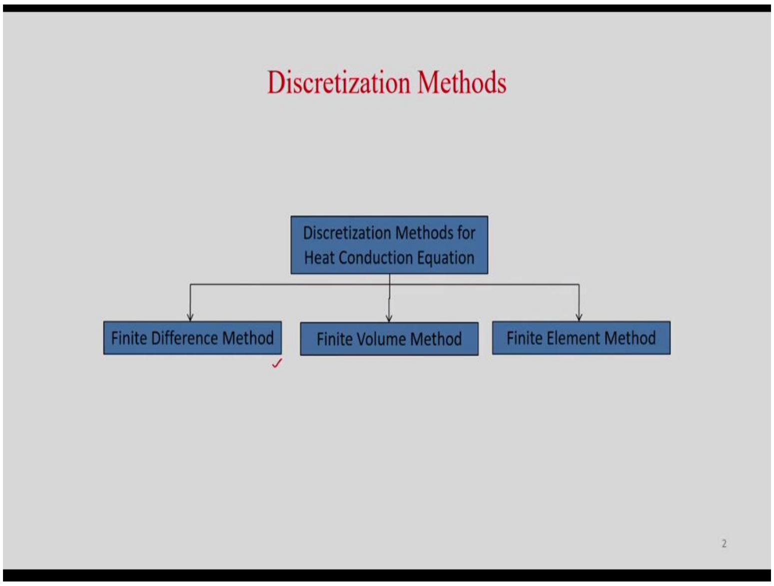
**Fundamentals of Conduction and Radiation**
**Amaresh Dalal and Dipankar N. Basu**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Guwahati**

**Lecture - 21**
**Steady Heat Conduction**

Hello everyone. So, today we will study numerical methods in conduction. So, earlier we have studied the two-dimensional heat conduction using analytical method, we used separation of variables method and also we have used graphical method. But today will use numerical methods where it is easy to apply in two dimensions and three dimensions even if you have heat generation inside the body.

So, we will use some numerical techniques to discretize the heat conduction equation. So mostly we will use finite difference method.

**(Refer Slide Time: 01:11)**



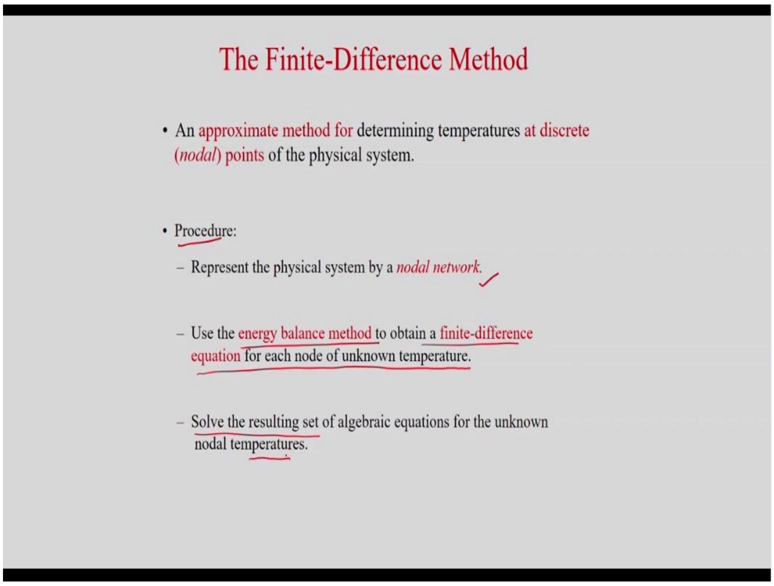

Here you can see, we can use either finite difference method or finite volume method or finite element method. In analytical method, the expression for temperature whatever we have derived, that is valid at any point x, y. So, we can find the temperature just putting the value of x, y. So, at any point you can find. But in numerical techniques what we do, we use some nodal points.

So, at these nodal points we calculate or discretize the governing equation and we get the algebraic equation. So, these are discrete equations which are valid in all those nodal points.

In finite difference method, generally we use Taylor series expansion and we find the gradients and we discretize the governing equation whereas in finite volume method, we use the integral form of the governing equation and we integrate that in a finite continuous control volume.

And in finite element method, we use weight function and integrate the governing equation in a node. So, in this study will use just finite difference method.

