Fundamentals of Conduction and Radiation
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Lecture - 23
Problem Solving using Energy Balance Method

Hello everyone, so in last lecture we have discretized the steady state heat conduction equation as well as the unsteady conduction equation using finite difference method as well as using energy balance method. We have also seen how the boundary conditions can be treated using energy balance method and we have also discussed in unsteady heat conduction equation the implicit scheme and the explicit scheme.

In explicit scheme obviously we have a time step restriction but in implicit scheme it is unconditionally stable. So today we will solve some problem using these techniques so let us solve the first problem.

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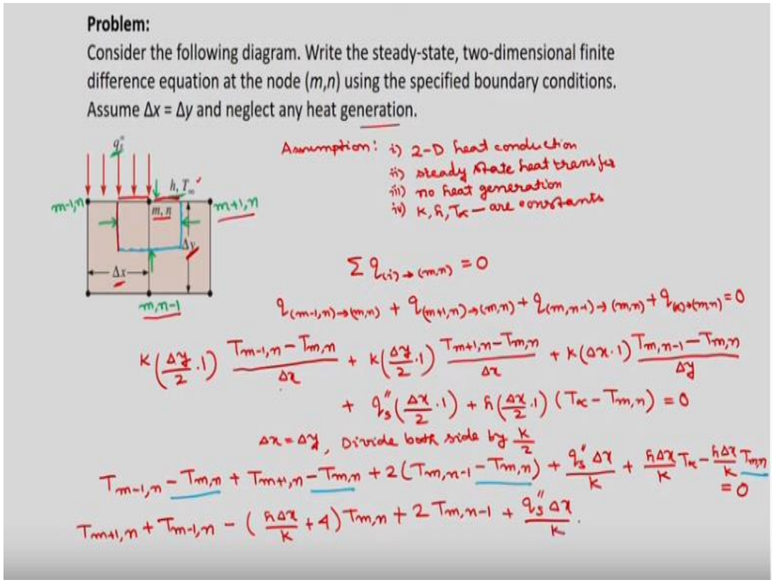

So in this diagram you can see that heat flux is given here and this is the main nodal point (m, n) and here some ambient temperature T_∞ and heat transfer coefficient h is there. And Δx and Δy, these are the x spacing and y direction spacing. So let us read first. Consider the following diagram. Write the steady state two dimensional finite difference equation at the node (m, n) using the specified boundary conditions. Assume $\Delta x = \Delta y$ and neglect any heat generation.

So here what are the assumptions we are taking here? So first assumption is it is two dimensional heat conduction. The second assumption we are taking that it is steady state heat transfer, and the third one is no heat generation okay and obviously constant properties.

So k, h, T_∞ are constants. So as we have already solved in last lecture, we will use the energy balance method and we will try to find the discretized equation at the point (m, n). So we will make a control volume halfway between the nodal points around the node (m, n). Then we will assume that all the heat fluxes are coming in okay.

So heat flux is coming from bottom left and right side. And in the top boundary halfway heat flux in coming in (q''_s) and halfway length will be convection. So the nodes will be if the central node is (m, n) so it will be (m-1, n), it will be (m+1, n) and it is (m, n-1) okay. So now let us write the energy balance okay.

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} = 0$$

So it will be

$$q_{(m-1,n)\to(m,n)} + q_{(m+1,n)\to(m,n)} + q_{(m,n-1)\to(m,n)} + q_{(\infty)\to(m,n)} = 0$$

So now you write so what is$q_{(m-1,n)\to(m,n)}$? We can write that using Fourier's law of heat conduction. So in this case we will use per unit depth. And your length will be your halfway length so it will be Δy/2. So it will be

$$q_{(m-1,n)\to(m,n)} = k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

Similarly, for $q_{(m+1,n)\to(m,n)}$ it will be

$$q_{(m+1,n)\to(m,n)} = k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

For $q_{(m,n-1)\to(m,n)}$ you have a full length so it will be

$$q_{(m,n-1)\to(m,n)} = k(\Delta x.1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Now let's write for $q_{(\infty)\to(m,n)}$. Here halfway length is $q''_s$and halfway is convection. So it will be

$$q_{(\infty)\to(m,n)} = q''_s\left(\frac{\Delta x}{2}.1\right) + h\left(\frac{\Delta x}{2}.1\right)(T_\infty - T_{m,n})$$

