Fundamentals of Conduction and Radiation
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Lecture - 04
Generalized Heat Diffusion Equation

Hello friends. How are you today? So in the previous lecture that is yesterday we discussed about the fundamentals of conduction and in this week we are looking to continue on that, that is the Fourier law of heat conduction that we have discussed I am going to develop on that and today we are going to discuss about the heat diffusion equation in Cartesian coordinate.

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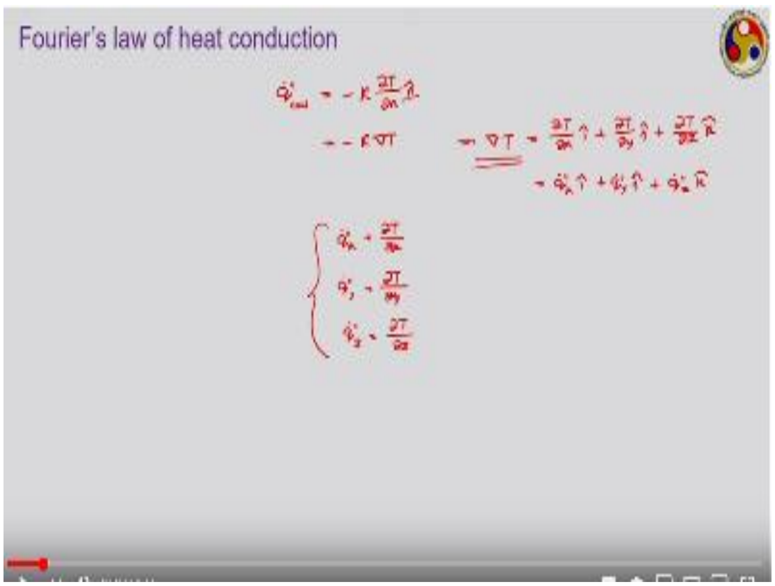

Now just to review the discussion on Fourier's law of heat conduction that we had in the first lecture, there we have seen that though we started with a very basic one-dimensional kind of form of Fourier's law of heat conduction but the most generalized representation of Fourier law of heat conduction is the conduction heat flux can be written as

$$\dot{q}"_{cond} = -K\frac{\partial T}{\partial n}\hat{n}$$

where this $\hat{n}$ represents the direction of normal to the surface and $\frac{\partial T}{\partial n}$ represent the temperature gradient in that particular direction. Or more general representation can be

$$= -K\nabla T$$

where this ∇T is the temperature gradient which can have all three possible coordinates like if we write in Cartesian coordinates it is

$$\nabla T = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}$$

Quite often we write this one as something like

$$= \dot{q}''_x\hat{\imath} + \dot{q}''_y\hat{\jmath} + \dot{q}''_z\hat{k}$$

Where

$$\dot{q}''_x = \frac{\partial T}{\partial x}$$

$$\dot{q}''_y = \frac{\partial T}{\partial y}$$

$$\dot{q}''_z = \frac{\partial T}{\partial z}$$

So these are the three components of heat fluxes which we can find and once we know the orientation of a particular surface that is once we know the direction of the normal of a particular surface then we can estimate the temperature gradient in each of the three directions and then combine to get the expression for this temperature gradient and multiplying that by the thermal conductivity we can get the expression for the corresponding conductive heat flux.

Actually the heat flux expression that I have written they are incomplete because all of them should be multiplied with the thermal conductivity and if we are talking about a non-isotropic material then the magnitude of thermal conductivity can also vary in all three possible directions and also another property that was introduced in the previous lecture and that will come back today that is the thermal diffusivity or α which is defined as the ratio of thermal conductivity and volumetric heat capacity that is $\rho * C_P$.

With this background with this background of Fourier's law of heat conduction let us try to develop something known as the heat diffusion equation.

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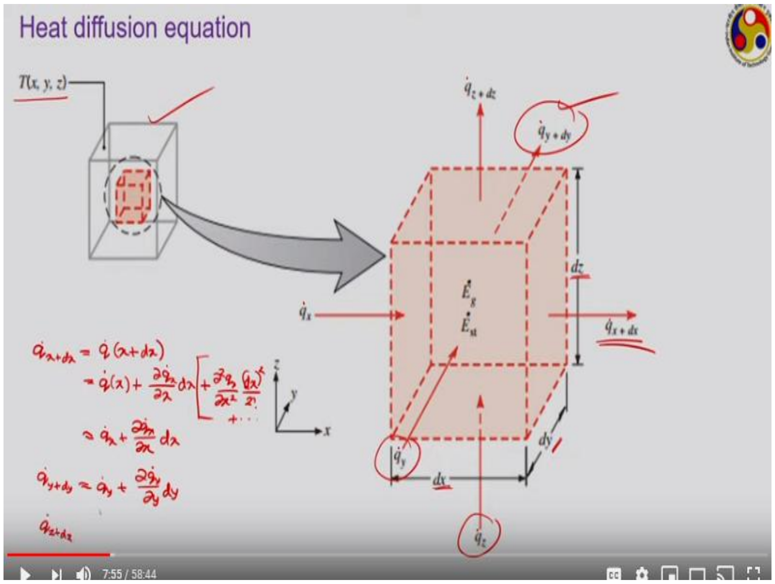

In this heat diffusion equation, I should make it very clear at the beginning that we are neglecting any kind of convective or radiative heat transfer. Just assume that say we are talking about the situation inside a solid, like this block shown here. We have a solid block just for the moment we assume that this is a solid block though the same situation prevails in liquid and gases also but just for ease of visualization we are assuming this one to be a solid block and we are talking about a point inside the block and we want to know the conductive heat fluxes that is acting on this block in all the three directions these are the coordinate directions XY and Z. We want to calculate the conductive heat fluxes that are acting at this particular point in all three directions and also you want to know the temperature distribution inside this body because of this conduction heat transfer.

