction; from the skin through the insulation to the outer surface. Remember here we are not talking about any kind of contact resistance at this point. It has been assumed that the contact resistance is 0 in this situation.

Then from the outer surface of the insulation heat can go to the surrounding following two modes, as some kind of fluid air or water is flowing over the surface at a lower temperature, so there will be heat loss by convection. Similarly, there will be also radiative heat loss. So this is the corresponding resistance circuit that you can think of. This is the conduction resistance corresponding to the fat layer.

This is the conduction resistance corresponding to the insulation. This is the convective resistance and this is the radiative resistance from the surface to the outer surrounding or outer air. Now we have to calculate each of the resistances separately. So how we can form or calculate the equivalent resistance in this case? It can be

$$R_{eq} = \frac{L_{sf}}{K_{sf}A} + \frac{L_{ins}}{K_{ins}A} + \frac{\frac{1}{hA} \times \frac{1}{h_r A}}{\frac{1}{hA} + \frac{1}{h_r A}}$$

Simplifying it,

$$R_{eq} = \frac{L_{sf}}{K_{sf}A} + \frac{L_{ins}}{K_{ins}A} + \frac{1}{A(h + h_r)}$$

Subscripts $_{sf}$ is for skin fat layer and $_{ins}$ is for the insulation layer. Now there are two convective and radiative resistances. Both of them are connected in parallel to each other because at this point the temperature is the outer surface temperature and on this side the temperature is the surface temperature. So the temperature across them are the same, but the amount of power transmitting through both of them they are different that's why you can assume them to be connected in parallel and if they are connected in parallel just the way you calculate the equivalent resistance for two electrical resistors connected in parallel, we can write them also. Now other values are all given, but we have to calculate the h_r.

How can we get the h_r? Here this h_r actually talking about the radiation heat transfer that is taking place from the exterior surface of the insulation to the surrounding. So we can write its expression to be as

$$\begin{aligned} \dot{q}_{rad} &= \varepsilon\sigma(T_e^4 - T_\infty^4)A \\ &= \varepsilon\sigma(T_e^2 + T_\infty^2)(T_e + T_\infty)(T_e - T_\infty)A \\ &= [\varepsilon\sigma(T_e^2 + T_\infty^2)(T_e + T_\infty)](T_e - T_\infty)A \\ &= h_r A(T_e - T_\infty)A \end{aligned}$$

So from this given information, we have ε here as 0.95, σ the Stefan Boltzmann constant, do you remember its value? The value of σ I told earlier also, it is $5.67 \times 10^{-8} \mathrm{W/m^2.K^4}$. So this value of σ that you can put here.

Area is not given, but for the moment we can assume area to be 1, because area is happening in each of the resistances. And then we have T_e and T_∞. T_∞ is 10 ºC but while doing this, we have to remember that the temperatures need to be expressed in Kelvin as we are talking about radiation. So let us convert T_i to K. So it is 35 ºC means it becomes 308 K. This is 10 ºC, so your T_∞ becomes 283 K.

So we know T_∞ or T_{sur}, but this T_e, the T exterior which is acting at this particular point, this is not known yet and unless that is known, it is not possible to calculate the value of this h_r. So we are assuming some value of h_r in this case. Let me provide you the value of h_r to be equal to 5.9 $W/m^2.K$. This value can be obtained through an iterative calculation. I am directly giving you the value.

Once you have the value h_r, then you can easily calculate the value of R_{eq}. So what will be your R_{eq}? The expression that I have written from there lets write it again,

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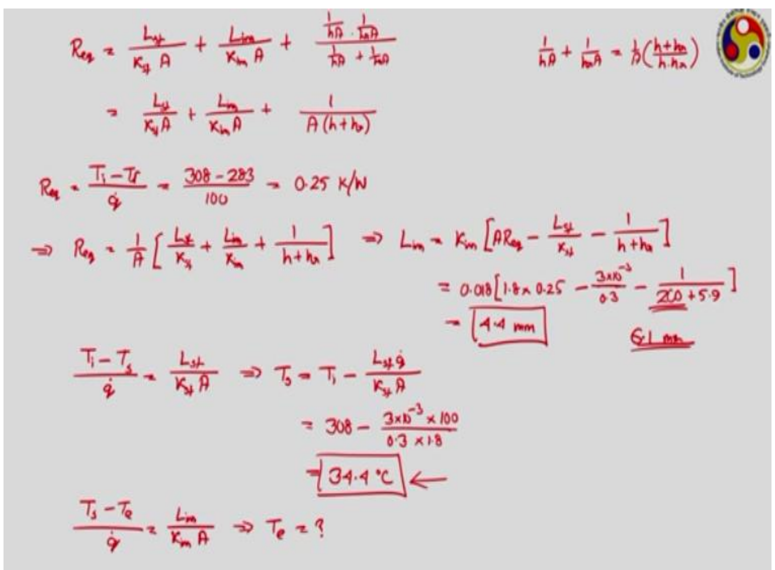

$$R_{eq} = \frac{L_{sf}}{K_{sf}A} + \frac{L_{ins}}{K_{ins}A} + \frac{1}{A(h + h_r)}$$

