Fundamentals of Conduction and Radiation

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Lecture - 05
Heat Diffusion Equation in Curvilinear Coordinates

Hello friends. Welcome to lecture number 3 of our second week where we are developing the fundamental equations for conduction. In the first lecture, you were introduced to the concept of thermal conductivity and thermal diffusivity and then in the previous lecture we develop the generalized form of the heat diffusion equation in Cartesian coordinate and today we shall be continuing with that to develop the equations in the curvilinear coordinate systems.

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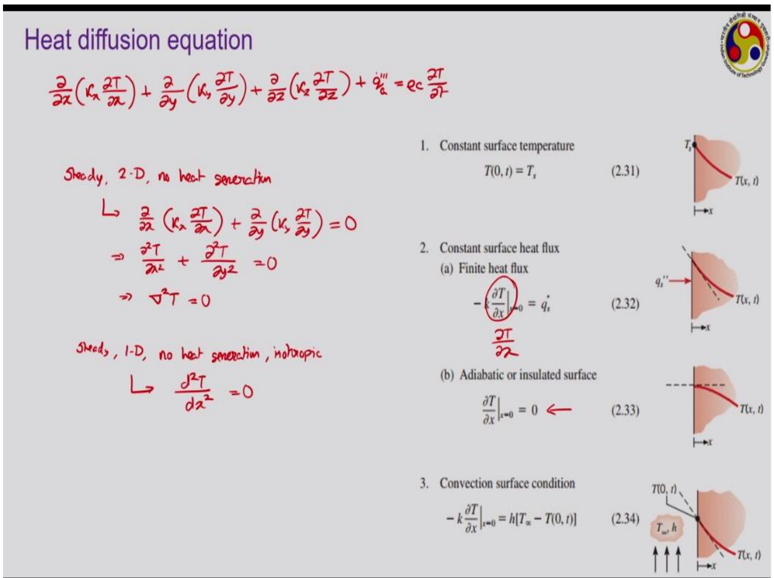

But before we do that just a recap of whatever we have done in the previous lecture. We have developed the generalized heat diffusion equation or generalized heat conduction equation; I mean assuming conduction as the only mode of heat transfer inside some kind of medium; we have developed the generalized energy transmission equation of a form like this.

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial T}{\partial z}\right)+\dot{q}'''_G = [\rho c]\frac{\partial T}{\partial t}$$

Quite often instead of c we can write also C_p, this refers to specific heat. For solids and liquids, we know that there is only one specific heat that to be considered but for gases we have to consider two different specific heats.

So, you can write this one as C_p but I shall be sticking with the notation just c only to denote this one to be the specific heat of importance. So, we already know the meaning of this 3 terms or rather the 5 terms involved there. The first 3 terms of the left hand side corresponds to the heat diffusion in all the 3 coordinate directions x, y and z. Here K_x K_y and K_z refer to the thermal conductivity in different directions.

If we are talking about an isotropic material then K_x K_y and K_z are equal to each other and thankfully most of the materials that we deal with are isotropic in nature but there may be certain exceptions like the example of graφte I have mentioned, its thermal conductivity along a particular layer and perpendicular to the layer they are widely different.

The fourth term $\dot{q}'''_G$ as I just mentioned, is a volumetric energy generation rate in certain limited cases like nuclear materials etc we may have this term of importance. But in most cases this is not present at all. And the last term, that is the term on the right hand side refers to the rate of change in energy content or time rate of change of energy content of the medium.

For steady state situation, this term = 0. And there are different boundary conditions also that I have mentioned. We may have broadly 3 kinds of boundary conditions. I repeat the diagrams that I am showing here. These diagrams I have taken from the book of Incropera and DeWitt as which is the primary reference book for this particular course.

So, the first kind of boundary condition where we have value of the temperature is specified which is called the Dirichlet boundary condition or boundary condition of the first kind. Second is the constant surface heat flux boundary condition, which is also referred as Neumann boundary condition where practically we have the temperature gradient. This$\frac{\partial T}{\partial x}$ that is specified, a special case of this one can be adiabatic boundary condition where temperature gradient = 0.

And a boundary condition of third kind can be convective or radiative boundary condition, where we go for a balance of conduction heat flux and convection or radiation or maybe both heat fluxes at a particular surface. So, depending upon what kind of problem you are dealing with we may have to deal with different kinds of boundary condition and also the 3 boundary conditions shown here.

But if you are solving a transient problem, we also need the initial condition that is we need to know the value of temperature at all the points or throughout the domain at time t=0, that is the time where you are starting your calculation and there may be several simplified form of the heat diffusion equation that we can have like a particular one of interest if we are dealing with a steady two-dimensional and no heat generation situation.

In that case, let us say z is the component that is not of importance. Then, in that case, the equation reduces to a simple form like this

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right) = 0$$

And if you are talking about an isotropic material that is K_x=K_y then an even simpler form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

or you can say the Laplacian of temperature = 0

$$\nabla^2 T = 0$$

This is the Laplace equation. Of course, we can get the Laplacian or Laplace equation in three-dimensional form also.