So T is a temperature T (x, y, z) represents the temperature at this x, y, z location and we have a zoom view of this. Just assume that around that point we are assuming an infinitesimally small volume just shown here. It is an infinitesimally small volume which is having the face length in the x direction aligned with the x coordinate is dx the one aligned with the y coordinate is dy and the one aligned with the z direction or z coordinate is dz.

I have taken this particular diagram from the book of Incropera's and DeWitt where they have not put the dot notations but for our convenience or just to be consistent with our notation let me put the dot here. This $\dot{q}_x$ represents the conduction heat flux which is going in the x direction and is entering this block from the face that is shown. Through the opposite face of that block $\dot{q}"_{x+dx}$ heat flux is going out. Similarly through the plane which is aligned with the xz direction $\dot{q}_y$ amount of heat flux entering through this side and $\dot{q}"_{y+dy}$ leaving the block from that side. And considering the two faces which are parallel to the xy plane; $\dot{q}"_z$ conduction heat transfer rate passing into the block and $\dot{q}"_{z+dz}$ is leaving through this. We are assuming this block to be infinitesimally small so that each of the faces are also very small and therefore we are considering the conduction heat transfer through each of them only by the corresponding normal components.

Now $\dot{q}_{x+dx}$ is leaving through this side. dx being extremely small we can expand this one following Taylor series. If we expand this one following Taylor series we can write in a conventional functional form

$$\dot{q}_{x+dx} = \dot{q}(x + dx)$$

$$= \dot{q}(x) + \frac{\partial \dot{q}}{\partial x} dx + \frac{\partial^2 \dot{q}}{\partial x^2} \frac{dx^2}{2} + \cdots$$

So if we neglect the terms from second order on words assuming dx to be extremely small then this one approximately is

$$= \dot{q_x} + \frac{\partial \dot{q_x}}{\partial x} dx$$

The same way we can write

$$\dot{q}_{y+dy} = \dot{q_y} + \frac{\partial \dot{q_y}}{\partial y} dy$$

$$\dot{q}_{z+dz} = \dot{q_z} + \frac{\partial \dot{q_z}}{\partial z} dz$$

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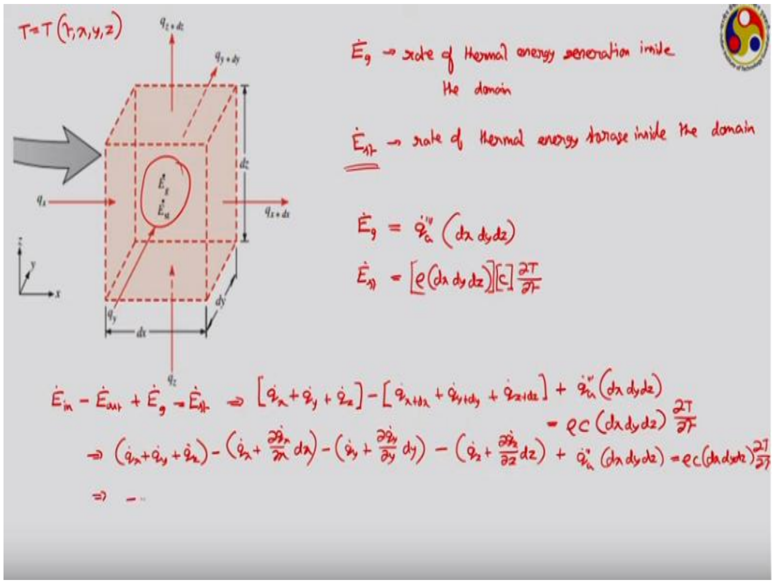

Now just look at there are two more terms that is shown on this figure. Here this $\dot{E}_g$ represents the rate of thermal energy generation inside this block per unit volume. Similarly, the other term $\dot{E}_{st}$ represents the rate of thermal energy storage per unit volume inside the domain or inside the control volume just to be consistent with the thermodynamic notation.

Because you can see there are six different heat transfer rates that we are considering as per the diagram shown here; three of them are entering the block, three of them are leaving the block and also there is an energy generation which can be present in certain limited cases like suppose if this block we are talking about is made up of some kind of nuclear fissionable material and some nuclear reaction is going on. Then definitely some energy will be generated. The energy released by fission reaction will be converted to thermal energy.

Similarly, all these 6 incoming and outgoing heat transfer rates plus the inner generation together if they are not properly balanced then that will lead to an a net increase or decrease in the total energy content of the block itself; that is given by this second term. Now if we put that triple prime notation to make it per unit volume

$$\dot{E}_g = \dot{q}'''_G(dxdydz)$$

where volume of the block is $(dxdydz)$

And from energy balance energy storage will be

$$\dot{E}_{st} = [\rho(dxdydz)][c]\frac{\partial T}{\partial t}$$

Where mass of the block is $\rho(dxdydz)$ where ρ represents the density. c is its specific heat and $\frac{\partial T}{\partial t}$ is time rate of change of its temperature. So at this particular node location whatever is the time rate of change of temperature that multiplied by the mass of the block into the specific heat of the block given by this c that gives you the rate of energy stored inside this because of the conduction heat transfer. I repeat again we are neglecting any convective or radiative heat transfer.