Now this is the R_{eq} that we are getting and it is mentioned in the problem that we want the rate of heat loss to be restricted to 100 W. So using this R_{eq}, we can write that

$$R_{eq} = \frac{T_i - T_\infty}{\dot{q}}$$

Now putting the values for these

$$R_{eq} = \frac{308 - 283}{100} = 0.25 \frac{K}{W}$$

So once you have got the values, then using that R_{eq} what we can write? R equivalent is equal to

$$R_{eq} = \frac{1}{A}\left[\frac{L_{sf}}{K_{sf}} + \frac{L_{ins}}{K_{ins}} + \frac{1}{(h + h_r)}\right]$$

$$\Rightarrow L_{ins} = K_{ins}\left[AR_{eq} - \frac{L_{sf}}{K_{sf}} - \frac{1}{(h + h_r)}\right]$$

Now what about the area? How much is the area that is given? The area information is again not explicitly given in the problem. Now in the true sense in the problem the value was given, I have omitted that because I want to discuss about that.

Like the way I have shown how to calculate the h_r, which may need an iterative calculation with finally assumed an value. For the A also, we have to take a value and generally the human the outer surface of the human body can have an area of approximately 1.8 m^2. So if we put that here,

$$0.018 = \left[1.8 \times 0.25 - \frac{3 \times 10^{-3}}{0.3} - \frac{1}{2 + 5.9}\right]$$

$$= 4.4\, mm$$

I have consider air here to be the surrounding fluid which has a heat transfer coefficient of 2 and h_r is 5.9. So this is the insulation thickness that we need for that suit.

And the second parameter that we have to calculate is the skin temperature. That is the temperature T_s at this particular point that we have to calculate. Now how can we calculate that one? We know the rate of heat transfer now. The same $\dot{q}$ is going to flow through all the layers, then how can we use that. So if we consider only that inner layer, then corresponding to that layer, we can write the conduction equation to be

$$\frac{T_i - T_s}{\dot{q}} = \frac{L_{sf}}{K_{sf}A}$$

From there we can write

$$T_s = T_i - \frac{L_{sf}\dot{q}}{K_{sf}A}$$

$$= 308 - \frac{3 \times 10^{-3} \times 100}{0.3 \times 1.8} = 34.4\ ℃$$

So you can see the temperature here was 35 ºC, and on the skin 34.4 ºC. So there is very little difference. However, following the same way, you can calculate this T_e also. And you will find obviously difference there. If we want to consider or repeat the same calculation considering water as the surrounding fluid, then you have to replace this 2 here by 200.

And if you do the calculation, you will find that in that case the required thickness of insulation will be approximately 6.1 mm. Please do this calculation and also I would request you to calculate the value of this T_e, the exterior temperature. How can you do that? If we consider the insulation layer only or heat transfer through the insulation layer, then

$$\frac{T_s - T_e}{\dot{q}} = \frac{L_{ins}}{K_{ins}A}$$

So from there you can get this exterior temperature. Just check what values you are getting corresponding to both the materials. What should be the value of this T_s with water as the working medium? This one only, the same, why? Look at the expression. In that expression, the convective heat transfer coefficient is not coming and as the $\dot{q}$ is same regardless of the exterior fluid, so the expression remains the same and the calculation procedure is same.

And you will be getting the same value of T_s and also the same value of T_e for both the cases or sorry same value of Ts, but the value of T_e may be different. Just calculate and check that out.

