Fundamentals of Conduction and Radiation
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology - Guwahati

Lecture – 22
Unsteady Heat Conduction

Hello everyone. So in last class we have derived the discretized algebraic equation for the steady state heat conduction equation using Finite Difference Equations as well as the energy balance method.

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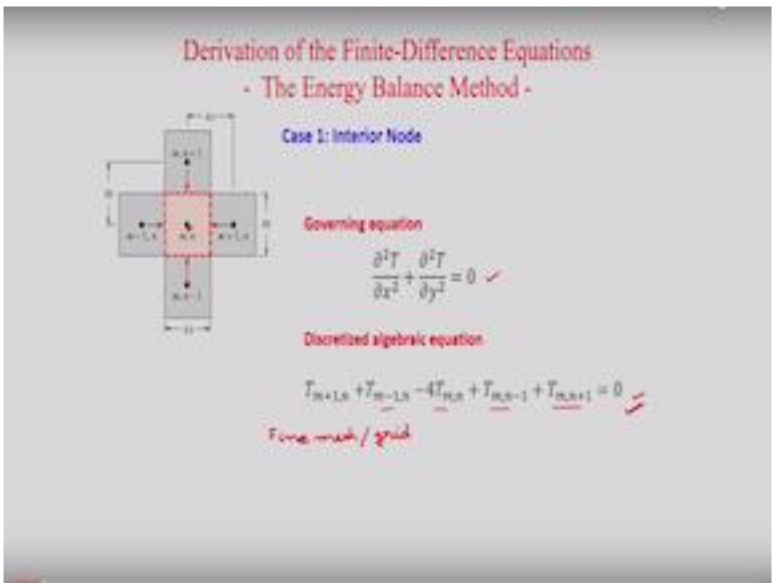

So let us see what we have done yesterday.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

So you can see that part of any interior point, if the node indices are m and n, then if this is the governing equation with the assumption that it is a two dimensional steady state heat conduction without heat generation, then either using finite difference approximation where we have used central difference scheme or using energy balance method, we can derive this equation.

$$\Rightarrow T_{m+1,n} + T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} = 0$$

This is the algebraic equation okay. The unknown temperature will be at the central node, which is $T_{m,n}$. And this unknown temperature you can find using the neighbor nodal points. So you can see that these are discrete points, ((m,n)), (m +1, n) or ((m,n)+1). So we are solving this discretized algebraic equation at those nodal points.

So the accuracy depends on the how refined mesh you have generated. So obviously if you use fine mesh or grid, then you will get good accuracy. So this, we are finding at the interior points, but these are boundary value problems. So we need to find the boundary condition as well.

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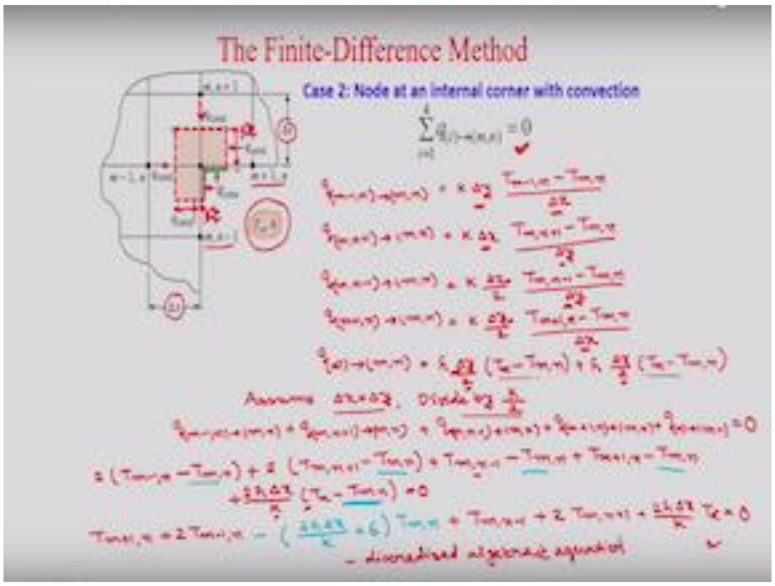

So now we will see in next slide for a node at an internal corner with convection. So you can see these are boundaries and where this boundary are actually open atmospheric condition where you have temperature T_∞ and the heat transfer coefficient h. So obviously the heat transfer will take place due to convection. Now at this corner point (n + 1), what will be the discretized algebraic equations. So let us derive it okay. We are considering the heat transfer from all the neighboring points to the central nodal poin$T_{m,n}$ as positive. So the energy balance equation without heat generation is

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} = 0$$

So you can write heat transfer q from (m –1, n) that is from the left neighbor point to ((m,n)) using Fourier’s law as

$$q_{(m-1,n)\to(m,n)} = k(\Delta y)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

This we derived in the last class where the heat transfer area is (Δy) and distance between the two nodal points is Δx. So from other points, let us say from ((m,n) + 1) to ((m,n)). So what would be the q? So it will be

$$q_{(m,n+1)\rightarrow(m,n)} = k(\Delta x)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

This also we derived in earlier class. So in both the cases you can see that the heat transfer area is either Δx or Δy.