So now if we write an energy balance over this block following the first law of thermodynamics for a control volume we can easily write that the rate of energy coming in minus rate of energy going out plus rate of energy generation has to be equal to rate of energy storage for this.

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

If the three quantities on left hand side are properly balancing each other than the right hand side will be 0. That is there will be no change in the net energy content of this. Now if we write their expressions for each terms

$$\Rightarrow [\dot{q_x} + \dot{q_y} + \dot{q_z}] - [\dot{q}_{x+dx} + \dot{q}_{y+dy} + \dot{q}_{z+dz}] + \dot{q}'''_G(dxdydz) = [\rho(dxdydz)][c]\frac{\partial T}{\partial t}$$

Now if we put the expressions for the three components that we wrote in the previous slide then we have

$$\Rightarrow [\dot{q_x} + \dot{q_y} + \dot{q_z}] - \left[\left(\dot{q_x} + \frac{\partial \dot{q_x}}{\partial x}dx\right) + \left(\dot{q_y} + \frac{\partial \dot{q_y}}{\partial y}dy\right) + \left(\dot{q_z} + \frac{\partial \dot{q_z}}{\partial z}dz\right)\right] + \dot{q}'''_G(dxdydz)$$
$$= [\rho(dxdydz)][c]\frac{\partial T}{\partial t}$$

So if we balance them together then we have

$$\Rightarrow -\left(\frac{\partial \dot{q_x}}{\partial x}dx + \frac{\partial \dot{q_y}}{\partial y}dy + \frac{\partial \dot{q_z}}{\partial z}dz\right) + \dot{q}'''_G(dxdydz) = [\rho(dxdydz)][c]\frac{\partial T}{\partial t}$$

So here we have done an energy balance over this infinitesimally small control volume that we are considering subjected only to conduction heat transfer and from there we have got some intermediate expression.

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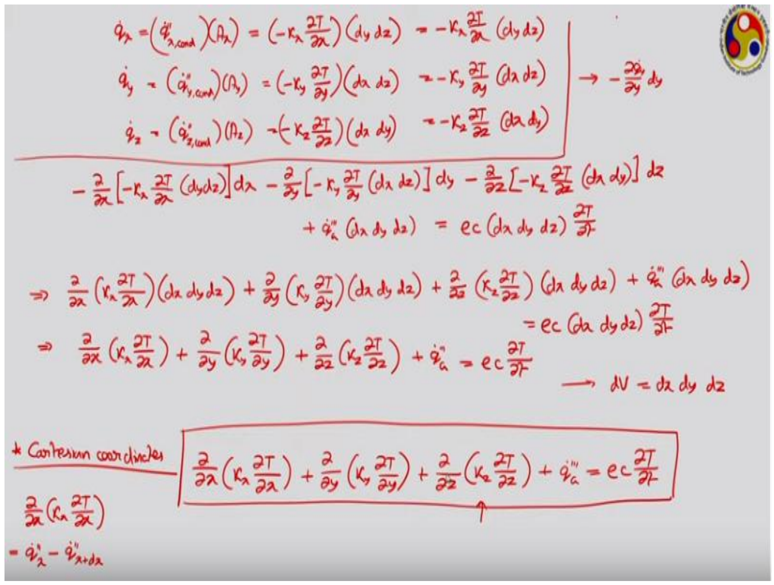

Now look at the three heat fluxes that are given to us. $\dot{q}_x$ is energy entering this block through this particular face via conduction only. Then using the Fourier's law of heat conduction we can write it to be the product of the corresponding conduction heat flux going in the x direction into the area of the plane having normal in the x direction.

$$\dot{q}_x = \left(\dot{q}''_{x,cond}\right)(A_x)$$

Here Ax represents the area which is having normal in the x direction. Now using the conduction equation what is the first term in the bracket the heat flux using Fourier's law of heat conduction and what is your Ax? It is the area which is aligned to the yz plane and having normal in the x direction. So this area is dy dz. So this becomes

$$= \left(-K_x \frac{\partial T}{\partial x}\right)(dydz)$$

$$= -K_x \frac{\partial T}{\partial x}(dydz)$$

Similarly in y and z direction,

$$\dot{q}_y = \left(\dot{q}''_{y,cond}\right)\left(A_y\right) = -K_y \frac{\partial T}{\partial y}(dxdz)$$

$$\dot{q}_z = \left(\dot{q}''_{z,cond}\right)(A_z) = -K_z \frac{\partial T}{\partial z}(dydz)$$

Now let us put these expressions back in previous equation

$$\Rightarrow -\left(\frac{\partial}{\partial x}\left(-K_x \frac{\partial T}{\partial x}(dydz)\right)dx + \frac{\partial}{\partial y}\left(-K_y \frac{\partial T}{\partial y}(dxdz)\right)dy + \frac{\partial}{\partial z}\left(-K_z \frac{\partial T}{\partial z}(dydz)\right)dz\right)$$
$$+ \dot{q}'''_G(dxdydz) = [\rho(dxdydz)][c]\frac{\partial T}{\partial t}$$

Now as we are assuming this block to be infinitesimally small then this dimensions dx dy and dz all can be assumed to be independent of each other, particularly if we are talking about a rigid block then these dimensions are not changing and they are all independent of each other that is this dx dy dz can come out of the corresponding differentials, also taking the minus signs out of this we can write

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right)(dxdydz) + \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right)(dxdydz) + \frac{\partial}{\partial z}\left(K_z \frac{\partial T}{\partial z}\right)(dxdydz) + \dot{q}'''_G(dxdydz)$$
$$= [\rho c(dxdydz)]\frac{\partial T}{\partial t}$$

So dxdydz represents the volume of this particular block that you are talking about and that can be cancelled out from this.