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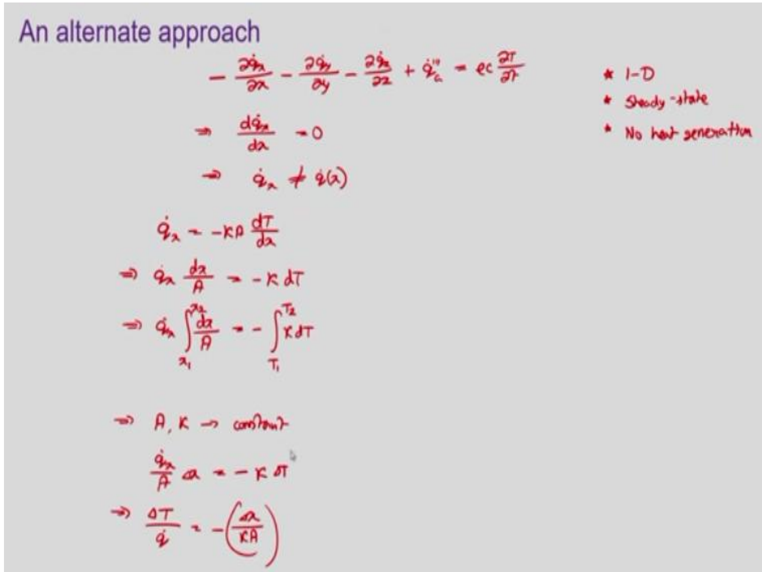

Now I am going to do something known as an alternate approach. The thermal conduction based or a thermal resistance based approach is very useful. As we have seen we can easily identify each of the thermal resistances, connect them in series and parallel and we can have an equivalent thermal resistance circuit to solve for. However, there is one important assumption that we have taken while doing this, that is we have assumed the thermal conductivity to be a constant.

But your if your thermal conductivity is not constant in the x direction, then if the thermal conductivity is not constant in the x direction or if your system is suffering from a change in the cross section area in the direction of heat transfer, then that thermal resistance concept becomes invalid and then we have to go for this alternate approach. Here I am going to take you back to the development process of that heat diffusion equation.

The generalized heat diffusion equation that we developed in a Cartesian coordinate in one of the earlier steps of that we got an expression of this form

$$-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} + \dot{q}'''_G = \rho c \frac{\partial T}{\partial t}$$

Now if we put our conditions, which we are always doing, that is if we restrict ourselves to one dimensional heat conduction under steady state and no heat generation.

Then what we are going to get? This equation becomes simpler the y and z direction can be neglected and the transient part also heat generation goes off, leaving us

$$\frac{d\dot{q}_x}{dx} = 0$$

Which gives $\dot{q}_x$ is not a function of x itself,

$$\dot{q}_x \neq \dot{q}(x)$$

That means $\dot{q}_x$ or heat transmission rate in the x direction actually remains to be a constant. So as long as you are talking about this Cartesian coordinate in plane wall the conduction heat transfer rate is a constant under steady state one dimensional scenario without any heat generation.

See here we are not talking about any variable thermal conductivity or constant thermal conductivity. It is equally applicable for both of them, because we are yet to introduce the Fourier's law of heat conduction. Then what is your Fourier's law of heat conduction. We know that as per Fourier's law

$$\dot{q}_x = -KA\frac{dT}{dx}$$

Of course we are using one minus dimensional conditions. So we are straight away writing $\frac{dT}{dx}$.

Now if we rearrange the terms a bit, then what we can do? Depending upon whether your parameters are variable with space coordinate or constant with the space, then maybe we can write this one to be something like

$$\dot{q}_x \frac{dx}{A} = -KdT$$

Because commonly K is a function of temperature, so we are clubbing K with this and now we are putting the integration sign.

$$\dot{q}_x \int_{x_1}^{x_2} \frac{dx}{A} = \int_{T_1}^{T_2} -KdT$$

And once we know how the area is varying with x and how K is varying with temperature you can easily do the calculation. If A is constant and K is also independent of temperature, then this one reduces to the earlier concept only.

In that case, what you are going to get? In that case you are going to have

$$\frac{\dot{q}_x}{A}\Delta x = -K\Delta T$$

And if you rearrange this one, then

$$\frac{\Delta T}{\dot{q}_x} = -\left(\frac{\Delta x}{KA}\right)$$

The negative sign is there, which indicates that the heat transfer is taking place in the direction of reducing temperature and this expression is nothing but the thermal resistance that we have used.

So the concept of thermal resistance can be employed only when we are talking about constant thermal conductivity and no area change in the direction of heat transmission, but when either of them or both are present, then it is better to go for this alternate approach, where we are making use of the information that the heat transfer rate is independent of x direction, as long as we are talking about steady state one dimensional heat conduction without any heat generation.