Here for both the cases the area will be half. So the final equation will be

$$k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}+k\left(\frac{\Delta y}{2}.1\right)\frac{T_{m+1,n}-T_{m,n}}{\Delta x}+k(\Delta x.1)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+q"_s\left(\frac{\Delta x}{2}.1\right)$$
$$+h\left(\frac{\Delta x}{2}.1\right)(T_\infty-T_{m,n})=0$$

Now let's simplify it okay assuming Δx=Δy, and also divide both side by k/2. So we will get

$$T_{m-1,n}-T_{m,n}+T_{m+1,n}-T_{m,n}+2(T_{m,n-1}-T_{m,n})+\frac{q"_s\Delta x}{k}+\frac{h\Delta x}{k}(T_\infty-T_{m,n})=0$$

If we simplify further

$$T_{m+1,n}+T_{m-1,n}-\left(\frac{h\Delta x}{k}+4\right)T_{m,n}+2T_{m,n-1}+\frac{q"_s\Delta x}{k}+\frac{h\Delta x}{k}T_\infty=0$$

So this is the discretized algebraic equation for the given condition okay. So if $T_{m,n}$ is unknown you can find from this discretized algebraic equation because all other neighbour terms if you know then you can easily find $T_{m,n}$.

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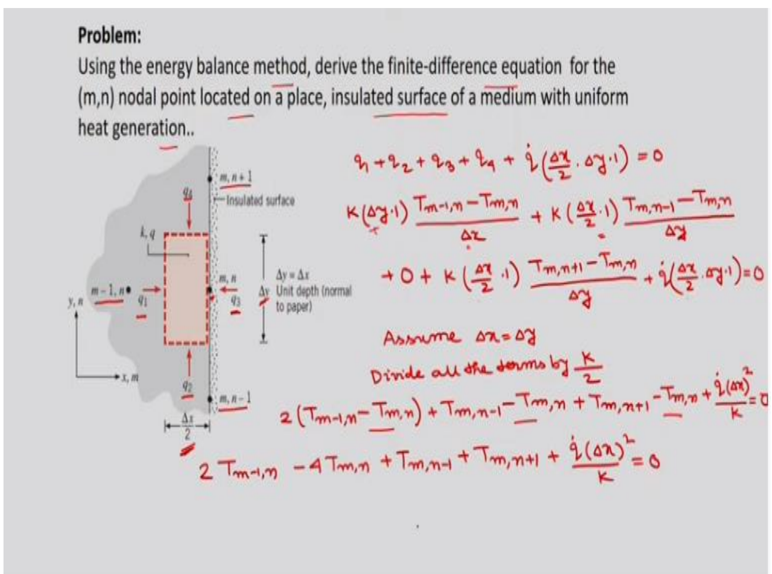

So let us solve the next problem. Let me read first. Using the energy balance method derive the finite difference equation for the (m, n) nodal point located on a place, insulated surface of a medium with uniform heat generation. So in this case we are considering uniform heat generation and the boundary surface is insulated that means it is adiabatic boundary condition and there will be no heat transfer across this boundary okay.

So with that you just use the energy balance method, whatever way earlier we have done similar way you just find but you remember that the boundary is adiabatic. So there will be no heat loss from the boundary surface. So you can see that T(m, n) is the main nodal point and these are all neighbour points okay (m, n-1), (m, n+1). So similar way we will write but there will be heat generation rate.

So the assumptions are, it is two dimensional heat conduction and steady state and thermal conductivity is constant okay. With those assumptions we are writing this energy balance method. So you can write

$$q_1 + q_2 + q_3 + q_4 + \dot{q}\left(\frac{\Delta x}{2}\,.\Delta y.1\right) = 0$$

So q_1, q_2, q_3, q_4 are the heat transfer from the 4 sides and $\dot{q}$ is the heat generation per unit volume inside the cell. So total heat generation will be $\dot{q}$ multiplied by the volume of the cell. So here you can see this is the dotted line we have taken. So that is your volume. So the height is your Δy and this is your Δx/2. So total volume will be $\left(\frac{\Delta x}{2}\,.\Delta y.1\right)$ as we are taking unit depth. So this is your energy balance we have done. And all the heat is coming in to the point (m, n) that we have made the assumption. So using Fourier's law of heat conduction and taking the heat transfer area and distance as we did earlier,

$$q_1 = q_{(m-1,n)\to(m,n)} = k(\Delta y.1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_2 = q_{(m,n-1)\to(m,n)} = k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