So we are getting the equation now as

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial T}{\partial z}\right) + \dot{q}'''_G = [\rho c]\frac{\partial T}{\partial t}$$

and this particular one can be considered as the generalized form of the heat diffusion equation where of course we have neglected in this equation dV represents the dxdydz that is the volume of this infinitesimally small block which cancels out.

So this is the generalized form of the heat diffusion equation or generalized form of the heat conduction equation where we have of course used the Fourier's law of heat conduction and we have neglected any kind of convective or radiative heat transfers but we have not put any other assumptions of course.

Nothing else has been assumed so far apart from the absence of any kind of convective and radiative heat transfer. Now what kind of equation is this? This equation you can see it is a partial differential equation which has both space and time dependences. It is a three-dimensional equation theK_x, $K_y$and $K_z$are the three components of the thermal conductivity for non-isotropic medium.

Only for isotropic cases they become equal to each other. One thing we have to remember that the equation that we have written that is in Cartesian coordinate system. So most of the system that we deal with they are generally Cartesian in nature or we can use a Cartesian coordinate system so this equation can always be used there in certain cases we may have to go for cylindrical coordinate or even limited number of cases we have to go for spherical coordinate system and those these two we shall be discussing in the next lecture.

Again I shall be developing the generalised diffusion equation for both of them. But for the moment, just continue with this heat diffusion equation in the Cartesian coordinates. So look at what terms we are getting. The first term that we have

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right)$$

From where have we got this one? If we just retrace back, the way we have derived this neglecting the area part then this basically is

$$= \dot{q}''_x - \dot{q}''_{x+dx}$$

Now the first term therefore represents the net conduction flux; similarly the second term represents the net conduction flux acting through the two faces having normal in the y direction and the third term represents the this term the net conduction flux that is entering the block through the areas having normal in the z direction. Then we have a volumetric heat generation. And this 4 together, 3 fluxes plus the volumetric energy generation together they give you the time rate of change of energy storage inside this block.

Now there are several situations when we may have can go for much simpler form. We hardly have to solve such a complicated equation.

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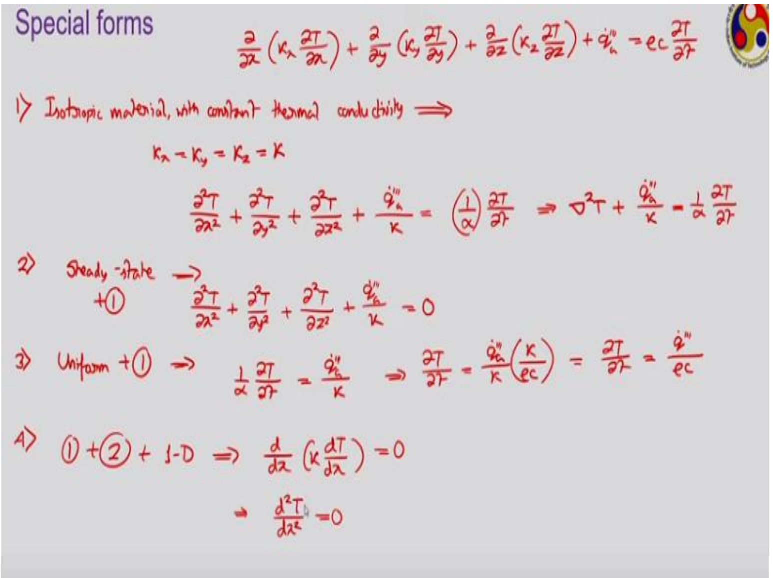

In most of the real life situations that we can deal with can have can offer several simplifications. Let us just repeat the generalized form and from that we shall try to see a few special forms or simpler forms. So the generalized heat diffusion equation was

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial T}{\partial z}\right)+\dot{q}'''_G = [\rho c]\frac{\partial T}{\partial t}$$

The first set of simplification that we can have if we are dealing with an isotropic material. Isotropic means the thermal conductivities direction independent that is your

$$K_x = K_y = K_z = K$$

Then we can drop the subscript and it becomes K.

As we have seen in the previous lecture thermal conductivity definitely varies with temperature but if the temperature range that we are concerned about is not very significant the variation of thermal conductivity can often be neglected. Then what happens. Now we are talking about an isotropic material and K is not having any kind of direction dependency and it is constant then you can take the K out of this then the equation now becomes just

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}'''_G}{K}=\frac{\rho c}{K}\frac{\partial T}{\partial t}$$

Now what is this $\frac{\rho c}{K}$. Its inverse $\frac{K}{\rho c}$ is the thermal diffusivity that we introduced in the previous lecture. So this is just $\frac{1}{\alpha}$ or reciprocal of thermal diffusivity in this situation. So this is simpler form

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}'''_G}{K}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

We can write this in a generalized form also that is

$$\nabla^2 T+\frac{\dot{q}'''_G}{K}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