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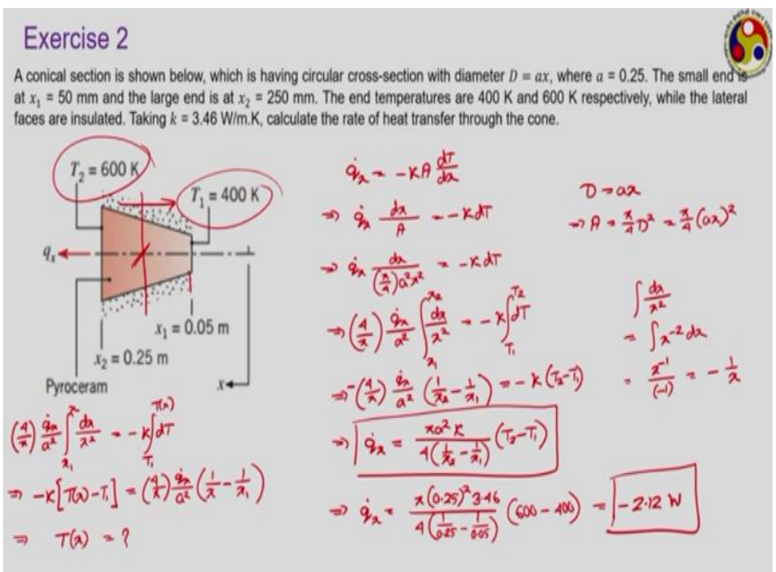

To demonstrate this, I have this problem for you. Here we are talking about a conical section which is having circular cross section in diametereter D = ax. So in the heat transfer direction, this is your x direction, x_1 is this point where the value is given to be 50 mm or 0.05 m, on the other end is 0.25 m. Temperature is given to be 400 and 600 K respectively.

So we have to calculate the rate of heat transfer through this cone. Here the thermal conductivity of the material is also given to be 3.46 W/ m K. Here the thermal conductivity is constant, that is independent of temperature, but the area is changing, because the diametereter is changing in the direction of heat transmission. Here the direction of heat transmission is actually, in this because from 600 to 400, it you will get transmitted.

But we are not talking about this; we are just sticking to the conventional x direction, which is from this smaller end to the larger end. So going back to the general expression

$$\dot{q}_x = -KA\frac{dT}{dx}$$

Or,

$$\dot{q}_x \frac{dx}{A} = -KdT$$

Now in this case, we know that the diameter is equal at any particular location, the diameter D is equal to

$$D = ax$$

The value of a is also given.

Then the cross sectional area at a particular location will be equal to

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}(ax)^2$$

If we put it back

$$\dot{q}_x \frac{dx}{\frac{\pi}{4}(ax)^2} = -KdT$$

A and K being constant then and also $\dot{q}_x$ is also constant, so we are writing this as

$$\left(\frac{4}{\pi}\right)\frac{\dot{q}_x}{a^2}\int_{x_1}^{x_2}\frac{dx}{x^2} = -K\int_{T_1}^{T_2} dT$$

Doing the integration and putting the limit

$$-\left(\frac{4}{\pi}\right)\frac{\dot{q}_x}{a^2}\left(\frac{1}{x_2} - \frac{1}{x_1}\right) = -K(T_2 - T_1)$$

Rearranging

$$\dot{q}_x = \frac{\pi a^2 K}{4\left(\frac{1}{x_2} - \frac{1}{x_1}\right)}(T_2 - T_1)$$

So you can put the limits here now. This is the expression for $\dot{q}_x$. We have got a closed form expression of $\dot{q}_x$. In this case, all the values are given. If you put the numbers for this particular problem, then $\dot{q}_x$ will come as

$$\dot{q}_x = \frac{\pi(0.25)^2 3.46}{4\left(\frac{1}{0.25} - \frac{1}{0.05}\right)}(600 - 400) = -2.12\, W$$

So this is the final solution that we are looking for. Why it is the minus sign? It indicates that the heat transmissions take place in the negative x direction and that is logical also, because your T_2 is higher than T_1.

So you can always expect the heat to get transmitted in this direction. The same is proved by this problem also. As long as you are consistent with your symbols, your notations, then your solution should also indicate the direction of heat transfer like it is doing here. If your interest is to calculate or get a closed form expression for the temperature at any intermediate plane, then how can you do that?