The third boundary is adiabatic, so

$$q_3 = 0$$

$$q_4 = q_{(m,n+1)\to(m,n)} = k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

So we can write

$$k(\Delta y.1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + 0 + k\left(\frac{\Delta x}{2}.1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + \dot{q}\left(\frac{\Delta x}{2}\,.\Delta y.1\right) = 0$$

So now you simplify it. So you just assume Δx =Δy okay and divide all the terms by k/2. So we will get

$$2\left(T_{m-1,n} - T_{m,n}\right) + T_{m,n-1} - T_{m,n} + T_{m,n+1} - T_{m,n} + \frac{\dot{q}\Delta x^2}{k} = 0$$

Simplifying it further

$$2T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} + \frac{\dot{q}\Delta x^2}{k} = 0$$

So this is the finite difference equation for the main nodal point (m, n) okay. So now in continuation with this problem, let us solve the next problem okay. So we will use this derived algebraic equation to solve the next problem okay.

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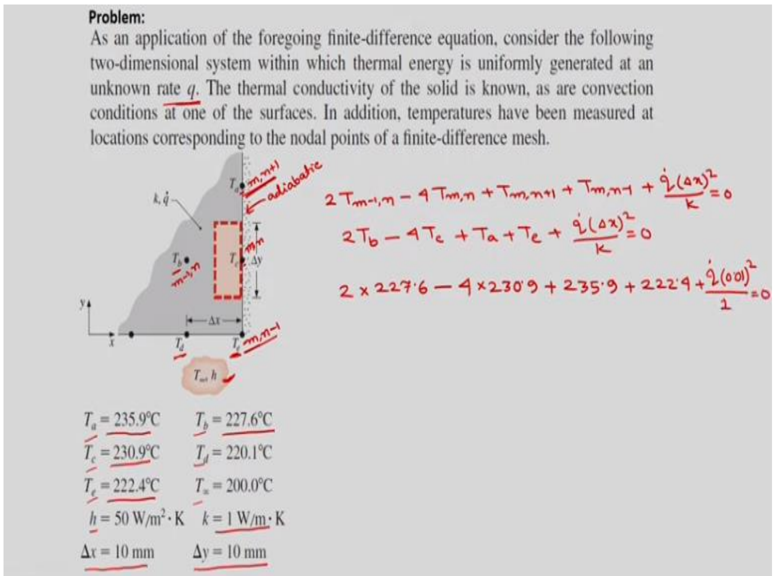

So you see this problem. As an application of the foregoing finite difference equation, consider the following two dimensional system within which thermal energy is uniformly generated at an unknown rate $\dot{q}$ okay. The thermal conductivity of the solid is known as are convection condition at one of the surface. In addition, temperatures have been measured at location corresponding to the nodal points of finite difference mesh.

So you can see in the image at (m-1, n) temperature is T_b, at (m, n+1) it is T_a, at (m, n-1) it is T_e and at (m, n) it is T_c. And this is your adiabatic surface, so obviously there will be no heat loss from there. And the properties are given are

$$T_a = 235.9\,°C, T_b = 224.6°C, T_c = 230.9°C, T_d = 220.1°C, T_e = 222.4°C\ T_\infty = 200.0°C$$

$$h = 50\frac{W}{m^2K}, k = 1\frac{W}{mK}, \Delta x = 10\ mm\ , \Delta y = 10\ mm$$

So here Δx=Δy and all the temperatures T_a, T_b, T_c, T_d and T_e and T_∞ are given okay. So if T_e and T_d are given, obviously we do not need to solve for those points. So these data whatever are given, h and T_∞ will not be used in the solution okay because already the temperature on this bottom surface T_e and T_d are already given.