But now you consider other 2 neighbor points like (m+1, n) or ((m,n)-1). You can see that heat transfer area is half of the earlier one okay. Because you have half channel, because this is your $\Delta y/2$ and this is your $\Delta x/2$, because this part is open as it is a corner point. So now if we write

$$q_{(m,n-1)\rightarrow(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Here we are writing the heat transfer area to be $\frac{\Delta x}{2}$ as it is half and the distance between these 2 points is obviouslyΔy. Now in the other point, from (m+1, n) to ((m,n)), it will be

$$q_{(m+1,n)\rightarrow(m,n)} = k\left(\frac{\Delta y}{2}\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

Here also the area is half and it is $\frac{\Delta y}{2}$ and the distance is Δx. So these are heat inflow that is taking place due to conduction okay and we have used Fourier's law of heat conduction. Now 2 half lengths are there where heat transfer is taking place due to convection okay. As those areas are open to atmosphere so due to convection there will be heat transfer and that now let us write. So we will write for the vertical bouondary

$$q_{(\infty)\rightarrow(m,n)} = h\left(\frac{\Delta y}{2}\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta x}{2}\right)\left(T_\infty - T_{m,n}\right)$$

$(\infty) \rightarrow (m,n)$ here means just the ambient to ((m,n)), As it is convection we have written heat transfer coefficient h, and the area here will be $\frac{\Delta y}{2}$. Now T_∞ is the ambient temperature. Now, here we are assuming obviously the region, inside this dotted line okay, the temperature is the average temperature and that lies at the center nodal point okay. That is $T_{m,n}$ ((m,n)). So along this surface where convection is taking place, obviously your temperature is maintained at $T_{m,n}$ because that is the average temperature.

So that assumption anyway we have considered so it is valid to write that temperature difference is $T_\infty - T_{m,n}$. Because $T_{m,n}$ is constant for this dotted line, and it is the average temperature and everywhere it is the same. So we can write $T_{m,n}$. Similarly from this surface, the horizontal corner surface if you consider there will be heat convection okay. So that is given by the 2nd term in the equation of $q_{(\infty)\rightarrow(m,n)}$. There the area will be Δx but all other things will be constant.

Now we will assume $\Delta x = \Delta y$ okay, or uniform grid in both x and y directions. So now if we go to the energy balance equation and do the summation of all these $q_{(i)\rightarrow(m,n)}$ where i represents the neighboring points, we can write

$$q_{(m-1,n)\rightarrow(m,n)} + q_{(m,n+1)\rightarrow(m,n)} + q_{(m,n-1)\rightarrow(m,n)} + q_{(m+1,n)\rightarrow(m,n)} + q_{(\infty)\rightarrow(m,n)} = 0$$

If we divide both sides by k/2 and write their expressions we will get the final expression as

$$2(T_{m-1,n} - T_{m,n}) + 2\left(T_{m,n+1} - T_{m,n}\right) + T_{m,n-1} - T_{m,n} + T_{m+1,n} - T_{m,n} + \frac{2h\Delta x}{k}\left(T_\infty - T_{m,n}\right) = 0$$

Let's simplify it and write as

$$T_{m+1,n} + 2T_{m-1,n} - \left(\frac{2h\Delta x}{k} + 6\right)T_{m,n} + T_{m,n-1} + 2T_{m,n+1} + \frac{2h\Delta x}{k}T_\infty = 0$$

So this is the discretized algebraic equation for the node at an internal corner with convection okay. So this case we have considered now take another type of boundary condition. So these all are, we are considering boundary conditions okay. Because when you are solving the interior points you need to solve for the boundary points as well depending on that boundary condition. So we are discretizing those boundary points.

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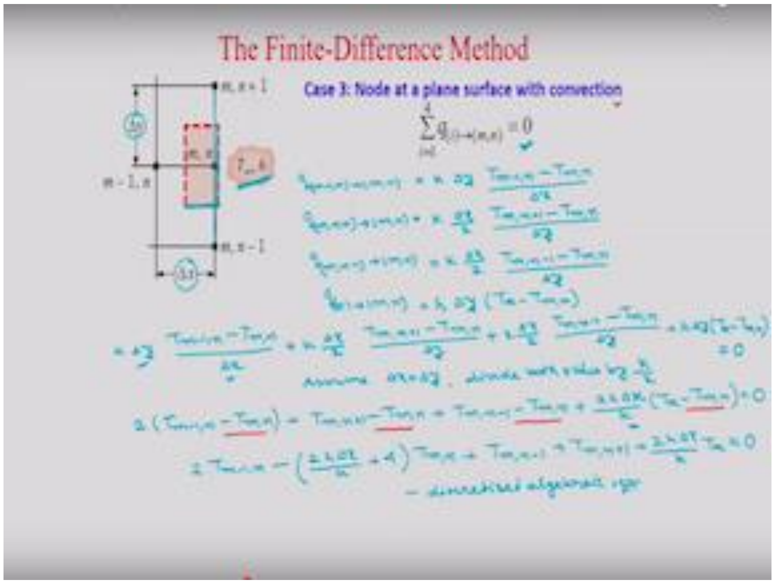

Now you consider node at a plane surface with convection. So this is your plane surface and this is open to convection okay. T_∞ and h okay. It is not a corner point. It is a plain surface and the heat convection is taking place. So heat transfer coefficient is h and temperature is T_∞ and it is obviously boundary points at $T_{m,n}$. So at this point now how we will find that temperature. So we will use same energy balance okay.