There also we can make use of the expressions. In that case, you can perform the integration something this way

$$\left(\frac{4}{\pi}\right)\frac{\dot{q}_x}{a^2}\int_{x_1}^{x}\frac{dx}{x^2} = -K\int_{T_1}^{T(x)} dT$$

In that case

$$\Rightarrow -K[T(x) - T_1] = \left(\frac{4}{\pi}\right)\frac{\dot{q}_x}{a^2}\left(\frac{1}{x} - \frac{1}{x_1}\right)$$

And now you can rearrange the terms to get a closed form expression for T at x. So this way you can solve using the alternate approach also when you are dealing with the variable cross section area or temperature dependent thermal conductivity. So far we have restricted our discussion only to plane walls that is only to the Cartesian coordinate, but in several scenarios we have to go to the cylindrical or spherical coordinate system as well. So let us check out the 1D steady state heat conduction scenario in a plane cylinder.

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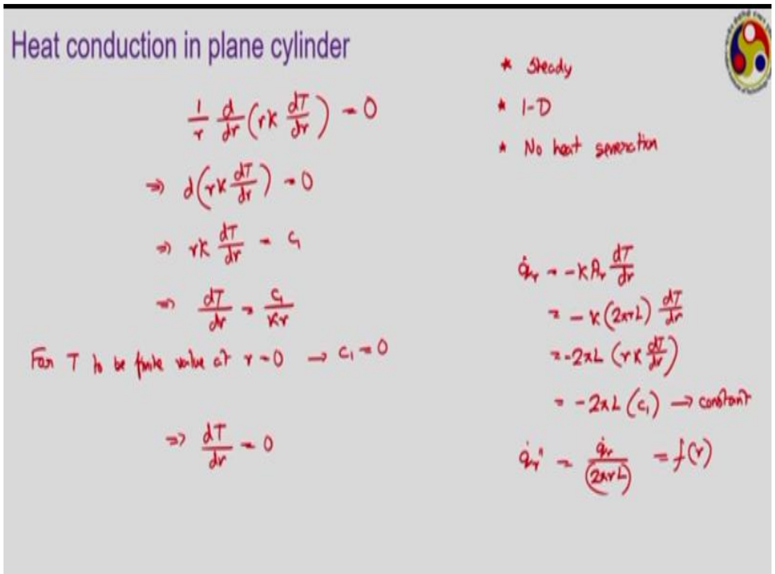

Like I have shown in the very first slide of yesterday's lecture, that under one dimensional scenario the see generalized heat diffusion equation in cylindrical coordinate gets used to a form like this

$$\frac{1}{r}\frac{d}{dr}\left[rK\frac{dT}{dr}\right] = 0$$

This is of course the same assumptions I should repeat again. We are doing this for steady state, one dimension and at the absence of any kind of heat generation. So we have

$$\Rightarrow \frac{d}{dr}\left[rK\frac{dT}{dr}\right] = 0$$

$$\Rightarrow rK\frac{dT}{dr} = C_1$$

With C_1 being a constant. Also we know that the rate of heat transmission in the cylindrical coordinate direction

$$\dot{q}_r = -KA_r\frac{dT}{dr}$$

Now what is your area for a cylinder? Let us take one cylindrical object something like this. This is the center line and we calculate r in this direction. So if the radius at this point is R and the length of the cylinder is L, then this total Ar can be represented as the peripheral area as $2\pi rL$. Putting the value

$$= -K(2\pi rL)\frac{dT}{dr}$$

Taking the constants out

$$= -2\pi L\left(rK\frac{dT}{dr}\right)$$

Putting the value from earlier developed relation

$$\dot{q}_r = -2\pi L(C_1)$$

As all are constant in this equation, this $\dot{q}_r$ is constant. It is not a function of r. So in the plane wall case we have seen that the rate of heat transfer remains constant. Similarly, in case of plane cylinder also as long as you are talking about 1D steady state heat conduction without any heat generation, rate of heat transfer is constant, but what about the heat flux.

In case of plane wall, we have seen that if there are no parallel resistor kind of case, then all the surfaces sharing the same area and accordingly the heat flux also remains constant. Only if we are talking about one layer being divided into parallel layers, we have a division of the heat flux and we have a division of the total power transmission according the value of heat flux may also change truly; but I am leaving this question open to you.

Do you really feel the value of heat flux will also change in that case? Value of power definitely changes but what about heat flux, just think on that, but that is not true for plane cylinder cases. In case of plane cylinder as you are moving from one layer to the another layer, the heat flux in the radial direction should be equal to

$$\dot{q}"_r = \frac{\dot{q}_r}{2\pi rL} = f(r)$$

So the value of heat flux that keeps on changing at every location, but the power remains the same. That is because of the change in the peripheral area. Now let us continue with this one. So we get

$$\frac{dT}{dr} = \frac{C_1}{Kr}$$

Now if the value of temperature has to be finite at r = 0, that is for T to have some finite value, because if you put r =0 here, you are getting dT/dr to be infinity at r = 0, but that is not possible. Therefore, dT/dr in order to for the T to have some finite value, then this C_1 has to be equal to 0. So we get

$$\frac{dT}{dr} = 0$$

But generally we do not need to analyze cylinder from this way. Rather we are more interested in hollow cylinders.