So whatever expression we have derived in the last slide so that directly you can use. So what we have derived in the last slide that is

$$2T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} + \frac{\dot{q}\Delta x^2}{k} = 0$$

So putting the temperatures

$$2T_b - 4T_c + T_a + T_e + \frac{\dot{q}\Delta x^2}{k} = 0$$

So whatever this ambient conditions h and T_∞ are given for the bottom wall, those are not required because the temperatures on those nodal points are known. So now you put all the values as you to find $\dot{q}$.

$$2 \times 227.6 - 4 \times 230.9 + 235.9 + 222.4 + \frac{\dot{q}(0.01)^2}{1} = 0$$

$$\Rightarrow \dot{q} = 1.01 \times 10^5\ W/m^3$$

So this is the solution using energy balance method we have found the unknown term $\dot{q}$. So let us solve the next problem.

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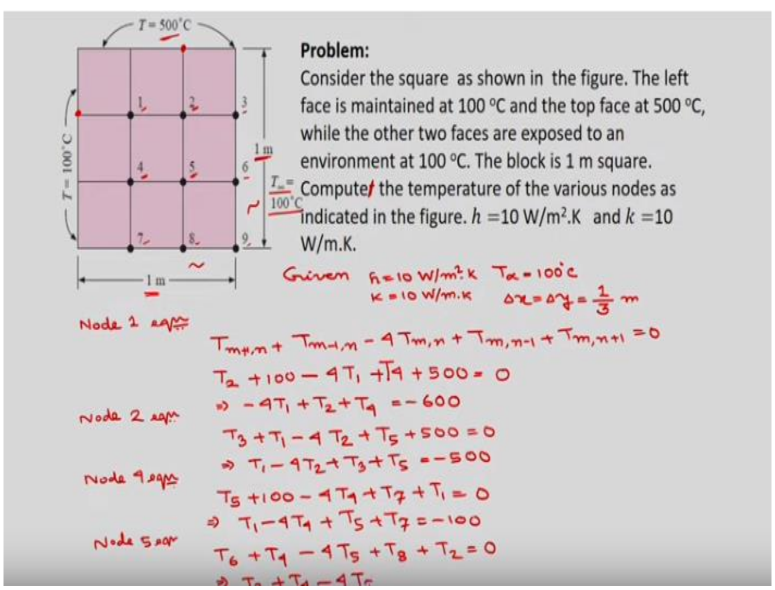

You see here, consider the square as shown in the figure. The left face is maintained at 100 ºC and the top face at 500 ºC while the other two faces are exposed to an environment at 100 ºC.

The block is 1 m square. Compute the temperature of the various nodes as indicated in the figure. The h is given 10 W/m^2.K and k is 10 W/m.K.

So you have a square 1 m by 1 m one solid square is there. Left hand side boundary is maintained at temperature 100 ºC. The top boundary is maintained at constant temperature 500 ºC and right hand side and bottom walls are exposed to the environment where h is given and T_∞ is 100 ºC and thermal conductivity is known. Now you have to find here the temperature of these points.

Here 9 nodal points are there. So these nine temperatures are unknown. So this you have to solve using energy balance method and you can construct 9 equations and if you use the direct method and get [A][T] = [C], C is the known term in the right hand side.

Given quantities are

$$h = 10\frac{\mathrm{W}}{\mathrm{m}^2.K} \ \ \mathrm{k} = 10\frac{\mathrm{W}}{\mathrm{m.K}}, \mathrm{T}_\infty = 100℃,$$

And you can see length and height are 1 m each so Δx =Δy. Here there are 3 points between edges. So Δx will be 1/3 m okay. And left side and top are maintained with the isothermal walls okay. So this is the given things. So at interior points like 1 2 4 5 it is easy to solve because you know that the value at that point is equal to 1/4 of all the neighbour points okay. So it let us write for node 1 okay.

$$T_{m+1,n} + T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} = 0$$

Assuming there is no heat generation with steady state heat conduction and two dimensional situation okay this is the equation. So this is the equation for any interior node. So for node 1 simply you can write putting the values of temperatures at corresponding nodes

$$T_2 + 100 - 4T_1 + T_4 + 500 = 0$$

$$\Rightarrow -4T_1 + T_2 + T_4 = -600$$

So you can see that here coefficient of T_1 is -4 coefficient of T_2 is 1 coefficient of T_4 is 1 and right hand side known term is - 600. So you can find the coefficients A and C from here okay. Similarly you can write for node 2 okay.

$$T_3 + T_1 - 4T_2 + T_5 + 500 = 0$$

$$\Rightarrow T_1 - 4T_2 + T_3 + T_5 = -500$$

Similarly, you write for node 4 okay

$$T_5 + 100 - 4T_4 + T_7 + T_1 = 0$$

$$\Rightarrow T_1 - 4T_4 + T_5 + T_7 = -100$$

Now for node 5

$$T_6 + T_4 - 4T_5 + T_8 + T_2 = 0$$

$$\Rightarrow T_2 + T_4 - 4T_5 + T_6 + T_8 = 0$$

So these all four equations we have written for the interior points. Now let us write for the boundary points okay.