If we are talking about steady state three-dimensional heat conduction situation without any heat generation and for isotropic material, then we shall be getting a three-dimensional form of the Laplace equation. And if we are talking about steady one-dimensional, no heat generation and isotropic, then it becomes even simpler, then your equation is just an ordinary differential equation of this form.

$$\frac{d^2 T}{dx^2} = 0$$

So, depending upon the order of the equation, we need multiple boundary conditions and these boundary conditions are of the 3 types that we have mentioned. Now, the Cartesian form of heat conduction equation or heat diffusion equation is very much applicable whenever you are dealing with a system where rectangular coordinate system is applicable;

like heat conduction through a block, maybe heat conduction through the walls of your room, heat conduction through a large cube of ice something like that or a long metallic bar etc.

But there may be several situations where we have to go for a curvilinear coordinate system; like suppose you are having a cylindrical bar or a cylindrical rod and you want to estimate the heat conduction through that. Now, because of the shape the rectilinear coordinate system or rectangular coordinate system rather, is not applicable there and we have to go for the cylindrical coordinate system.

And therefore, we need a modification in the basic heat diffusion equation that we have developed here. And that is why let us check out the heat diffusion equation in cylindrical and later on in spherical coordinate system.

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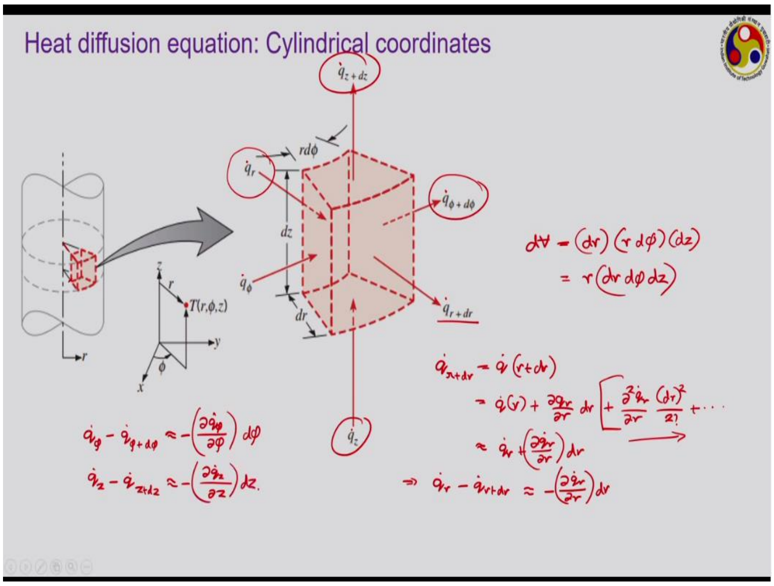

So, first we have the cylindrical coordinate system. This is the problem statement that is given to us. We are having a cylinder just like shown here and we want to estimate or we want to develop the heat diffusion equation inside the cylinder. So, just the same way we have done for rectangular coordinate system from the given domain, we have assumed an infinitesimally small volume.

Similarly, here also an infinitesimally small volume within the cylinder that has been identified and the zoomed view of that one is shown on my right. Now, the coordinate system is important, look at the coordinate system that is shown here; x, y, z is a traditional Cartesian coordinate system but here we are going for r, φ, z. Here z is the same because the z coordinate direction remains unaltered.

If we take the projection of a point one on the x-y plane, then r refers to the distance of this projected point from the origin of the coordinate system on this x-y plane. Like in this particular diagram r is shown here but it could have been shown here also. r refers to this vector which starts from here and finishes off here that is; it is entirely on the x-y plane.

And φ is the angle this r is making with the x-coordinate direction. So, now we have a zoomed view of this particular volume that we are talking about. Let us try to identify the dimensions of this one. For rectangular coordinate system, it is very easy. In this case, we have to be careful, it is not at all difficult but just carefully observe all the terms and try to relate or try to express the length of each of the faces or each of the edges in terms of r, φ and z.

We now know the definitions of r, φ and z. I repeat again here z refers to like, if this is the point then if we take a projection of that one on the x-y plane, then we arrive at this particular point. Then, z refers to the vertical distance between the original point and the projection that is this particular distance is your z, which is parallel to the z-coordinate direction and r refers to this particular vector.

Again, r refers to the distance of this projected point from the origin on the x-y plane and φ is the angle this position is making or this r is making with the x-coordinate direction. So, for the infinitesimally small volume, let us say this position refers to the radius r, so it starts from point r and let us say it finishes at point r+dr. Accordingly, this dimension is dr. Here the dimension is r, here the dimension is r+dr, accordingly your length of this edge is dr.

Then, how can we identify the others? It is straight forward we can easily say this one = dz or rather it is shown on the other side, this one = dz because this is parallel to the z-coordinate direction and we have to identify now this particular one. We know that this particular length is r and then we are having an arc of radius r that is traversing an angle of dφ in the azimuthal direction.