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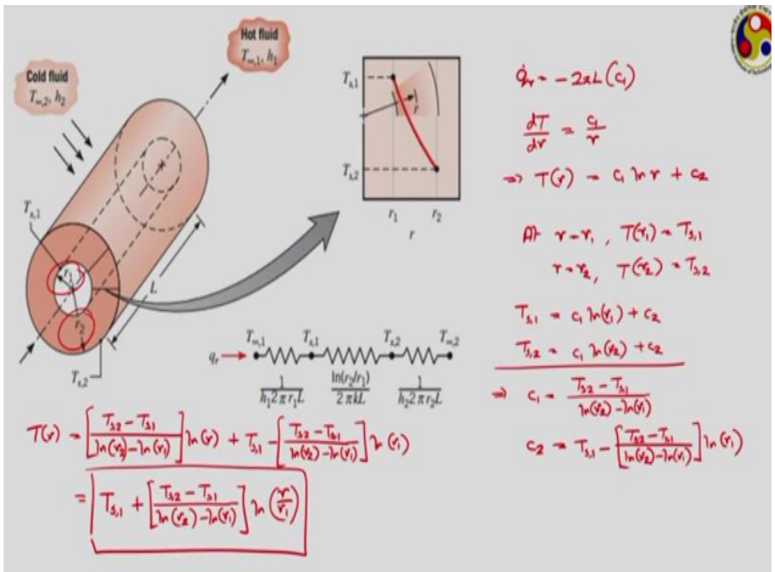

Just like the configuration shown here. Here we are talking about an hollow cylinder or an annular section where the inner radius is r_1, outer radius is r_2 and we are going to now analyze for this particular one. So we have just seen that

$$\dot{q}_r = -2\pi L(C_1)$$

And

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Here actually K also I am considering as a part of C_1. So accordingly T if we integrate this one is going to be

$$T(r) = C_1 lnr + C_2$$

So if we are talking about a solid cylinder, then $C_1 = 0$, but if you are talking about a hollow cylinder like the one shown here, then of course C_1 and C_2 both will be present there. So to get the values say

$$at\ r = r_1, T = T_{s,1}$$

$$at\ r = r_2, T = T_{s,2}$$

So putting the numbers,

$$T_{s,1} = C_1 lnr_1 + C_2$$

$$T_{s,2} = C_1 lnr_2 + C_2$$

Combining these two, we can write that

$$C_1 = \frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}$$

And

$$C_2 = T_{s,1} - \left[\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right] lnr_1$$

So the expressions for C_1 and C_2 both we have now. Let us take it back to the expression for T(r) that we had.

$$T(r) = \left[\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right] lnr + T_{s,1} - \left[\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right] lnr_1$$

$$= T_{s,1} + \left[\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right] \ln\left(\frac{r}{r_1}\right)$$

So you get a solution for the temperature distribution within the cylinder. But now can we make use of the thermal resistance concept here? Of course we can. For that let me erase these things. For that I have to make use of the expression for that heat flux or rather power that I had originally. So we have seen that the expression for $\dot{q}_r$ at any location can be given as

$$\dot{q}_r = -2\pi L(C_1)$$

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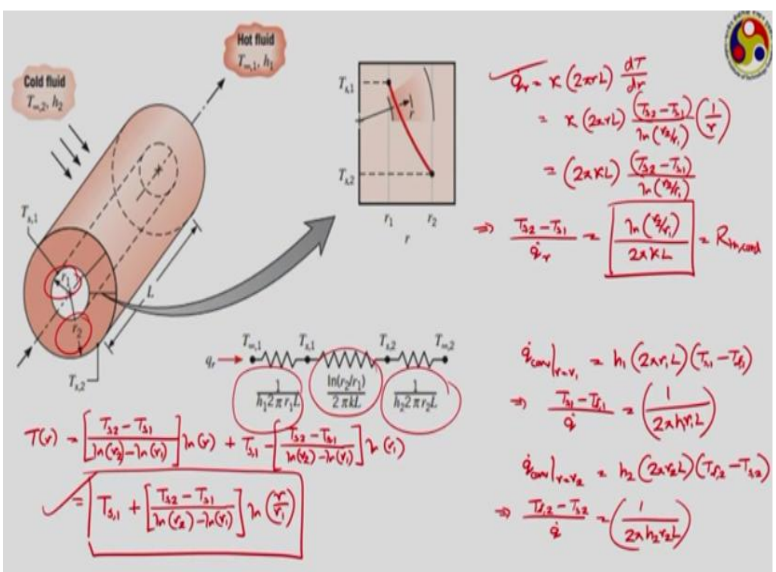