That is one common representation. If we are dealing with a situation where we are bothered only about steady state,. What do you mean by steady state? You must have encountered this term in your fluid mechanics. Steady state refers to there is no time variation. So if we are talking about steady state equation for isotropic material with constant thermal conductivity then what happens the time variation goes off then your equation becomes just

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}'''_G}{K}=0$$

Of course we are considering steady state plus the assumptions of isotropic material with constant thermal conductivity. And if we have supposed unsteady situation but uniform that is no variation in the space coordinates. So if we assume uniform plus the assumption number one then what we are going to have.

$$\frac{\dot{q}'''_G}{K}=\frac{1}{\alpha}\frac{\partial T}{\partial t}\Rightarrow\frac{\partial T}{\partial t}=\frac{\dot{q}'''_G}{K}\left(\frac{K}{\rho c}\right)\Rightarrow\frac{\partial T}{\partial t}=\frac{\dot{q}'''_G}{\rho c}$$

This is possible when you are talking about a material having extremely high thermal conductivity so that whatever temperature gradient that is created at a certain location that immediately vanishes or the energy is immediately dispersed to the surrounding.

So that more or less an uniform temperature profile is maintained. However with time it keeps on varying then we can have this situation. And the simplest possible case that we can have where we are considering an isotropic material plus we are considering steady state plus we are considering one dimension.

Actually the consideration of steady state along with the condition 1 gives us a heat transfer situation with x is the only important direction and if K is independent of space then equation becomes something like this

$$\frac{d^2T}{dx^2} = 0$$

So it's a very simple second order ordinary differential equation which can be solved with suitable boundary conditions. So there are several special forms that we get. Though this is the generalized form of the heat diffusion equation but quite often you may not have to go for such complicated equation. We can have several kinds of simplifications and simpler forms.

So whatever scenario you are dealing with just carefully look what kind of simplifications you can do and accordingly simplify your equation and if you are lucky you may well end up with very simple equation something like this. Now whatever may be the situation you are dealing with either partial differential equations or even the simplest case of 1d steady state heat conduction problem or in a differential equation.

Now even when you are dealing with ordinary differential equation like this one, still it is a second-order equation and to solve this mathematically you need to have how many boundary conditions? Two boundary conditions are required. So let us quickly check the different kinds of boundary condition that you may encounter.

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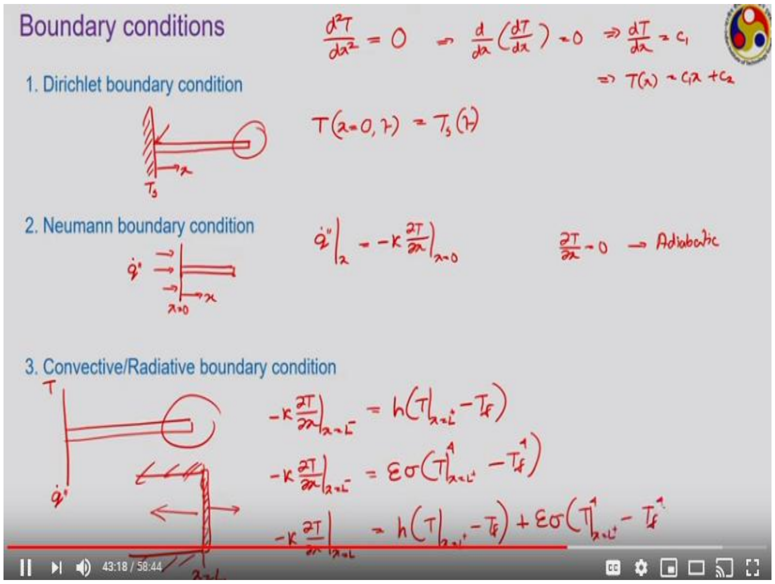

We take a very simple situation the last situation that we talked about that is one-dimensional steady state heat conduction through an isotropic material. Then your corresponding form of the heat diffusion equation reduces to

$$\frac{d^2T}{dx^2} = 0$$

Actually one assumption that I have not written that I should have added here also that is no heat generation.

Heat generation is absent in most of the situations. Very rarely we find a practical application of conduction, like the example of nuclear or fissionable materials that I have given there only can have this significant heat generation. Otherwise it can be neglected in most situations. So the equation that we have written here is one-dimensional steady state heat conduction equation with constant properties and no heat generation.

If we want to solve this we will need two boundary conditions. Because if you solve this let us write this as

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) = 0$$

So if you integrate it once with respect to x we are getting

$$\frac{dT}{dx} = C_1$$

Where, C_1 is a constant. If you integrate it once more you are going to get

$$T(x) = C_1 x + C_2$$

So there are two constants coming into picture C_1 and C_2 and therefore we need at least two boundary conditions.

So the first possible kind of boundary condition is known as a Dirichlet boundary condition or called the boundary condition of the first kind. Dirichlet boundary condition refers to when the temperature is specified; actually the term Dirichlet boundary condition is very general one and it is applicable to any kind of fluid flow situations.

It is a general the name Dirichlet boundary condition refers to whatever is the variable for which you are looking to get the solution the magnitude of the variable itself is specified at some boundary. Like you can say some surface temperature you are solving in a heat conduction problem. Let us take a problem like this. We have a surface maintained at some temperature T_s and we are assigning a long metal rod to this and we want to know the distribution of temperature through this particular rod.