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} = 0$$

So you can write

$$q_{(m-1,n)\to(m,n)} = k(\Delta y)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

This is the same as we did for the earlier case. Now for other point (m,n + 1) to (m,n), we have half length right, because your area is half or $\frac{\Delta x}{2}$. So it will be

$$q_{(m,n+1)\to(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

Similarly from (m,n-1), you can write similar expression because it is also half length

$$q_{(m,n-1)\to(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Now convection is taking place so we will write $q_{(\infty)\to(m,n)}$ okay. So now we have only 1 plane surface whose area is Δy as this vertical distance is Δy okay. So,

$$q_{(\infty)\to(m,n)} = h\Delta y\left(T_\infty - T_{m,n}\right)$$

So now all summation you write.

$$k(\Delta y)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\Delta y\left(T_\infty - T_{m,n}\right) = 0$$

Now assume $\Delta x = \Delta y$, and divides both sides by k/2. Then we will get

$$2\left(T_{m-1,n} - T_{m,n}\right) + T_{m,n+1} - T_{m,n} + T_{m,n-1} - T_{m,n} + \frac{2h\Delta x}{k}\left(T_\infty - T_{m,n}\right) = 0$$

So now you can rearrange, all the coefficient of $T_{m,n}$ you can take in one place. So you can write

$$2T_{m-1,n} - \left(\frac{2h\Delta x}{k} + 4\right)T_{m,n} + T_{m,n-1} + T_{m,n+1} + \frac{2h\Delta x}{k}T_\infty = 0$$

So this is the final discretized algebraic equation for the condition of a node at a plane surface with convection. So second type of boundary condition we have discussed. Now another type of boundary condition we will consider next.

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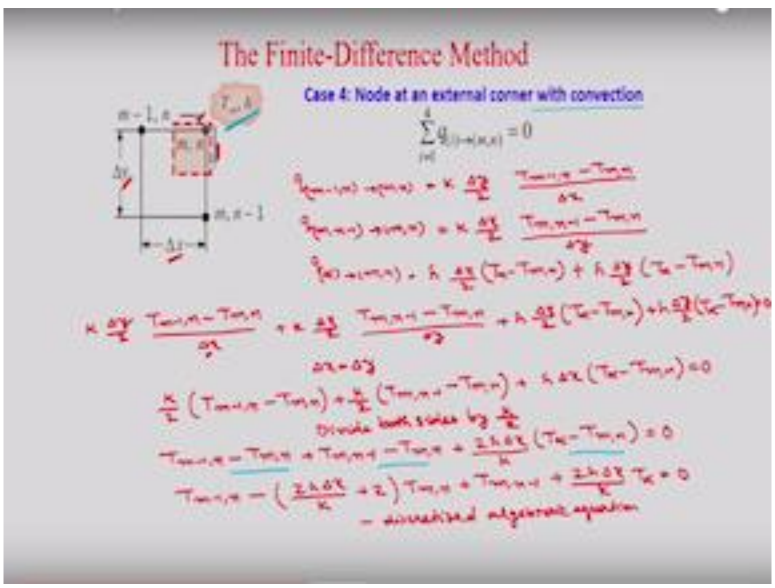

So that is your node at an external corner with convection. So you can see here, it is an external corner node, $T_{m,n}$ and this side and this side are open to atmosphere and heat convection is taking place there, with ambient condition T_∞ and h okay. So in this case let's derive the heat transfer rate from (m-1, n) to (m,n). So area in this case it will be half okay. So it will be $\frac{\Delta y}{2}$, and the distance will be Δx okay. So it will be

$$q_{(m-1,n)\to(m,n)} = k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

Similarly

$$q_{(m,n-1)\rightarrow(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Now all other neighbor nodes are not there. Only the heat convection is taking place. Here, we have 2 surfaces for convection, but heat transfer area will be half of the total area in both. So that you can consider the top surface that will be $\frac{\Delta x}{2}$ as we are considering per unit width and for the right side surface it will be $\frac{\Delta y}{2}$ okay. So that will be

$$q_{(\infty)\rightarrow(m,n)} = h\left(\frac{\Delta x}{2}\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta y}{2}\right)\left(T_\infty - T_{m,n}\right)$$

So all the heat transfer to the nodal point (m,n) we have written. Now your summation will be zero. So that you write

$$k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h\left(\frac{\Delta x}{2}\right)\left(T_\infty - T_{m,n}\right) + h\left(\frac{\Delta y}{2}\right)\left(T_\infty - T_{m,n}\right) = 0$$

Now let us assume that uniform grid so you can write $\Delta x = \Delta y$ okay. So it will be

$$\left(\frac{k}{2}\right)\left(T_{m-1,n} - T_{m,n}\right) + \left(\frac{k}{2}\right)\left(T_{m,n-1} - T_{m,n}\right) + h(\Delta x)\left(T_\infty - T_{m,n}\right) = 0$$

Now divide both side by k/2 okay.

$$T_{m-1,n} - T_{m,n} + T_{m,n-1} - T_{m,n} + \frac{2h(\Delta x)}{k}\left(T_\infty - T_{m,n}\right) = 0$$

So now let us simplify it further, so all the $T_{m,n}$ term will take together.

$$T_{m-1,n} - \left(\frac{2h(\Delta x)}{k} + 2\right)T_{m,n} + T_{m,n-1} + \frac{2h(\Delta x)}{k}T_\infty = 0$$

This is the discretized algebraic equation for the case of a node at an external corner with convection. So this type of boundary condition you may face while solving some of the problem. So along with the discretized equation for the interior point you need to solve these discretized boundary values too.