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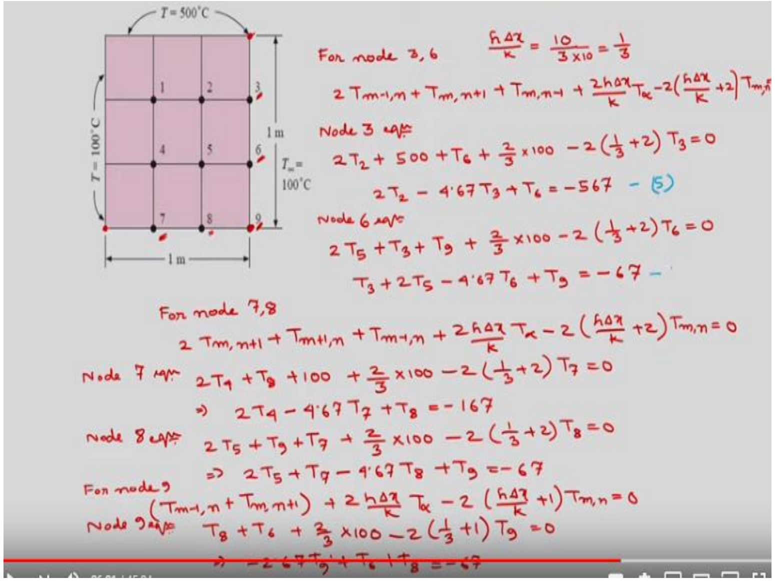

So boundary points if you remember, we have already done for a flat surface with heat convection. So we can write for node 3 and 6. So for node 3 and 6 we can write

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} + \frac{2h\Delta x}{k}T_\infty - \left(\frac{2h\Delta x}{k} + 4\right)T_{m,n} = 0$$

So already we have derived this equation. Let us calculate $\frac{h\Delta x}{k}$.

$$\frac{h\Delta x}{k} = \frac{10 \times \left(\frac{1}{3}\right)}{10} = \frac{1}{3}$$

So now you let us write for node 3

$$2T_2 + 500 + T_6 + \frac{2}{3} \times 100 - \left(\frac{2}{3} + 4\right)T_3 = 0$$

$$\Rightarrow 2T_2 - 4.67T_3 + T_6 = -567$$

Similarly you can write for node 6

$$2T_5 + T_3 + \frac{2}{3} \times 100 - \left(\frac{2}{3} + 4\right) T_6 = 0$$

$$\Rightarrow T_3 + 2T_5 - 4.67T_6 + T_9 = -67$$

Similarly 7 and 8 are also exposed to convection at the bottom flat surface. So for them we can write after deriving

$$2T_{m,n+1} + T_{m+1,n} + T_{m-1,n} + \frac{2h\Delta x}{k} T_\infty - \left(\frac{2h\Delta x}{k} + 4\right) T_{m,n} = 0$$

So for node 7

$$2T_4 + T_8 + 100 + \frac{2}{3} \times 100 - \left(\frac{2}{3} + 4\right) T_7 = 0$$

$$\Rightarrow 2T_4 - 4.67T_7 + T_8 = -167$$

And for node 8

$$2T_5 + T_9 + T_7 + \frac{2}{3} \times 100 - \left(\frac{2}{3} + 4\right) T_8 = 0$$

$$\Rightarrow 2T_5 + T_7 - 4.67T_8 + T_9 = -67$$

So only we are left in the corner point T_9 okay. So as it is a corner point with convection

$$T_{m-1,n} + T_{m,n+1} + \frac{2h(\Delta x)}{k} T_\infty - \left(\frac{2h(\Delta x)}{k} + 2\right) T_{m,n} = 0$$

Putting the temperatures

$$T_8 + T_6 + \frac{2}{3} \times 100 - \left(\frac{2}{3} + 2\right) T_9 = 0$$

$$\Rightarrow -2.67T_9 + T_6 + T_8 = -67$$

So we have 9 unknowns because T_1 to T_9 there are 9 unknown temperatures and 9 equations okay. So obviously all this coefficient you can put in the A matrix and whatever in the right hand side that you put in the C matrix and this is the temperature vector at T1, T2, T3 to T9.