So putting that into use, we have

$$\dot{q}_r = -2\pi L\left(\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right)$$

If we rearrange them

$$\frac{T_{s,2} - T_{s,1}}{\dot{q}_r} = \left(\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi L}\right)$$

Another way we can do it just to get a better feel, just start from the basics, because this expression actually is not good representation. It is dependent on the C_1 expression.

Then $\dot{q}_r$ at any location will be

$$\dot{q}_r = -2\pi L\left(rK\frac{dT}{dr}\right)$$

$$= K(2\pi rL)\left[\left(\frac{T_{s,2} - T_{s,1}}{lnr_2 - lnr_1}\right)\left(\frac{1}{r}\right)\right]$$

The expression of dT/dr you can get by differentiating T(r) expression with respect to r.

$$= (2\pi KL)\left[\left(\frac{T_{s,2} - T_{s,1}}{ln\, r_2 - lnr_1}\right)\right]$$

Why I am repeating the same thing? I did it earlier but I erased and I have done it again because the expression for C_1 that we have used the K was included there, which gave a wrong expression actually.

That is why it is always better to be starting from the basic principles and which is at this form of the heat flux. Now if we rearrange it again

$$\frac{T_{s,2} - T_{s,1}}{\dot{q}_r} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi KL} = R_{th,cond}$$

This is the thermal resistance that you are getting associated with the conduction in a plane cylinder. So the nature of the expression is definitely different for plane walls, you got just L/KA, very straightforward expression. Here we are getting a logarithmic expression involving both radius r_1 and $r_{2,}$ the length of the cylinder L and also a thermal conductivity of the corresponding material K. What will be the convective resistance on any one of the surface? The rate of convective heat transfer at r equal to r_1, how much will be that. If h_1 is the corresponding heat transfer coefficient, it will be

$$\dot{q}_{conv}|_{r=r_1} = h_1(2\pi r_1 L)\left(T_{s,1} - T_{\infty.1}\right)$$

Rearranging

$$\frac{T_{s,1} - T_{\infty,1}}{\dot{q}} = \frac{1}{2\pi r_1 h_1 L}$$

This is the convective thermal resistance. Similarly, at the outer surface

$$\dot{q}_{conv}|_{r=r_2} = h_1(2\pi r_2 L)\left(T_{s,2} - T_{\infty,2}\right)$$

$$\Rightarrow \frac{T_{s,2} - T_{\infty,2}}{\dot{q}} = \frac{1}{2\pi r_2 h_2 L}$$

So we can see that the system that is shown in this case a hollow cylinder with a fluid passing through the inner surface or passing to the inner hole and another fluid flowing over the outer surface can be represented as a combination of 3 resistances. This is the corresponding conduction resistance $\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi KL}$ and this is the convective thermal resistance on the inner side $\frac{1}{2\pi r_1 h_1 L}$. This is the convective thermal resistance on the outer side $\frac{1}{2\pi r_2 h_2 L}$.

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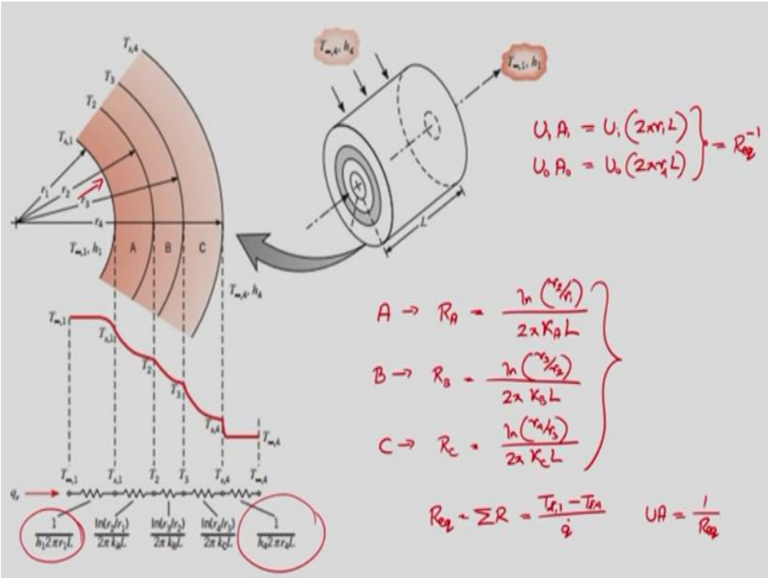