So, the length of this arc can be equal to rdφ, assuming φ to be sufficiently small. So, accordingly we are getting the 3 edges, this one is dr, this is dz and this is rdφ. Then, the volume, say d~~V~~, I am crossing this ~~V~~ to indicate that I am referring to volume here. v we shall be using for velocity if required. So, it will be equal to assuming all these dimensions dr, dz and d φ to be sufficiently small.

Then, we can write its volume to be nearly equal to dr in one coordinate direction, then rdφ in the other coordinate direction and dz in the third coordinate direction. So, accordingly we have the volume

$$d\cancel{V} = (dr)(rd\varphi)(dz)$$
$$= rdrd\varphi dz$$

This is the volume of this infinitesimally small block. Now, look at the directions, here the plane which is having its normal in the r direction that is a plane which is having its normal in the r direction that is this particular one is being subjected to an heat flux of $\dot{q}_r$, not heat flux rather heat transfer rate of $\dot{q}_r$.

Through its opposite face $\dot{q}_{r+dr}$is going out. The face having its normal in the z direction, it is having $\dot{q}_z$entering from this side, $\dot{q}_{z+dz}$leaving from the other side and the third face, third pair of faces this $\dot{q}_\varphi$ is entering from this and $\dot{q}_{\varphi+d\varphi}$ leaving out from this particular face. Now, following Taylor series $\dot{q}_{r+dr}$can be expanded as

$$\dot{q}_{r+dr} = \dot{q}(r+dr)$$
$$= \dot{q}(r) + \frac{\partial \dot{q}_r}{\partial r}dr + \frac{\partial^2 \dot{q}_r}{\partial r}\frac{(dr)^2}{2!} + \cdots$$

If we neglect everything from second order onwards then this one can be approximated to be equal to

$$= \dot{q}_r + \frac{\partial \dot{q}_r}{\partial r}dr$$

Accordingly

$$\dot{q}_r - \dot{q}_{r+dr} \cong -\frac{\partial \dot{q}_r}{\partial r}dr$$

Just the same way following similar methodology we have already done the same thing for in Cartesian coordinate system.

So, I can quickly write

$$\dot{q}_{\varphi} - \dot{q}_{\varphi+d\varphi} \cong -\frac{\partial \dot{q}_{\varphi}}{\partial \varphi} d\varphi$$

$$\dot{q}_{z} - \dot{q}_{z+dz} \cong -\frac{\partial \dot{q}_{z}}{\partial z} dz$$

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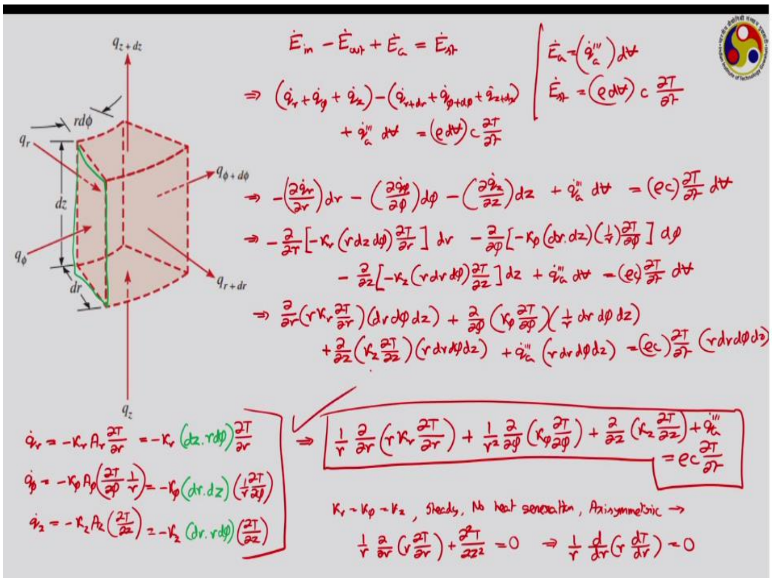

Now, we have to write an energy balance over this control volume. So, following the same procedure we are writing an simple energy balance that is

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{g} = \dot{E}_{st}$$

Here we are talking about rate. This is one important difference with thermodynamics. In thermodynamics, we always write them in general form. That is energy coming in minus energy going out plus energy generation equal to change of energy content.

But here we are writing everything in rate form that is the rate of energy coming in minus rate of energy going out plus rate of energy generation = rate of time rate of change of stored energy content. Now, rate of energy generation is not shown here in this diagram but we shall be following the procedure that what we have done earlier also. Accordingly energy generation how we can write it?