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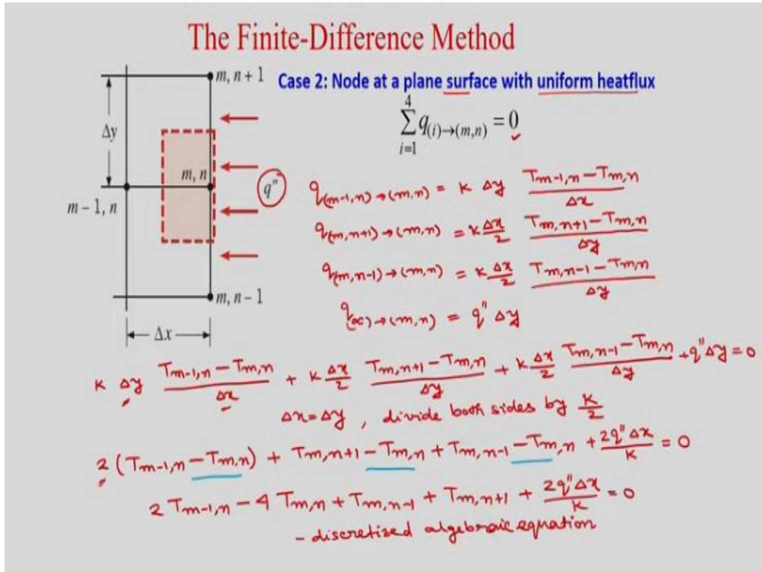

So now another type of boundary condition we will consider for a node at a plane surface with uniform heat flux okay. So you have a plane surface where uniform heat flux is there. So you can see uniform heat flux $q"$ we are considering on this surface okay, so for this particular nodal point (m,n) we will write

$$\sum_{i=1}^{4} q_{(i)\rightarrow(m,n)} = 0$$

Now let's write the heat transfer values

$$q_{(m-1,n)\rightarrow(m,n)} = k(\Delta y)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

This is same as we did earlier. The area we can consider full. But for the top and bottom surface the area will be half as you can see in the figure. So it will be

$$q_{(m,n+1)\rightarrow(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

Similarly,

$$q_{(m,n-1)\rightarrow(m,n)} = k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

And now there is a heat flux $q"$ which is acting on the area Δy. So that you can write

$$q_{(\infty)\rightarrow(m,n)} = q"\Delta y$$

So now you sum it up, so it will be

$$k(\Delta y)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + \left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + q"\Delta y = 0$$

And again we will assume uniform grid so $\Delta x = \Delta y$ okay and divide both sides by k/2. So let's just write the simplified form this final equation.

$$2\left(T_{m-1,n} - T_{m,n}\right) + T_{m,n+1} - T_{m,n} + T_{m,n-1} - T_{m,n} + \frac{2q''\Delta x}{k} = 0$$

Now all the $T_{m,n}$ term we will write together.

$$2T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} + \frac{2q''\Delta x}{k} = 0$$

This is the final discretized algebraic equation for the condition of a node at a plane surface with uniform heat flux. So today we have discussed different types of boundary conditions and we have tried to discretize the boundary condition at that nodal point $T_{m,n}$. So if you are solving the interior points, along with that equation you need to solve this boundary condition okay. So now all these conditions whatever we have considered all are steady state heat transfer. So only we have the considered governing equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

But there are many situations where unsteady heat conduction takes place. Already you have solved 1D unsteady heat transfer and you have the analytical solution for that. But, when you solve 2D unsteady heat conduction, it is more complicated. So you do not have the analytical solution or exact solution for that. So you can go for either energy balance method or finite difference method for these kind of problem as analytical solutions are not directly available. So now we will consider two dimensional heat conduction with no heat generation. So let us consider that.

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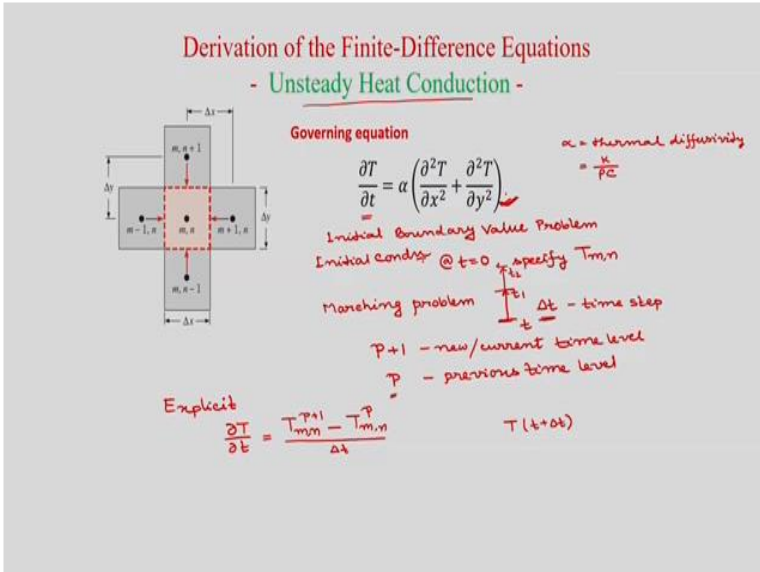

So now we are considering unsteady heat conduction where governing equation will be

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

What is α? α is you know thermal diffusivity right, already you have derived thermal diffusivity, where

$$\alpha = \frac{k}{\rho C}$$

where k is the thermal conductivity of the solid, ρ is the density and C is the heat capacity of the solid. So this is the two dimensional unsteady heat conduction equation with no heat generation and with constant properties as we have considered. This you can solve using finite difference approximation as well as the energy balance method. We will here consider the finite difference approximation and we will discretize this equation for an interior point. But you can discretize this as a homework using energy balance method as well as for the different boundary conditions.