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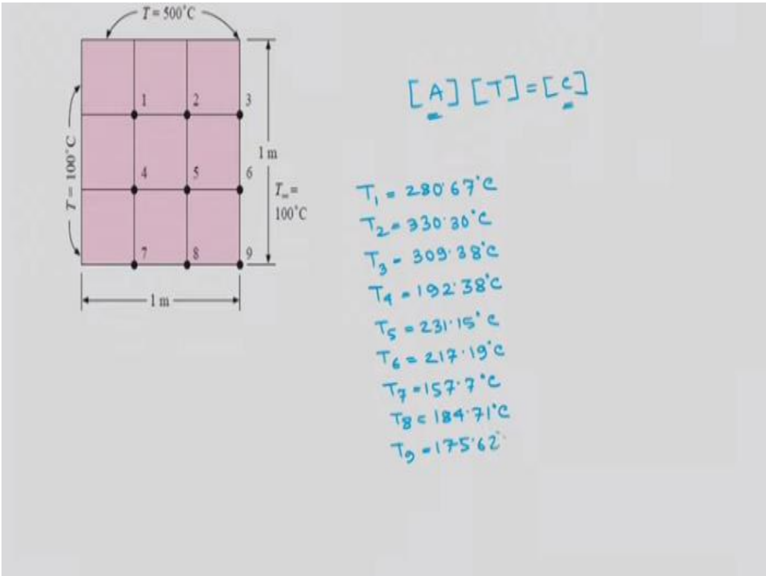

So you can write like

$$[A][T] = [C]$$

So if you solve this you will get the temperature values T_1 as 280.67 ºC, T_2 you are going to get 330.30 ºC, T_3 you are going to get 309.38 ºC, T_4 will be 192.38 ºC, T_5 will be 231.15 ºC, T_6 will be 217.19 ºC, T_7 will be 157.7 ºC, T_8 will be 184.71 ºC and T_9 will be 175.62 ºC okay.

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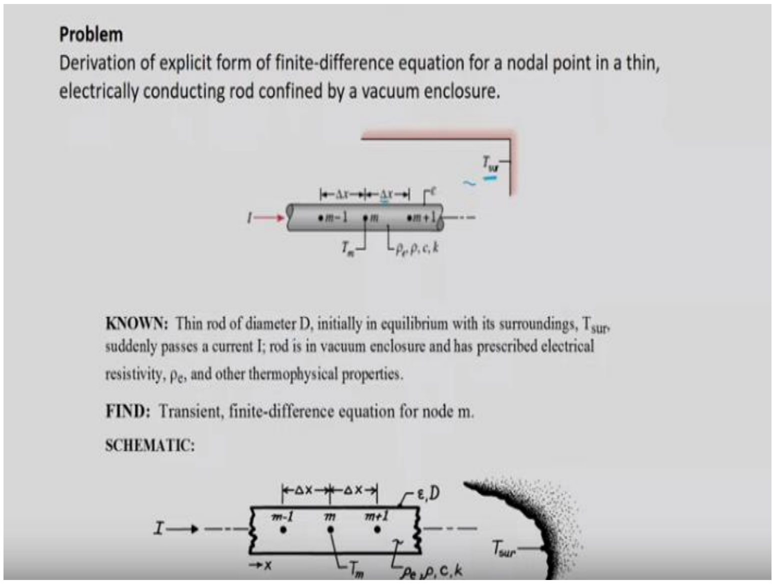

So at last now we will solve this unsteady heat conduction problem. So you can see the problem is derivation of explicit form of finite difference equation for a nodal point in a thin electrically conducting rod confined by a vacuum enclosure. So there is a vacuum enclosure you can see here. So this is one 1D rod where temperature T_m and your resistivity is ρ_e, ρ , c, k is the density thermal capacity and thermal conductivity, these are the Δx and ϵ is the emissivity of the surface.