**(Refer Slide Time: 02:36)**

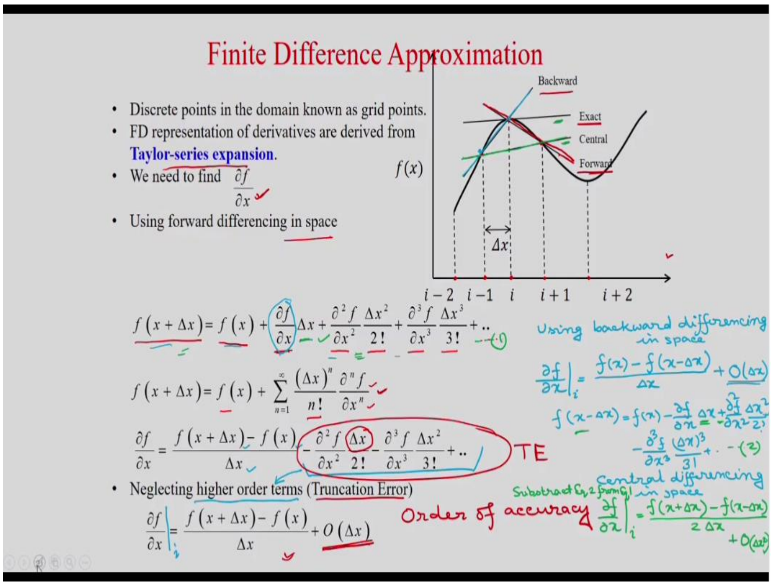



So, you can see that finite difference method is an approximate method for determining temperature at discrete or nodal points of the physical system. So, for that we need to divide the domain into some finite nodal points. So, those are known as grid. So, at every grid points we will solve the heat conduction equation and using some discretization technique. So, what we will do? You can see the procedure.

So, procedure is, we represent the physical system by a nodal network. Then, we can use either Taylor series expansion and we can discretize the equation using finite difference method and finally we can solve the final discrete algebraic equation. Or also we can use the energy balance method to obtain a finite difference equation for each node of unknown temperature.

Then, solve the resulting set of algebraic equations for the unknown nodal temperature. So, today we will find the discrete algebraic equation by using both Taylor series expansion as well as using the energy balance method.

**(Refer Slide Time: 03:56)**




So, before going to the discretizing it let us see how you find the gradient at any point. So, here you can see in this graph, this f(x) how it varies along x okay. So, these are different nodal point okay. This is point i we have given, here i+1, this is i+2 and this is i-1 and i-2. So, these are indexes in x direction.

So, f(x) is varying like this. So, you can see f(x) this black bold line, so this is your function, so let us say at ith point we want to find the gradient okay. So, that means it is a tangent we want to find, so that is $\frac{\partial f}{\partial x}$ okay. So, how we will do? So, if you just draw a tangent at this point, so that will be your exact value of that gradient, but we can use some numerical approach and we can find that if we divide this into discrete points in x direction i + 1, i - 1, i - 2 and i + 2 then you can see we can use these points and we can find say if we use forward point i + 1 and this i then using that we can find some gradient if you join this line, so you will get one gradient and that gradient is forward difference okay and backward points if you consider i-1, so this will be backward approximation and if you use this forward and backward both then it is central approximation.

So, now let us see using forward differencing in space if you use Taylor series expansion, then what we can write? So, we can write

$$f(x + \Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{\Delta x^2}{2!} + \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^3}{3!} + \cdots$$

Assuming uniform spacing in x direction.

So, this is the Taylor series expansion. So, using that let us see if we can find the gradients. So, you can write

$$f(x+\Delta x) = f(x) + \sum_{n=1}^{\infty} \frac{\Delta x^n}{n!}\frac{\partial^n f}{\partial x^n}$$

Now from the 1st equation we can write by rearranging it

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{\partial^2 f}{\partial x^2}\frac{\Delta x}{2!} - \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^2}{3!} + \cdots$$

So, this you can see that the gradient $\frac{\partial f}{\partial x}$ we have approximated using forward difference approximation $\frac{f(x+\Delta x)-f(x)}{\Delta x}$. So $\Delta x$ is the distance between these two points and $f(x+\Delta x)$ is the value at i + 1 point and f(x) is the value at ith point and the additional terms are known as higher order terms. So, these all terms in the right hand side are known as higher order terms. So, these higher order terms are known as truncation error, because when we approximate the first derivative $\frac{\partial f}{\partial x}$ and we are writing as $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ that means we are neglecting the other terms which are known as higher order term and that is known as truncation error, TE okay.