Now here you can see it is an initial boundary value problem. As you have a temporal term which is $\frac{\partial T}{\partial t}$. To solve this type of problems you need initial condition that means at t=0, at all interior nodes you have to specify the temperature. And boundary values, already you know that you need to specify the boundary condition and accordingly you need to discretize the equation.

So that means initial condition you need. So at t=0, you have to specify the T at all nodal points okay. Then after that you are marching in time okay. So you are actually from t to $t + \Delta t$ or t to t_1 then t_1 to t_2; that way you are marching in the time direction.

So it is a marching problem okay. Initial value problem or marching problem where if you have a specified at t some values then at time t_1, you are again finding the values then at t_2 again you are finding the value. So that way you are marching in time direction and you are finding the values at different time levels okay.

And the time between these two time points we will consider as Δt okay. So that is known as time step. So that means we will give the increment in the time direction as Δt okay. Like you have discretized in space coordinate Δx and Δy. Similarly, in time direction you need some points so that at those points you will find the temperature.

And here the index we will use P okay. Where, P+1 is the new or current time level okay, and P will be the previous time level. So we are going from P to P+1 and we are trying to find the temperature at any nodal point (m, n) at time level P+1. So P is the index we are using in time direction, like we have used m and n in x and y direction as the indices. Similarly, at time level we are using P.

So P+1 is the current time level at which we are interested to find the temperature and P is already known because we have already calculated the value at time level P. So, all the values of interior nodal points and boundary points are known at time level P. Using those values now we will find the temperature at P+1. So now here anyway the spacial discretization we have done for steady state heat conduction equation that you know. We have used central differencing scheme which is a second order spacial accuracy $(\Delta x)^2$ and $(\Delta y)^2$. So now that we will not describe here. But now time derivative is there. So now you can discretize this time derivative either using forward time difference or backward time difference. Depending on that you will get either explicit scheme or implicit scheme. So first we will use forward time stepping and we will discretize that equation. So it is known as explicit discretization.

We can use Taylor series expansion for $T(t + \Delta t)$ and just like spacial discretization we can get first derivative of temperature with time using forward differencing

$$\frac{\partial T}{\partial t} = \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}$$

Here, P is the index in time and m and n are in space. So this is the forward time derivative. But for the spacial discretization we will use central difference okay.

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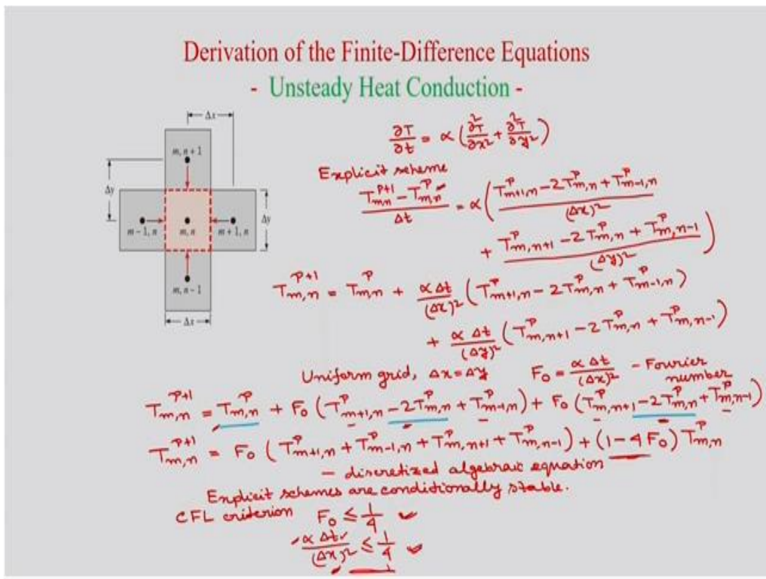

So we had

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

So now we are using explicit scheme that is forward time discretization and central differencing for space. So the discretized equation will be

$$\frac{T_{m,n}^{P+1} - T_{m,n}^{P}}{\Delta t} = \alpha \left(\frac{T_{m+1,n}^{P} - 2T_{m,n}^{P} + T_{m-1,n}^{P}}{(\Delta x)^2} + \frac{T_{m,n+1}^{P} - 2T_{m,n}^{P} + T_{m,n-1}^{P}}{(\Delta y)^2} \right)$$

Here in the space discretization we are using values from the previous time level so that represents explicit scheme but if we use the values from the current time level P+1 then it will be backward differencing and we can call it implicit scheme. Here we assumed that we know the all the nodal values from the previous time step. So obviously you have already calculated at time step P. Hence all P time step temperatures are known. So we can write this

$$T_{m,n}^{P+1} = T_{m,n}^{P} + \frac{\alpha \Delta t}{(\Delta x)^2} \left(T_{m+1,n}^{P} - 2T_{m,n}^{P} + T_{m-1,n}^{P} \right) + \frac{\alpha \Delta t}{(\Delta y)^2} \left(T_{m,n+1}^{P} - 2T_{m,n}^{P} + T_{m,n-1}^{P} \right)$$

So this equation if you see in the right hand side at different nodal points means the main nodal point (m,n) as well as all the neighboring nodal points (m+1, n) or (m-1, n) or (m, n+1) or (m,n-1), at all these points temperatures are known okay. So only unknown is $T_{m,n}^{P+1}$ and that we are trying to find here. So let's now assume that you have a uniform grid in both x and y direction.