I am not talking about the second boundary condition, but the boundary condition at this point can be a Dirichlet boundary condition because the magnitude of temperature is specified. So a common form of Dirichlet boundary condition can be written as T at location x equal to some reference value. If we start calculating x from here and assign this point as x equal to 0 then at x equal to 0 on time t is temperature is equal to T_s which can also be a function of time.

$$T(x = 0, t) = T_s(t)$$

So if Ts is not changing with time then this is of course a fixed boundary condition this kind of boundary condition is very easy to handle known as the Dirichlet boundary condition. The boundary condition of the second kind called the Neumann boundary condition. Neumann boundary condition refers when the temperature gradient is specified and not the value of temperature, at the location that we are looking to identify.

Like the same situation we have a surface and we are assigning a rod to this surface but this surface is receiving some $\dot{q}''$ amount of heat from certain source may be solar radiation or may be from an electrical heater or from some source it is getting this heat flux. Then how we can calculate what is specified? The temperature value at the surface is not given.

So this is our x direction and this position is x=0. At x=0 the value of temperature is not given but what is given is the heat flux and using conduction then we can write this

$$\dot{q}''\Big|_{x} = -K\frac{dT}{dx}\Big|_{x=0}$$

This form is called the Neumann boundary condition is also a very common boundary condition quite similar to a Dirichlet boundary condition.

Quite often you will find problems where you have Dirichlet boundary condition specified at one end and Neumann condition at the other end. A special case we can get where this

$$\frac{dT}{dx} = 0$$

That is we are talking about an adiabatic surface through which no heat flux is getting or no heat flux is allowed to pass through. Adiabatic boundary is only a special case of the Neumann boundary condition.

And third one is called the convective or radiative boundary condition. Here let us talk about the other end of this pipe. So this is the surface through which we are assigning or we are connecting this tube. At this end you may have the temperature specified or you may have some heat flux specified. Accordingly we can have a Neumann or Dirichlet boundary condition. If the temperature specified is a Dirichlet boundary condition. If the heat flux is specified or it is mentioned to be adiabatic that is a Neumann boundary condition.

But let us say this other end is open to atmosphere. It is open to atmosphere means if we zoom up this portion; say from this tip energy is being transferred to the surrounding because this portion is open. Let us say these portions are insulated and heat is not allowed to pass through the sides. Heat is allowed to pass only through our tip and the tip is open to surrounding air. And if the temperature of this tip, this face; that is I am talking about this particular face only if temperature of this face and the surrounding air are different then there will be convective heat transfer. As air is flowing over the surface and if the air temperature is low, air will pick up heat from here. That boundary condition is known as the convective boundary condition. How can we get it, let us say this location is referred as x =L.

Now what will happen on the left hand side of this x = L? That is I am talking about on this side what is the mode of heat transfer? Mode of heat transfer is conduction what is the mode of heat transfer here mode of heat transfer is convection. And under steady state condition these two should balance each other.

That is whatever you have the conduction heat flux on the left hand side will be equal to convection on the right hand side, which can be written as

$$-K\frac{\partial T}{\partial x}\bigg|_{x=L^-} = h(T|_{x=L^+} - T_\infty)$$

This condition is known as the convective boundary condition where we are assigning or when we are equating the conduction heat flux with the corresponding convective heat flux.

In certain cases instead of convection we may have radiation. In the same we can write it using a Stefan Boltzmann condition

$$-K\frac{\partial T}{\partial x}\bigg|_{x=L^-} = \varepsilon\sigma(T^4|_{x=L^+} - T_\infty^4)$$

Of course do not forget to convert these temperatures to the absolute temperatures because these are all absolute temperatures.

So this way we can easily assign a convective and radiative boundary condition also. These are all very common boundary conditions and you will encounter numerous heat transfer scenario in this course itself where we have to use either any one or maybe a combination of 2 or 3 boundary conditions. In fact the third one that we are talking about you may find a situation where both convection and radiation going on like one diagram we have seen in the previous lecture.

In the previous lecture where we have seen that a surface is receiving energy by radiation but it is also losing energy by its own emission and convection. If both convective and radiative heat transfers are important then in that case, we have to put all 3 types of heat fluxes together. That is

$$-K\frac{\partial T}{\partial x}\bigg|_{x=L^-} = h(T|_{x=L^+} - T_\infty) + \varepsilon\sigma(T^4|_{x=L^+} - T_\infty^4)$$

In this case all three modes of heat transfer are coming into picture but unless the temperature at this $x = L^+$ is extremely high radiation generally is quite insignificant but convection can be very dominant in several scenarios.

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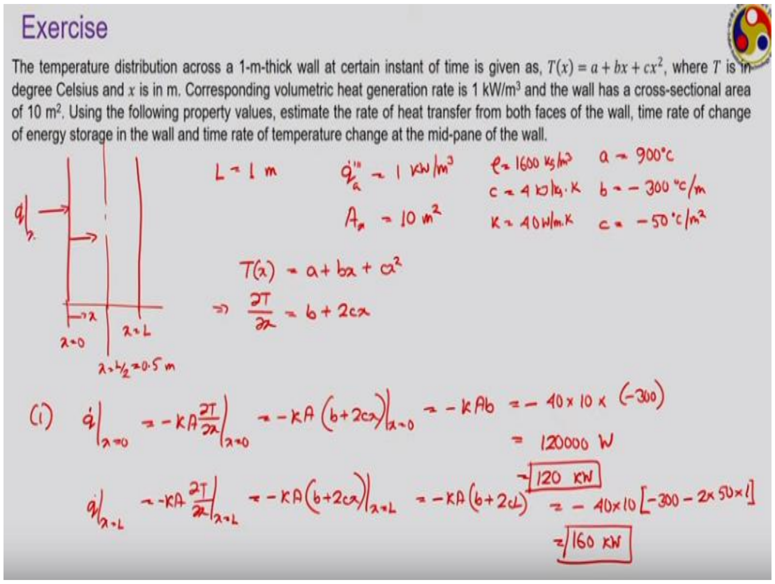

So let us use this idea to solve one exercise problem now. Here the problem is given that the temperature across a 1 m thick wall at a certain instant of time is given. It is given that the temperature distribution is following a quadratic law like this and the corresponding volumetric heat generation rate is given.