So, you can see in the truncation error, the order of $\Delta x$ in the first term will be the accuracy of the finite difference scheme. So, you can see here in the truncation error first term contains $\Delta x$. So, the order of accuracy of this forward differencing scheme is $\Delta x$ okay. So, whatever the first term in the truncation error, whatever $\Delta x$ order will be there that will be your order of accuracy of that finite difference scheme.

So, as you are using forward finite difference scheme, so we can see that first term contains $\Delta x$, so its order of accuracy is $\Delta x$ okay. So you can see this line will represent this forward differencing okay. So, you can see that it contains one forward point and one its own point i. So, we write in this way

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

So, now if we use backward differencing in space what you can write in this case, you can see the backward differencing scheme so we are using this point and this point. So you can write

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f(x) - f(x-\Delta x)}{\Delta x} + O(\Delta x)$$

Here, $f(x - \Delta x)$ actually it is at point i – 1. if you see its truncation error contains the first term $\Delta x$, so order of accuracy will be $\Delta x$. How can you do it? So, just use Taylor series expansion

$$f(x - \Delta x) = f(x) - \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{\Delta x^2}{2!} - \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^3}{3!} + \cdots$$

So if you rearrange this and find $\frac{\partial f}{\partial x}$ as we did previously it will be order of $\Delta x$.

Similarly, if you use central difference in space, you can see here two points we are considering at $f(x + \Delta x)$ and $f(x - \Delta x)$. So now if you subtract values of $f(x + \Delta x)$ and $f(x - \Delta x)$ we will get

$$\begin{aligned} f(x + \Delta x) - f(x - \Delta x) \\ &= f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{\Delta x^2}{2!} + \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^3}{3!} + \cdots \\ &- \left\{ f(x) - \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{\Delta x^2}{2!} - \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^3}{3!} + \cdots \right\} \end{aligned}$$

Rearranging it, we will get

$$f(x + \Delta x) - f(x - \Delta x) = 2\left\{ \frac{\partial f}{\partial x}\Delta x + \frac{\partial^3 f}{\partial x^3}\frac{\Delta x^3}{3!} + \cdots \right\}$$

Here the $\Delta x^2$ terms will cancel out as they will have opposite signs. Now, dividing both sides by $2\Delta x$.

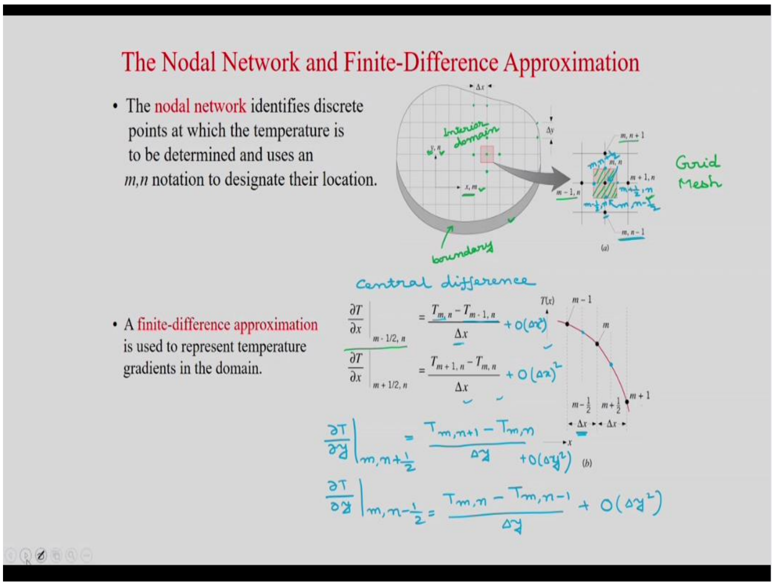
$$\Rightarrow \frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(\Delta x^2)$$

As the $\Delta x^2$ terms cancel out we will get the TE in the order of $\Delta x^2$ here. So it will be the second order accurate. And $2\Delta x$ will be the distance between $x + \Delta x$ and $x - \Delta x$ . As you can see the difference of value of the function $f$ at these two points we are using on the numerator.

So, you can see that in finite differencing scheme if you use forward and backward difference that will give order of accuracy 1 okay or first order accurate. And when we use the central differencing scheme, then it will be second order accurate. So in this graph you can see this central difference scheme line whatever you have drawn, this is actually mostly parallel to the exact solution.