And if you make this assumption that $\Delta x = \Delta y$ you can further simplify it. So now let's define Fourier number

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$$

It is dimensionless number and about this you have already learnt in a transient heat conduction. So you can write it

$$T_{m,n}^{P+1} = T_{m,n}^{P} + Fo\left(T_{m+1,n}^{P} - 2T_{m,n}^{P} + T_{m-1,n}^{P}\right) + Fo\left(T_{m,n+1}^{P} - 2T_{m,n}^{P} + T_{m,n-1}^{P}\right)$$

So we have written in terms of Fourier number. Now all the coefficient of $T_{m,n}$ you take it together. So if you write that

$$T_{m,n}^{P+1} = Fo\left(T_{m+1,n}^{P} + T_{m-1,n}^{P} + T_{m,n+1}^{P} + T_{m,n-1}^{P}\right) + (1 - 4Fo)T_{m,n}^{P}$$

So it will be final discretized algebraic equation for the interior points for a unsteady heat conduction equation.

You can see that you can easily solve because $T_{m,n}^{P+1}$ is the only unknown. So $T_{m,n}^{P+1}$ is in the left hand side. Right hand side all the terms are known, all the neighboring points and the nodal point at time level P, but there is a time restriction. Because explicit schemes are conditionally stable okay, it is not unconditionally stable okay. Implicit schemes are sometimes unconditionally stable but explicit scheme are conditionally stable only. So for that reason you cannot choose Δx any value okay. Then during the solution, it may diverge okay. Otherwise it will oscillate, and it will diverge. So for that there is the Courant Friedrichs Lewy number, that is known as CFL criteria okay. The condition is that for the explicit scheme for this two dimensional case, the coefficient of $T_{m,n}^{P}$ should be positive. So you should have

$$Fo \leq \frac{1}{4}$$

$$\Rightarrow Fo = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

So this is the time restriction. Now you can see that α, thermal diffusivity is constant for a particular solid. And if you have already done the meshing or made the grid then Δx and Δy are constants. And you cannot change them later. So only possible way you can satisfy this criterion is you change Δt okay. So Δt you have to choose from this condition okay. So because Δx you know, α you know, so Δt you choose such a way that this condition is satisfied okay. Then you will not get any problem in the convergence.

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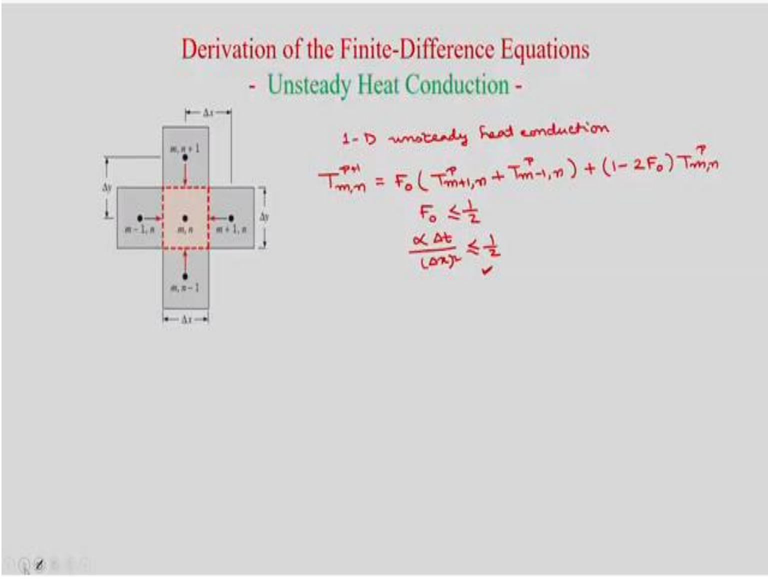

Now let's say we choose 1D unsteady heat conduction. Then we can write

$$T_{m,n}^{P+1} = Fo\left(T_{m+1,n}^{P} + T_{m-1,n}^{P}\right) + (1 - 2Fo)T_{m,n}^{P}$$

This we can derive using the same discretization scheme. So here CFL condition will be

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

So when you are choosing the time step Δt to solve these governing equations for given Δx and thermal diffusivity. Then you need to choose Δt so that this condition is satisfied. For two dimensional $Fo \leq \frac{1}{4}$ and for 1D it will be $Fo \leq \frac{1}{2}$.

So now you can see that we have written the discretized equation for interior points. Similar way for any boundary point you can find depending on the different boundary condition you can write the discretize equation. Then along with all these interior points discretize algebraic equation and the boundary points discretized equation you can solve it using different iteration techniques. So just one I will discuss here.