So radiation is taking place. So heat loss will take place due to radiation and this is the surrounding temperature T_{sur}, and rod diameter d is given so your perimeter will be πd okay and if you consider Δx as a length, so $\pi d \Delta x$ will be the volume okay. So obviously there will be heat generation because this is a small rod and current is passing so I^2R will be your heat generation rate.

So now you can see here so this is the Δx, ϵ and this is the main point m and if you consider I is passing through it so due to that there will be heat generation.

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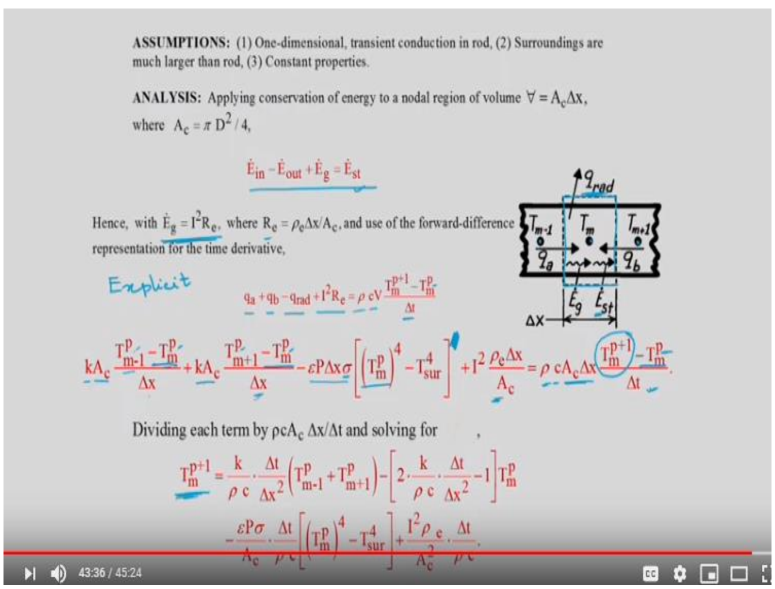

So if you use the energy balance equation you can write

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$

As it is a unsteady conduction equation obviously there will be heat storage right, heat stored will be there. Cross section of this wire will be$A_c = \frac{\pi d^2}{4}$. So heat generation will be

$$\dot{E}_{gen} = I^2 R_e$$

Where resistance will be $R_e = \rho_e \Delta x / A_c$.

So now if this is the control volume we have drawn okay, it is a one dimensional case okay so q_a is coming from (m – 1) to m , q_b is coming from (m +1) to m and heat loss is taking place due to radiation q_{rad} okay and there will be heat generation $\dot{E}_{gen} = I^2 R_e$ and there will be storage okay. So it is a unsteady problem so obviously you can write from this equation energy balance you can write

$$q_a + q_b - q_{rad} + I^2 R_e = \rho c V \frac{\left(T_m^{p+1} - T_m^p\right)}{\Delta t}$$

q_{rad} is heat loss is taking place and that means it is going out, so it will be $-q_{rad}$. And your heat generation is taking place$I^2 R_e$. And what is the heat storage that is $\rho c V \frac{\left(T_m^{p+1} - T_m^p\right)}{\Delta t}$, where p+1 is the current time level and p is the previous time level. And, Δt is the time step. If we write now using Fourier's law for conduction and Steffan Boltzman law for radiation we can get

$$kA_c \frac{T_{m-1}^p - T_m^p}{\Delta x} + kA_c \frac{T_{m+1}^p - T_m^p}{\Delta x} - \epsilon P \Delta x \sigma \left[\left(T_m^p\right)^4 - T_{sur}^4\right] + \frac{I^2 \rho_e \Delta x}{A_c} = \rho c A_c \Delta x \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

For q_a cross sectional area is A_c and heat is going form (m – 1) to m so it will be $kA_c \frac{T_{m-1}^p - T_m^p}{\Delta x}$. We are using explicit discretization here, so only one is unknown term and all other known terms we are considering for easier calculation.