So there is a volumetric heat generation that is

$$\dot{q}'''_G = 1\,\frac{kW}{m^3}$$

and the wall has a cross section area of 10 m^2. And this area is having its normal in the x direction that is we are talking about. Actually this particular area for the wall we are not considering for analysis.

This is a one dimensional heat conduction scenario that we are talking about. So heat is getting transferred from one wall to the other wall and certain informations are given and we have to use this information to estimate the rate of heat transfer from both the faces of the wall, time rate of change of energy storage in the wall and time rate of temperature change in the midplane of the wall. So mid plane corresponding to x=L/2; that is 0.5 m.

Certain values are given which I have noted down separately. It is given that a is equal to 900 ºC. Temperature is given in ºC and x is in m. So b is given as -300. What should be the unit of b? Any term in this equation all the 4 terms should have the temperature unit of Celsius.

Now b is multiplied with the length scale then it should be ºC/m, and c is given as -50 ºC/m^2. Certain properties are also given for the corresponding wall material. Its density is given as 1600 kg/m^3. Its specific heat is given as c or cp whatever you would like to write = 4 kJ/kg.K and thermal conductivity K is written as 40 W/m.K.

So these are the set of informations given. We have to estimate first the rate of heat transfer from both the faces of the wall. So we know that at any location x the temperature is given as

$$T(x) = a + bx + cx^2$$

Then what will be dT/dx? Here we cannot consider steady state because later on we have to calculate the time rate of change of energy then partial derivative of temperature with x will be

$$\frac{\partial T}{\partial x} = b + 2cx$$

So the first part of the problem we have to calculate the rate of heat transfer from both the faces of the wall. Then rate of heat transfer as only conduction heat transfer that is happening inside to this

$$\dot{q}|_{x=0} = -KA\frac{\partial T}{\partial x}\bigg|_{x=0}$$

$$= -KA(b + 2cx)|_{x=0} = -KAb = -40 \times 10 \times (-300) = 12000\ W$$

$$= 120\ kW$$

What is the unit of this quantity? This is the heat transfer rate that we are talking about. So this we can write in a concise form 120 kW. Similarly

$$\dot{q}|_{x=L} = -KA\frac{\partial T}{\partial x}\bigg|_{x=L}$$

$$= -KA(b+2cx)|_{x=L} = -KA(b+2cx) = -40 \times 10 \times (-300 - 2 \times 50 \times 1)$$

$$= 160\ kW$$

So if we combine all these numbers I have noted the value, it is coming to be 160 kW. So from the left face the rate of heat transfer is 120 kW and the right face it is 160 kW. but both of them are coming to be positive. But remember we are talking about heat transfer rate which can have a direction. Here both of them are coming to be positive, that indicate, through both faces the rate of heat transfer is in the positive x - direction.

So this is your $\dot{q}|_{x=0}$ direction, for this case also $\dot{q}|_{x=L}$ this is the direction. So 120 kW of heat is entering through x =0 via conduction and 160 kW is leaving by conduction again through the other face. Now I have to estimate the time rate of change of energy storage in the wall.

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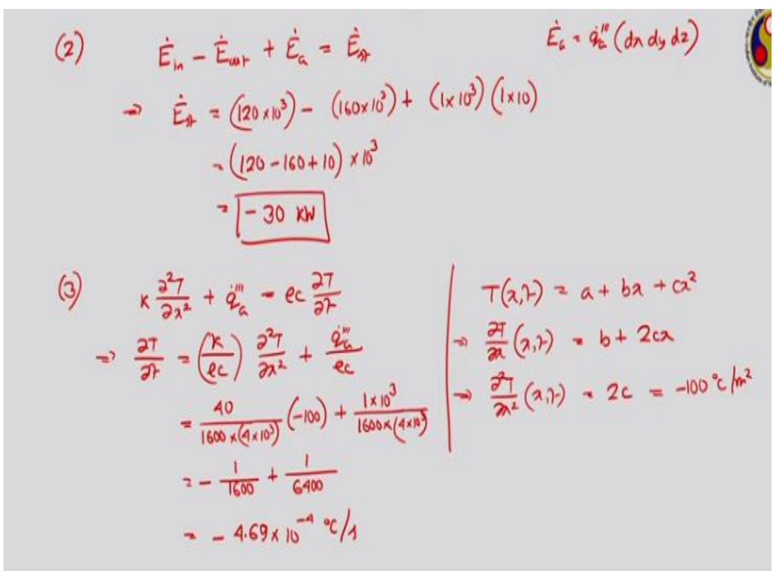

So you have to calculate the time rate of change of energy storage. If we write the energy balance equation

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

Now, what is our $\dot{E}_{in}$? Energies can enter only from that x = 0 face which is 120 kW. $\dot{E}_{out}$ is energy going out through the x = L face which you have calculated to be 160 kW.