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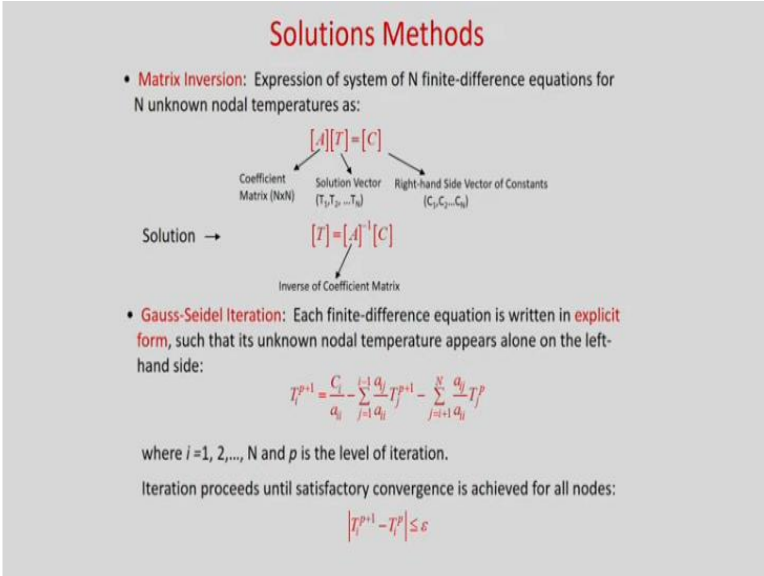

So we are writing this discretization equation for any nodal point (m,n). So there will be many points right (m,n). So if there are N number of points then obviously you will get N number of equations. N number of points mean in the x direction and in the y direction, if you consider total number of nodal points as N then we will get N number of algebraic equation. And that if you write then you can write in a matrix format and you can write in the form

$$[A][T]=[C]$$

Where, A is the coefficient matrix. So for all the neighboring points and the nodal points you have some coefficient. So all that coefficients you can bring it in the A matrix. T is the unknown. It is the solution vector and C is the right hand side vector of constants okay. So which are already known okay. So if it is known term, so that you can take it right hand side.

Now we will use implicit scheme which actually unconditionally stable. So you do not have any time restriction.

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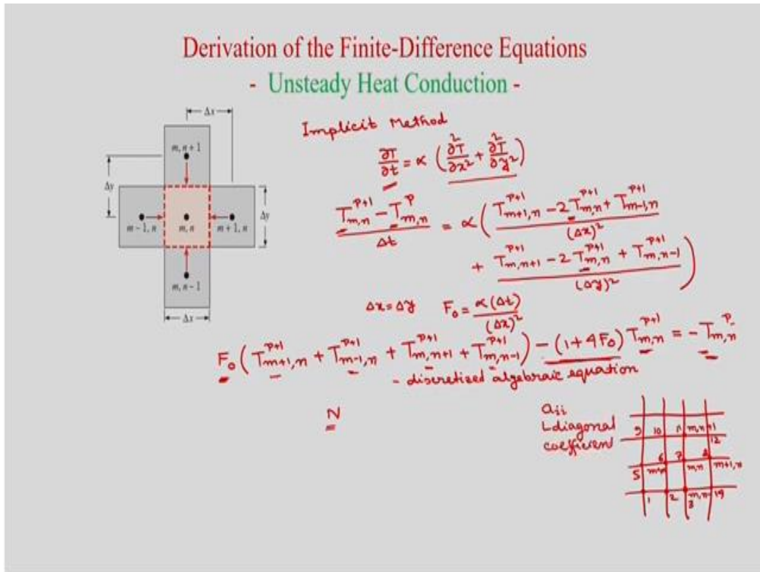

So let us use implicit method okay. So we will discretize the governing equation which is your unsteady heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

So similarly we will use for first order derivative $\frac{\partial T}{\partial t}$ backward difference and this spacial discretization we will use central difference okay. Earlier in the explicit scheme, for the time derivative we have used forward difference. But in this case, we will use backward difference and we will discretize like this. So it will be

$$\frac{T_{m,n}^{P+1} - T_{m,n}^{P}}{\Delta t} = \alpha\left(\frac{T_{m+1,n}^{P+1} - 2T_{m,n}^{P+1} + T_{m-1,n}^{P+1}}{(\Delta x)^2} + \frac{T_{m,n+1}^{P+1} - 2T_{m,n}^{P+1} + T_{m,n-1}^{P+1}}{(\Delta y)^2}\right)$$

So earlier case in explicit, you remember that we used only time level P in the right hand side, which is the previous time level. But in this case, we are using all these neighbor temperature and also the main point temperature at current time level P+1, which is unknown. So this is known as implicit scheme. So you can see here there are more than one unknown. But in the explicit scheme only one unknown was there $T_{m,n}^{P+1}$ and all other neighbor nodes were at time level P. So those were known.

Let's discretize and write in a simple form. So if you take $\Delta x = \Delta y$ and $Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$ we will get

$$Fo\left(T_{m+1,n}^{P+1} + T_{m-1,n}^{P+1} + T_{m,n+1}^{P+1} + T_{m,n-1}^{P+1}\right) - (1 + 4Fo)T_{m,n}^{P+1} = T_{m,n}^{P}$$

So now all the unknown terms we have written in the left hand side and the known term, which is at time level P, we have written in the right hand side. So this is the final discretized algebraic equation for the unsteady heat conduction equation using implicit discretization. So if you remember in the explicit scheme left hand side only one term was unknown. So that was $T_{m,n}^{P+1}$ and all right hand side term was at time level P. So it was easy to solve. But in this case now we have left hand side all are unknown terms.

So for a given node if you discretize the equation you are going to get 5 unknowns. You can see that (m+1, n), (m-1, n), (m, n+1), (m, n-1) and (m,n). So at this 5 locations, 4 neighbor points and at nodal point you have unknown terms. So those we have written in the left hand side. And right hand side we have kept which is known. So this equation is for any interior point okay.

So you can see that all these terms are having some coefficients like this is Fourier number here. And $T_{m,n}$ which is the main nodal point also known as diagonal term. For diagonal term coefficient is $1-(1+4Fo)$. And in right hand side this is known term. Because you have already calculated at time level P.