We can write this to be equal to

$$\dot{E}_{g} = (\dot{q}'''_{G}) d\forall$$

Similarly, change in stored energy content can be written as

$$\dot{E}_{st} = (\rho d\forall) c \frac{\partial T}{\partial t}$$

So, putting everything together, rate of energy coming in through the 6 faces

$$\left(\dot{q}_r + \dot{q}_\varphi + \dot{q}_z\right) - \left(\dot{q}_{r+dr}\dot{q}_{\varphi+d\varphi}\dot{q}_{z+dz}\right) + \left(\dot{q}'''_G\right)d\forall = (\rho d\forall)c\frac{\partial T}{\partial t}$$

So, now we can combine the expressions that we have written here,

$$\Rightarrow -\frac{\partial \dot{q}_r}{\partial r}dr - \frac{\partial \dot{q}_\varphi}{\partial \varphi}d\varphi - \frac{\partial \dot{q}_z}{\partial z}dz + \left(\dot{q}'''_G\right)d\forall = (\rho c)\frac{\partial T}{\partial t}d\forall$$

Here, there are 3 primes ($\dot{q}'''_G$), please be careful while writing because we are talking about per unit volume. You can easily expand $d\forall$ also; I am just not expanding that to save some space. Now, we need to know about the 3 components, $\dot{q}_r$ $\dot{q}_\varphi$ $\dot{q}_z$, or we have to relate them using the Fourier's law of heat conduction with the temperature gradient in the corresponding direction.

Then, what we can write? What will be your$\dot{q}_r$? $\dot{q}_r$ as per the Fourier's law of heat conduction may will be equal to

$$\dot{q}_r = -K_r A_r \frac{\partial T}{\partial r}$$

What will be q dot φ? That will be

$$\dot{q}_\varphi = -K_\varphi A_\varphi \left(\frac{\partial T}{\partial \varphi}\frac{1}{r}\right)$$

Else it is dimensionally not consistent otherwise. Of course, we can easily get this one from variable conversion. But one thing you can immediately identify φ being an angle, it is dimensionless, because its unit is radian in SI system. But you must have one per unit length scale to be dimensionally consistent and that is why this $\frac{1}{r}$ coming into picture. Another way of visualizing this one is whenever you are writing this gradient, in the numerator it will always be ∂T but in the denominator it will be the length scale associated with that particular direction.

Like in the r direction, the length scale was dr. Accordingly we are writing ∂r. In the φ direction, the length scale was rdφ, accordingly we are writing this $r\partial\varphi$ and similarly in the z direction, the dimension is dz only, so it is much easier to write

$$\dot{q}_z = -K_z A_z \frac{\partial T}{\partial z}$$

Now, we have to write the areas as well. Ar refers to the area which is having its normal in the r direction. The one having normal in the r direction that is actually on the backside. So, you can clearly identify the dimensions of this. You have dz on one side and rdφ on the other side.

Similarly, if we want to express the area in the φ direction, then for φ direction, this is the one having its normal in the φ direction. So, this area in this case will be equal to $dr \times dz$ and finally the area having normal in the z direction, how can you estimate that? So, this is the area that I am talking about, so what is dimension? You have dr on one side and rdφ on the other side. So, now let me complete the expressions.

$$\dot{q}_r = -K_r A_r \frac{\partial T}{\partial r} = -K_r (dz.rd\varphi) \frac{\partial T}{\partial r}$$

$$\dot{q}_r = -K_\varphi A_\varphi \left(\frac{\partial T}{\partial \varphi}\frac{1}{r}\right) = -K_\varphi (dr.dz) \left(\frac{\partial T}{\partial \varphi}\frac{1}{r}\right)$$

$$\dot{q}_z = -K_z A_z \frac{\partial T}{\partial z} = -K_z (dr.rd\varphi) \frac{\partial T}{\partial z}$$

I am putting it back here in the original expression.

$$\Rightarrow -\frac{\partial}{\partial r}\left[-K_r (dz.rd\varphi)\frac{\partial T}{\partial r}\right] dr - \frac{\partial}{\partial \varphi}\left[-K_\varphi (dr.dz)\left(\frac{\partial T}{\partial \varphi}\frac{1}{r}\right)\right] d\varphi - \frac{\partial}{\partial z}\left[-K_z (dr.rd\varphi)\ \frac{\partial T}{\partial z}\right] dz$$
$$+ \left(\dot{q}'''_G\right) d\forall = (\rho c)\frac{\partial T}{\partial t} d\forall$$

So, now we can easily see which terms we can take out from each of the differentials. The terms which are independent of the differentials like this dz and dφ are independent of r in the first term. And all the minus signs we can cancel out. Here we can put the expression of $d\forall$ too. So it remains

$$\Rightarrow \frac{\partial}{\partial r}\left[rK_r \frac{\partial T}{\partial r}\right] drd\varphi dz + \frac{\partial}{\partial \varphi}\left[K_\varphi \left(\frac{\partial T}{\partial \varphi}\right)\right]\frac{1}{r} drd\varphi dz + \frac{\partial}{\partial z}\left[K_z\ \frac{\partial T}{\partial z}\right] rdrd\varphi dz$$
$$+ \left(\dot{q}'''_G\right) rdrd\varphi dz = (\rho c)\frac{\partial T}{\partial t} rdrd\varphi dz$$