We have an energy generation in this case which is given as 1 kW/m^3. We have to multiply this with the volume because we know that

$$\dot{E}_g = \dot{q}'''_G(dxdydz)$$

So how much is the volume here we are talking about? It's a wall which is having a thickness of 1 m and the area of each of the faces is equal to 10 m^2.

Then we are getting

$$\dot{E}_{st} = (120 \times 10^3) - (160 \times 10^3) + (1 \times 10^3)(1 \times 10)$$
$$= (120 - 160 + 10) \times 10^3 = -30\ kW$$

So the block is actually losing energy or I should say that the total energy content of the block is decreasing following this that at this particular rate despite the energy generation. It is receiving energy from one of the faces at a rate of 120 kW. Also at a rate of 10 kW energy is getting generated, but it is losing 160 kW of energy or at a rate of 160 kW from the other face. Accordingly its total energy content is continuously reducing.

Now come to part 3. In part 3 we have to calculate the time rate of temperature change at the mid-plane of the wall.

So let us go to the heat diffusion equation. The general heat diffusion equation for one-dimensional isotropic material we can write as

$$K\frac{\partial^2 T}{\partial x^2} + \dot{q}'''_G = \rho c\frac{\partial T}{\partial t}$$

We are writing the heat diffusion equation but in one-dimensional version assuming an isotropic material that is with constant K. So the K has come out I should not write d. I should stick to the partial derivative notation because T in this case is varying both with time and x direction.

Accordingly, the time rate of change of temperature at any particular location can be written as

$$\frac{\partial T}{\partial t} = \left(\frac{K}{\rho c}\right)\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}'''_G}{\rho c}$$

Now what was the temperature profile that was given in the previous slide? T was given and $\frac{\partial T}{\partial t}$ we calculated but the second derivative we have not done. We had T as a function of x and t which was given as

$$T(x,t) = a + bx + cx^2$$

$$\Rightarrow \frac{\partial T}{\partial x}(x,t) = b + 2cx$$

Now we need the second derivative.

$$\frac{\partial^2 T}{\partial x^2}(x,t) = 2c = -100\ \frac{℃}{m^2}$$

And c being a constant we can directly put the magnitude to be equal to minus of 100 ºC/m^2. So we know the magnitude of this quantity so if we put it now k ρ and c all values are given k is equal to 40, ρ equal to 1600, c is equal to 4 kJ/kg.K. So

$$\frac{\partial T}{\partial t} = \frac{40}{1600 \times (4 \times 10^3)}(-100) + 1 \times \frac{10^3}{1600 \times (4 \times 10^3)}$$

$$= -\frac{1}{1600} + \frac{1}{6400} = -4.69 \times 10^{-4}\ \frac{℃}{s}$$

Note that I am writing all of the magnitudes in basic SI unit so that there is no confusion with kilo or mega or any such kind of units. 4.69 into 10 to the power minus 4.

It is an extremely small rate and also you can see that this though we are supposed to calculate this particular quantity at the mid plane but actually we are getting this $\frac{\partial T}{\partial t}$ as a constant number. That means anywhere in the domain from x =0 to x =L at all the faces the rate or at all the planes the time rate of change of temperature is like this that is the temperature is continuously decreasing at all the points.

So that is how we can use the generalized law of generalized form of the heat diffusion equation in conjunction with the Fourier law of heat conduction.

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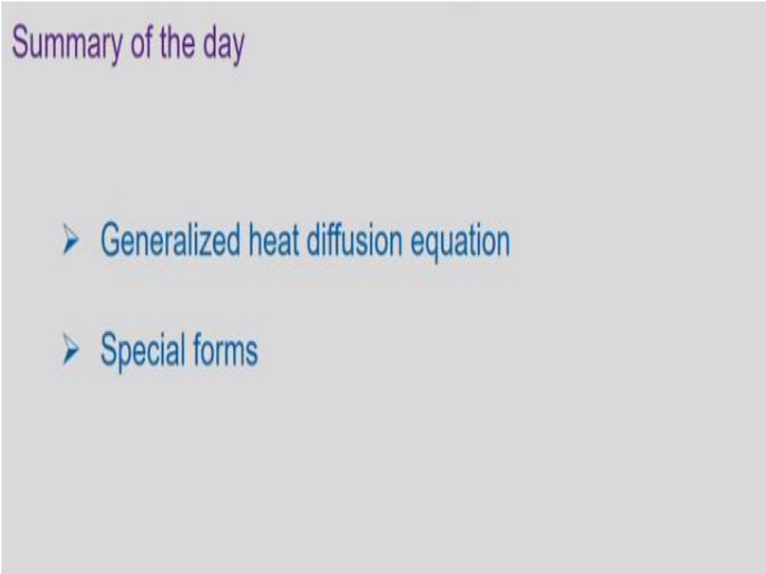

So that is where I would like to stop today. We have talked about the generalized heat diffusion equation and its version in Cartesian coordinate that we have derived. Then a few special forms like for isotropic materials for steady state for uniform materials or the very special case of one-dimensional steady state heat conduction with constant properties as 0 heat generation that was developed.

Then we talked about the boundary conditions. So that is it for the day in the next class I shall be developing the equations in cylindrical and spherical coordinate as well and then we shall be seeing the applications of those equations through quite a few numerical examples. Till then please revise this lecture and try to solve a few more numerals sticking to the Cartesian coordinate system. Thank you very much.