So this equation now you can solve using different methods. So say if you have a grid like this. So say let us say this is your (m,n). So it is (m+1, n). This is your (m-1, n) and this is your (m, n+1) and this is your (m, n-1) okay. So for each nodal point you are going to get this equation. So for (m,n) I have written. So if N numbers of points are there in the domain then you are going to get N equations and how many unknowns will be there. There will be N unknowns okay.

And you can include the boundary points as well but different discretized equation you have to use like we have discussed today. So if you write all this unknown terms there will be N number of unknowns and you are going to solve for N equations. So you can use some direct method where you can construct a matrix and use it or you can use some Gauss Seidel method or Jacobi method.

So let us discuss how we will solve okay.

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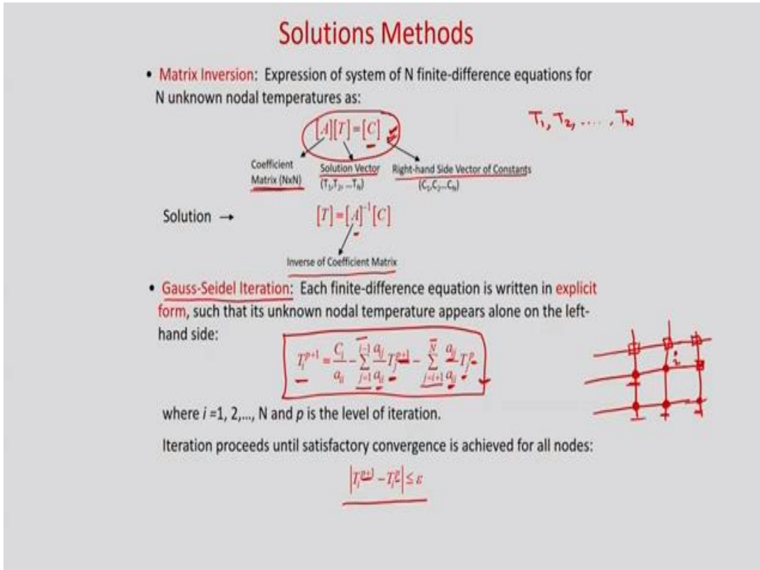

So right hand side whatever we are writing that is known term. So if we use that we can write in this form okay.

$$[A][T] = [C]$$

So all the coefficient you can keep in the A matrix okay. This is known as coefficient matrix okay and its sides will be NXN and temperature vector you will get okay. So that is solution vector which is unknown. So there we will write T_1 to T_N as the solution vector. And C, C is the right hand side vector, which is already known okay. That we are telling for each equation C_1, C_2 up to C_N you are going to get. Now you can write

$$[T] = [A]^{-1}[C]$$

When solving by direct method $[A]$ we have to take to the right hand side we will write the inverse of [A] matrix. So if you have N number of unknown points then it will be a NXN matrix. So making its inverse will be very difficult.

So another way you can find that is known as Gauss minus Seidel Iteration okay. So each finite difference equation is written in explicit such that its unknown nodal temperature appears alone on the left hand side. So you can see that all available temperature we will write at P + 1 and at known temperature is at P level. So in earlier equation this is your C right. It is your C and these are all A matrix we will find. And this is the diagonal term. So this is known as a_{ii}, because it is (m, n) point. So it is a called diagonal coefficient. So whatever best available values are there that you can take using T_j^{p+1}. So you see if you have this grid and if you are solving this point say it is, let us say i point.

So obviously you have already calculated at this point, this point and this point okay. But unknown are this point, because you have not found the temperature. So in this way whatever already you have found that you take in the current time level P +1 and whatever is unknown that you take at the previous time level okay, that is P.

So using that if you iteratively solve then you are going to solve actually $[A][T] = [C]$ matrix. So we can write in this way

$$T_i^{P+1} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{p+1} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{p}$$

You can see T_i^{P+1} for the main nodal point that is T_i and the coefficient is a_{ii} that is the diagonal coefficient. So you have to divide that, so $\frac{C_i}{a_{ii}}$ and C_i is the known term. So left hand side of the nodal term there were all neighbor points, and so those points you writing j = 1 to i -1, because j = 1 to i -1 means it will involve these points where already you have solved.

So those are available at P+1 time level. So we are writing that at P+1 time level. So a_{ij} is the coefficient of each neighbor points and a_{ii} already we have divided which is the diagonal coefficient. Summation of j = i +1 to N means those terms are not solved yet. So those temperatures are available at time level P, previous time level \P. So that we are writing T_j^p where a_{ij} is again the coefficient of temperature and a_{ii} is the diagonal coefficient.

So that way you can solve. And this is known as Gauss minus Seidel method. So it is easy to solve in this way because for each you do not need to build a matrix A and C only for each nodal point, you can solve this equation okay. Because whichever is known term already at P +1. So you take that and whatever is unknown, all not solved; that you take at time level P and iteratively you solve it unless it is converged.

And this convergence we can write

$$\left|T_i^{P+1} - T_i^P\right| \leq \epsilon$$

P+1 is the current time level and P is the previous time level. If the difference between these two temperatures is much smaller than that means it is converged okay. So that condition you can use for this Gauss minus Seidel iteration to avoid the construction of A matrix and also its inverse is very difficult to find. So this method you can use for the implicit scheme. So today I will stop here. In the next class we will solve few example problems. Thank you.