Now, let us divide the entire equation by this $d\forall$ that is rdrdφdz. Then, what we are left with? We are left with let me write in a proper way

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left[rK_r \frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left[K_\varphi \left(\frac{\partial T}{\partial \varphi}\right)\right] + \frac{\partial}{\partial z}\left[K_z\ \frac{\partial T}{\partial z}\right] + \left(\dot{q}'''_G\right) = (\rho c)\frac{\partial T}{\partial t}$$

This is the generalized heat diffusion equation in cylindrical coordinate system. Just carefully look at the 3 terms and try to compare with the one that we have developed in the previous class in relation with the Cartesian coordinate system. Here the first term refers to the heat conduction and heat diffusion in the radial direction, the second term refers to the heat diffusion in the azimuthal direction and the third one refers that same in axial direction. Then, we have the traditional heat generation term and the transient term or time rate of change of energy content of the system. And as we have divided everything by volume, so each of the terms basically having dimension of per unit volume or rather heat transfer rate per unit volume. In special cases of isotropic materials, we have

$$K_r = K_\varphi = K_z = K$$

So, if we assume an isotropic material plus steady state and no heat generation, then this equation leads to much simpler form.

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

And another even simpler form and also very well used form that is when we are talking about an axisymmetric material or axisymmetric system. Axisymmetric refers to the variation in the φ direction; variation is the azimuthal direction can be neglected.

In that case, that $\frac{\partial}{\partial \varphi}$ part that goes off. We have variation only in r and z direction.

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + \frac{\partial^2 T}{\partial z^2} = 0$$

And this particular form of heat diffusion in cylindrical coordinate system is very much well used because quite often we can neglect the variation in the φ direction, we just have to consider the variation in the r and z direction.

And we can reach even simpler situation where we are talking about purely one dimensional heat conduction situation and the primary direction being the radial one, in that situation even the z direction is also not there and your equation is even simpler. Then we are talking about temperature variation only in the r direction. So, in that case, it will be

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] = 0$$

As we are talking about one-dimensional transfer, so from partial derivative notation we can easily go to the ordinary derivative notation. So, this is the generalized form of heat diffusion equation in the cylindrical coordinate system. Please try to derive this one just following the method that I have discussed here.

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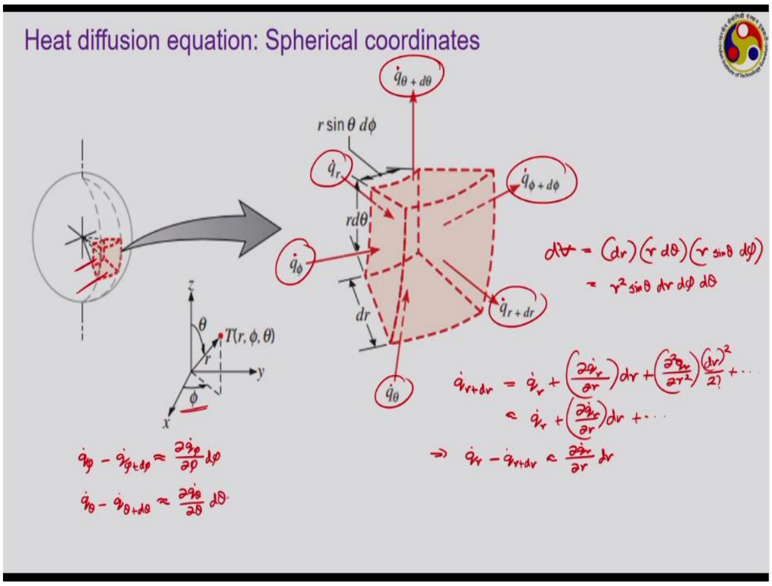

Now, let us try to repeat the same procedure in spherical coordinate system. The spherical coordinate system, you may encounter when we are dealing with a spherical kind of object, something like say the full storage tank of a space shuttle etc., where we generally have a spherical kind of vessel to store the fuels and a few other scenarios also you may encounter while solving numerical problems later on.

So, here we are talking about a sphere and we have got infinitesimally small volume of that, look at the coordinate system again, now the standard x, y, z coordinates system is there and here we are changing a bit from there, from the cylindrical one. So if this is your point, then here r refers to the radial position of the point. So, r is just direct distance of the point itself from the origin.

Now, once we are taking the projection of this r on the x-y plane that is I am talking about this particular projection which is the projection of r on the x-y plane. Then, the angle it is making with the x-coordinate direction is called φ and the angle r itself is making directly with z is called θ or we are denoting it as θ. Here φ is the azimuthal direction, θ is the polar direction.

So, r refers to the radial position of this one that is the distance of the point from the centre of the coordinate system. θ refers to the angle this radius is making with the z direction and φ refers to quite similar to the cylindrical coordinate system. If we take a projection of this r on the x-y plane, then the angle it is making with x coordinate system is called φ. So, now let us try to identify the dimensions for this infinitesimally small volume.