The same thing can easily be extended to the composite cylinder. So when the cylinder is having multiple layers, look at them. There are three layers plus there are two convections on inner and outer surfaces. So for the layer A, we can write that the resistance corresponding to layer A, it has an inner radius of r_1 and outer radius of r2. So it will be equal to

$$R_A = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi K_A L}$$

For layer B,

$$R_B = \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi K_B L}$$

Similarly, for material C

$$R_C = \frac{\ln\left(\frac{r_4}{r_3}\right)}{2\pi K_C L}$$

These are the three conduction resistances. On the inner side where the radius is r_1 we are having convective heat transfer characterized by the coefficient h_1. This is the corresponding convective heat transfer coefficient, convective thermal resistance.

These are convective thermal resistance on the outer side. So we have a combination of five resistances. So we can easily calculate R equivalent as a combination of these five resistances and that can be expressed as the ultimate temperature difference, that is

$$R_{eq} = \sum R = \frac{T_{\infty,1} - T_{\infty,2}}{\dot{q}} = \frac{1}{UA}$$

Now you can see here while the R_{eq} is same for all the surfaces, the value of U can keep on changing, because the value of area also changes.

Like if we want to define the value of this overall heat transfer coefficient corresponding to the inner surface this one, then the value of corresponding area like U corresponding the inner surface into A corresponding to the inner surface, if we want to define this way, then

$$U_i A_i = U_i(2\pi r_1 L)$$

Whereas if we want to define the overall heat transfer coefficient corresponding the extreme outer surface

$$U_o A_o = U_o(2\pi r_4 L)$$

But,

$$U_i A_i = U_o A_o = R_{eq}^{-1}$$

So as r_1 and r_4 are different according to the value of U_I and U_O also will be different. U_I will be larger than the value of U_O. So overall heat transfer coefficient may have different magnitude depending on your choice of the area. It is not that much relevant for plane walls, but when you are into the cylindrical coordinate, then this is very much important.

I shall be explaining a bit further on this curvilinear coordinate system in the next lecture. We shall be solving numerical problems involving composite cylinders and also we shall be dealing the situation of spheres and we shall be calculating the thermal resistance corresponding to the spherical coordinate system. So today we have discussed about the concept of contact resistance and we have learned an alternative way of using the conduction heat transfer analysis.

When the thermal conductivity and cross section area are not constants and then we have discussed about the heat conduction with plane and composite cylinders.

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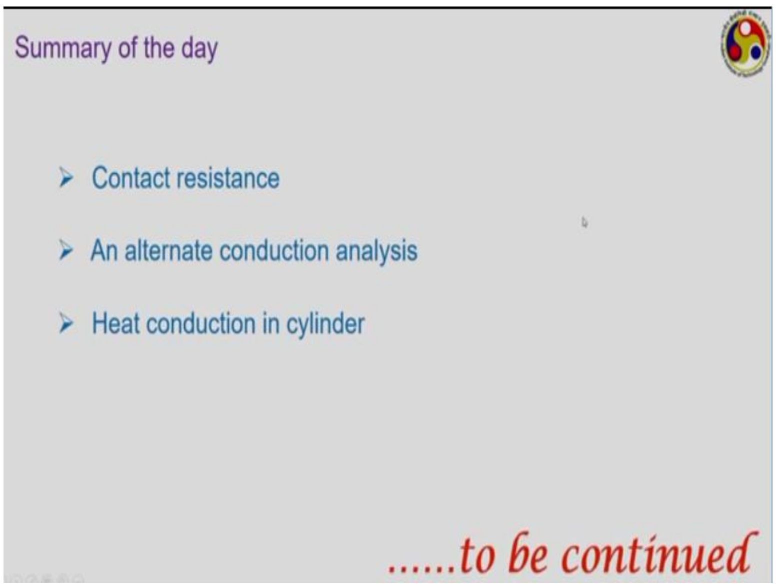

So that is it for the day. I shall be continuing tomorrow with the next part of this lecture. Thank you very much.