So q_b is from (m+1) to m so it will be $kA_c \frac{T_{m+1}^p - T_m^p}{\Delta x}$, because the distance between two points Δx and q_{rad} we can find the heat loss due to radiation using Stefan Boltzmann law between the surface and surrounding. So that you can write $\epsilon P \Delta x \sigma \left[\left(T_m^p\right)^4 - T_{sur}^4\right]$.

So here ϵ is the emissivity of the surface okay and σ is the Stefan Boltzmann constant okay. And now $I^2 R_e$. So $R_e = \frac{\rho_e \Delta x}{A_c}$, so that we have written. And in the last term volume will be $A_c \Delta x$, which is cross-sectional area multiplied by distance between nodes.

So now here unknown term is T_m^{p+1} and all are you see p terms. Explicit method we have used okay. So obviously now if you take T_m^{p+1} in the left hand side and all known terms you take in the right hand side, then you make it simple. And also divide both sides by $\frac{\rho c A_c \Delta x}{\Delta t}$, we will get

$$T_m^{p+1} = \frac{k}{\rho c}\frac{\Delta t}{\Delta x^2}\left(T_{m-1}^p + T_{m+1}^p\right) - \left[2\frac{k}{\rho c}\frac{\Delta t}{\Delta x^2} - 1\right]T_m^p - \frac{\epsilon P\sigma}{A_c}\frac{\Delta t}{\rho c}\left[\left(T_m^p\right)^4 - T_{sur}^4\right] + \frac{I^2\rho_e}{A_c^2}\frac{\Delta t}{\rho c}$$

So this is the expression for the temperature at point m for one dimensional transient heat conduction. Here heat loss is taking place due to radiation and heat generation is taking place due to electricity flow okay and also there will be heat storage. So from there we have just energy balance we have derived this equation but obviously there will be time restriction, time step restriction, because it is an explicit scheme.

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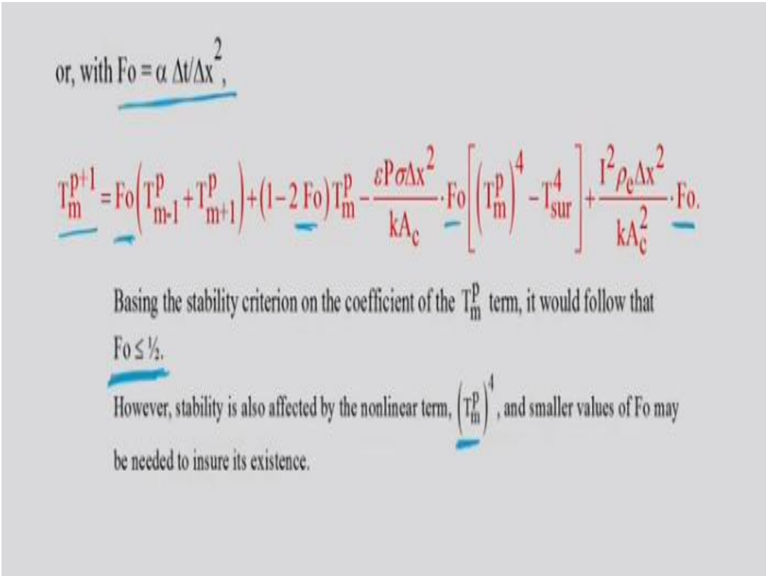

And if you define Fourier number as

$$Fo = \frac{\alpha\Delta t}{\Delta x^2}$$

Then you can write

$$T_m^{p+1} = Fo\left(T_{m-1}^p - T_{m+1}^p\right) - [1 - 2Fo]T_m^p - \frac{\epsilon P\sigma\Delta x^2}{kA_c}Fo\left[\left(T_m^p\right)^4 - T_{sur}^4\right] + \frac{I^2\rho_e\Delta x^2}{kA_c^2}Fo$$

So all these coefficients we have written in the Fourier number okay you can see all these coefficients. As it is a one dimensional heat conduction so your time step restriction will be

$$Fo \leq \frac{1}{2}$$

Which, earlier we have seen.

But as it is nonlinear problem as we have $\left(T_m^p\right)^4$. Sometime this condition will be too stringent so Fourier number we have to take much less than half for better convergence okay. So you can actually solve this equation for a given situation okay. So now you can solve some of the problems okay from any textbook and practice it and anyway we will give some assignments. Thank you.