If this position refers to r and this r+dr, then this dimension is quite straight forward. It is dr only because here this particular face or particular line is at a distance r, at radial location r from the center. This is at a distance of r+dr, so, you are having a distance of dr along this. Then, in the z direction the projection of r on the vertical plane or vertical dimension is rdθ.

And to get the final one, we have to talk about the projection of r on the x-y plane and the angle φ it is making with this from where we get this to be equal to rsinθdφ. So, these are the dimensions for this infinitesimally small control volume. Then, its volume should be equal to

$$d\forall = (dr)(rd\theta)(rsin\theta d\varphi)$$
$$= r^2 sin\theta dr d\varphi d\theta$$

Now, look at this, it has got 6 faces as usual. So through two of the faces that is the faces which are having normal in the r direction, it is having $\dot{q}_r$amount of heat is coming from this side and $\dot{q}_{r+dr}$ leaving from this side. Through the face is having normal in the φ direction, $\dot{q}_\varphi$ is coming to this face and $\dot{q}_{\varphi+d\varphi}$is the one leaving from the other face. And for the last two, $\dot{q}_\theta$is coming from the bottom face and $\dot{q}_{\theta+d\theta}$is leaving from the top face.

So, just following the Taylor series expansion, we can write

$$\dot{q}_{r+dr} = \dot{q}_r + \left(\frac{\partial \dot{q}_r}{\partial r}\right) dr + \left(\frac{\partial^2 \dot{q}_r}{\partial r^2}\right)\frac{(dr)^2}{2!} + \cdots$$
$$\Rightarrow \dot{q}_r - \dot{q}_{r+dr} = -\left(\frac{\partial \dot{q}_r}{\partial r}\right) dr$$

neglecting all terms from second order onwards we can write

$$\Rightarrow \dot{q}_r - \dot{q}_{r+dr} \cong -\left(\frac{\partial \dot{q}_r}{\partial r}\right) dr$$

Just the same way we can

$$\Rightarrow \dot{q}_\varphi - \dot{q}_{\varphi+d\varphi} \cong -\left(\frac{\partial \dot{q}_\varphi}{\partial \varphi}\right) d\varphi$$
$$\Rightarrow \dot{q}_\theta - \dot{q}_{\theta+d\theta} \cong -\left(\frac{\partial \dot{q}_\theta}{\partial \theta}\right) d\theta$$

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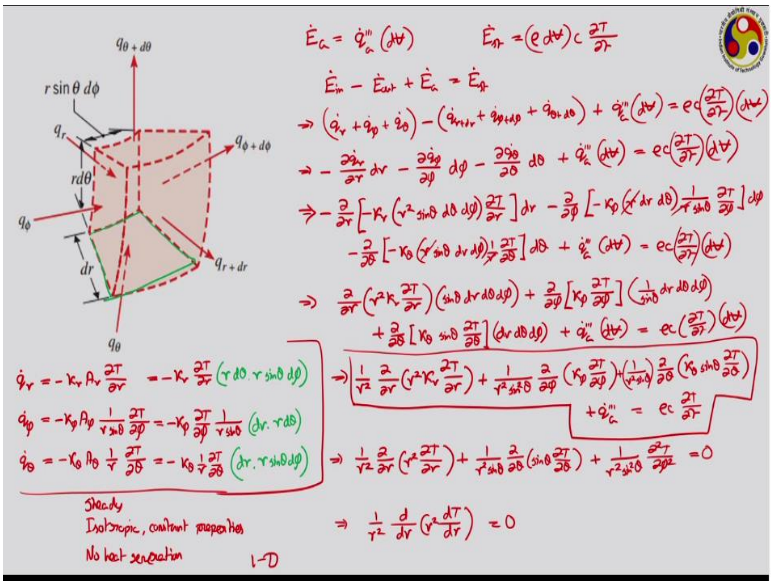

Now, we have to write the energy balance expression, so just following the earlier principle, the volumetric rate of energy generation can be represented as

$$\dot{E}_g = (\dot{q}'''_G)d\forall$$

And rate of energy storage can be written as

$$\dot{E}_{st} = (\rho d\forall)c\frac{\partial T}{\partial t}$$

So, we can write the energy balance that is

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

Putting the values of all the energies

$$(\dot{q}_r + \dot{q}_\varphi + \dot{q}_\theta) - (\dot{q}_{r+dr}\dot{q}_{\varphi+d\varphi}\dot{q}_{\theta+d\theta}) + (\dot{q}'''_G)d\forall = (\rho c)\frac{\partial T}{\partial t}d\forall$$

Then, using the expression that we wrote from here we can directly write

$$-\left(\frac{\partial \dot{q}_r}{\partial r}\right)dr - \left(\frac{\partial \dot{q}_\varphi}{\partial \varphi}\right)d\varphi - \left(\frac{\partial \dot{q}_\theta}{\partial \theta}\right)d\theta + (\dot{q}'''_G)d\forall = (\rho c)\frac{\partial T}{\partial t}d\forall$$

So, now we have to get this 3 transfer rates through each of the faces.

So, $\dot{q}_r$ will be equal to what? It will be equal to

$$\dot{q}_r = -K_r A_r \frac{\partial T}{\partial r}$$

But $\dot{q}_\varphi$, how we can write$\dot{q}_\varphi$? I mentioned you can get the full expression from the coordinate system conversion or just look at the dimension of the control volume in this particular direction. Like in the r direction, the dimension is only dr, so we have only dr in the denominator.

In the φ direction, what is the dimension? rsinθdφ. So, it will be equal to

$$\dot{q}_{\varphi} = -K_{\varphi} A_{\varphi} \frac{1}{rsin\theta} \frac{\partial T}{\partial \varphi}$$

Similarly, $\dot{q}_{\theta}$ will be equal to

$$\dot{q}_{\theta} = -K_{\theta} A_{\theta} \frac{1}{r} \frac{\partial T}{\partial \theta}$$

Now, you have to get the 3 areas. So, look at the areas, firstly the one having it's normal in the r direction. So, which one is this? This is the area that I am talking about. So, what will be the dimension of this area? Its dimension is rdθ in one direction and rsinθdφ in the other direction. Now the one having normal in the φ direction. So, this is the one having normal in the φ direction. So, you have dr on one side and rdθ on the other side. And final one is the one having normal in the θ direction. This is the area that I am talking about, so it is having dr on one side and rsinθdφ on the other side. So, we are putting it back here. Writing all exprssions

$$\dot{q}_{r} = -K_{r} A_{r} \frac{\partial T}{\partial r} = -K_{r} \frac{\partial T}{\partial r}(rd\theta . rsin\theta d\varphi)$$

$$\dot{q}_{\varphi} = -K_{\varphi} A_{\varphi} \frac{1}{rsin\theta} \frac{\partial T}{\partial \varphi} = -K_{\varphi} \frac{\partial T}{\partial \varphi} \frac{1}{rsin\theta}(dr . rd\theta)$$

$$\dot{q}_{\theta} = -K_{\theta} \frac{1}{r} \frac{\partial T}{\partial \theta}(dr . r\, sin\theta d\varphi)$$

Putting all in the previous expression

$$-\frac{\partial}{\partial r}\left[-K_{r} \frac{\partial T}{\partial r}(rd\theta . rsin\theta d\varphi)\right] dr - \frac{\partial}{\partial \varphi}\left[-K_{\varphi} \frac{\partial T}{\partial \varphi} \frac{1}{rsin\theta}(dr . rd\theta)\right] d\varphi$$
$$- \frac{\partial}{\partial \theta}\left[-K_{\theta} \frac{1}{r} \frac{\partial T}{\partial \theta}(dr . r\, sin\theta d\varphi)\right] d\theta + \left(\dot{q}'''_{G}\right) d\mathcal{V} = (\rho c) \frac{\partial T}{\partial t} d\mathcal{V}$$

Cancelling the same terms in numerator and denominator and cancelling the negative signs

$$\frac{\partial}{\partial r}\left[r^{2} K_{r} \frac{\partial T}{\partial r}\right](sin\theta dr d\theta d\varphi) + \frac{\partial}{\partial \varphi}\left[K_{\varphi} \frac{\partial T}{\partial \varphi}\right]\left(\frac{1}{sin\theta} dr d\varphi d\theta\right) + \frac{\partial}{\partial \theta}\left[K_{\theta} sin\theta \frac{\partial T}{\partial \theta}\right](dr d\theta d\varphi)$$
$$+ \left(\dot{q}'''_{G}\right) d\mathcal{V} = (\rho c) \frac{\partial T}{\partial t} d\mathcal{V}$$

Remember in the third term, if we want to take sin θ out, we have to differentiate, so better keep the sin θ inside. What is $d\mathcal{V}$ here? This is your $d\mathcal{V}$,

$$dV\!\!\!\!-\; = r^2 sin\theta dr d\varphi d\theta$$

So, instead of writing the form and making the expression even longer, I am just keeping it in $dV\!\!\!\!-$ form.

And in the next line, I am dividing everything by this $dV\!\!\!\!-$. So, if we divide by that then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 K_r \frac{\partial T}{\partial r}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \varphi}\left[K_\varphi \frac{\partial T}{\partial \varphi}\right] + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left[K_\theta sin\theta \frac{\partial T}{\partial \theta}\right] + \left(\dot{q}'''_G\right) = (\rho c)\frac{\partial T}{\partial t}$$

So this is the form of the generalized heat diffusion equation in spherical coordinate system. Of course, looks quite complicated but if you try to derive that just the way I have done, then you will get the hold of the entire thing. It will not be very difficult for this. Again, quite often we may have to deal with isotropic materials. So, if we try to develop a simpler form, let us put a few assumptions. So, let us write say steady then isotropic material and no heat generation. In that case, the K can be taken out of these brackets, let us stay isotropic and also constant properties.

In that case, $K_r = K_\varphi = K_\theta$ and we take them out of the differentials. So the equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial T}{\partial r}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left[sin\theta \frac{\partial T}{\partial \theta}\right] = 0$$

And if you are dealing with a 1-D or 2-D heat transfer situation, it can be even simpler, like if we are talking about just a plane 1-D heat transfer situation where radial direction is the one of importance then this becomes even simpler

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial T}{\partial r}\right] = 0$$

This kind of equation we shall be dealing in the next week and we shall be talking about purely 1-D heat conduction situation.

So, we have now developed the heat diffusion equation in all the 3 coordinate directions that is starting with the Cartesian we have developed for cylindrical and spherical coordinate system as well. And I hope you have got the procedure or at least if you have followed it carefully then you have got the procedure how we have developed this, but my request is try to derive the equations on your own so that you don't face any problem later on.

But at the same time, I should also mention that quite often at least as a part of the course, you don't have to deal with such big form of equations at least, you always will be using only 1-D or 2-D form of the equations. But still it is always good to know what the basic equation looks like. And that is what we are given here and the discussion in relation with the boundary condition that I have done for with regard to the Cartesian coordinate system, they are also valid here.

The same way we can easily imply the Dirichlet, Neumann or convective kind of boundary condition in cylindrical and spherical coordinate system as well. You just have to remember the expressions for this heat fluxes or powers, heat transfer rates; and like if we are talking about the Dirichlet boundary condition is very simple; at a given location the value of temperature is given.

If you are talking about Neumann boundary condition, then basically this power is given because the temperature gradient $\frac{\partial T}{\partial r}$, $\frac{\partial T}{\partial \varphi}$ or $\frac{\partial T}{\partial \theta}$ or may be this $\dot{q}_r$, $\dot{q}_\varphi$or $\dot{q}_\theta$ they are given. So, from there we can get the idea about how to solve the temperature or if you are talking about any convective boundary condition, then the power expression that we have written here that conduction heat transfer rate will be equal to the corresponding convection or radiative heat transfer rates.

So, accordingly we can apply any kind of boundary condition depending on whatever kind of problem we are dealing with. So, that takes me to the end of the second module.

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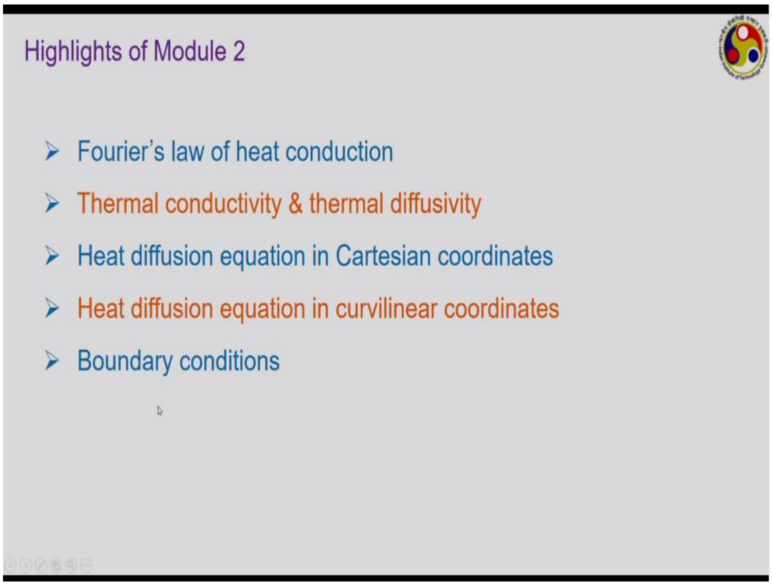

I was tempted to solve a few numerical problems here but I am not doing that because as we have developed only the generalized form of the conduction or heat diffusion equation corresponding to numerical problem may be quite difficult to solve as we have to deal with either very complicated equations or we may have to simplify the equations.

Now, in the next week, we shall be talking about 1-D heat conduction, where we shall be developing or I should say we shall be reducing these equations; all the 3 versions into their corresponding one-dimensional counterpart and therefore it will be much easier to deal with corresponding numerical problems. So, I shall be doing that only in the next week but in this week what we have done?

If we want to have a brief summary, we started with a generalized form of the Fourier's law of heat conduction, then talked in detail about thermal conductivity as a property and also introduced you to the thermal diffusivity, then we developed the heat diffusion equation in Cartesian coordinates. Today, we have developed the same equation in curvilinear coordinates both spherical and cylindrical coordinates.

And also we have discussed about 3 possible kinds of boundary condition that we can encounter while solving the heat diffusion equation. So, I would like to thank you for your attention while listening this. Please solve the assignment problems, try to solve a few problems from the book and also I request again try to develop the equations on your own so that you don't face any problem later on. Thank you very much.