So, it will give more correct evaluation of that gradient $\frac{\partial f}{\partial x}$ okay rather than the forward and backward difference and obviously you can see that order of accuracy is higher in central difference scheme so it will give more accurate result compared to the other schemes okay. So, with this introduction you will be able to find the derivative in the heat conduction equation, so let us see that.

**(Refer Slide Time: 15:14)**




Here, you see this is the interior domain and these are all boundary. So, now here first what you have to do, you have to divide into few discrete points. Those are known as nodal points which we have done. You see these lines we have drawn. So, at interior point, if you consider one nodal point, so it is having 4 neighbor points right. If (m, n) is the main point then you have (m + 1, n) in right side, left side you will have (m - 1, n) and on the top it will be (m, n + 1) and at the bottom (m, n – 1). So, (m, n) are the indices of that main nodal point. So, obviously if you consider any interior points, you will have 4 neighbors except near to the boundary points. And the distance between these neighboring points will be $\Delta x$ in the horizontal direction and $\Delta y$ in the vertical direction.

And you can see when we will find some value T at (m, n) point, that is actually average value in this hatched domain okay. So, now we have discretized or we have got the discrete points. So those are known as grid or mesh. So, in this case, you can see we have considered any distance between two nodal points in x direction is $\Delta x$ and in y direction it is $\Delta y$ and m indices we are representing in the x direction and n we are representing the index of this y direction.

So, now let us in the horizontal direction represent the central point between (m, n) and (m+1, n) as (m+1/2, n) and the central point between (m, n) and (m-1, n) as (m-1/2, n). These two points will lie on the boundary of our hatched region. Now let's take central differencing scheme to find the value of $\frac{\partial T}{\partial x}$. So it will be

$$\left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2},\ n} = \frac{T_{m.\ n} - T_{m-1,\ n}}{\Delta x} + O(\Delta x^2)$$

And

$$\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2},\ n} = \frac{T_{m+1.\ n} - T_{m,\ n}}{\Delta x} + O(\Delta x^2)$$

Here we are taking central differencing scheme to get higher order accuracy. We have followed similar discretization approach as before. $\Delta x$ is the distance between the two points whose difference we are taking on the right side that is between (m, n) and (m+1, n) which is same as between (m, n) and (m-1, n). So, now similarly you can write taking similar assumptions in vertical direction defining points (m, n+1/2) and (m, n-1/2) which again lie on the boundary of our hatched region. Taking central differencing

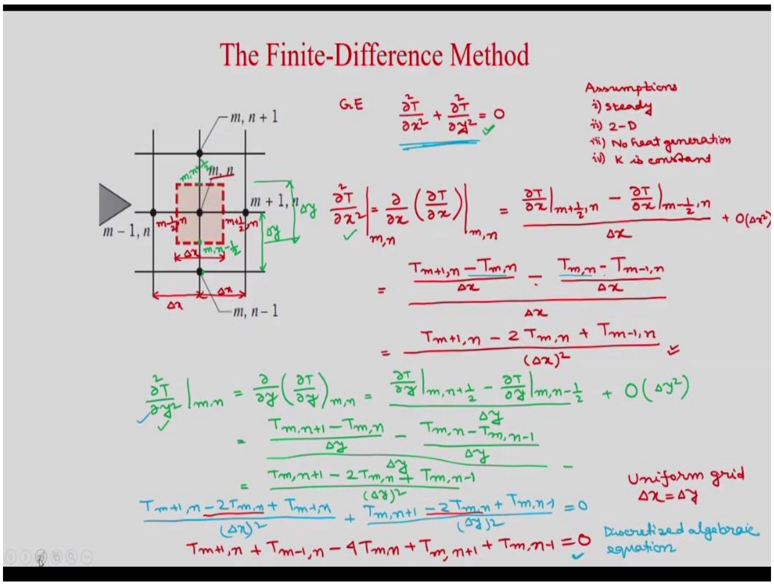
$$\left.\frac{\partial T}{\partial y}\right|_{m,\ n+\frac{1}{2}} = \frac{T_{m.\ n+1} - T_{m,\ n}}{\Delta y} + O(\Delta y^2)$$

And

$$\left.\frac{\partial T}{\partial x}\right|_{m,\ n-\frac{1}{2}} = \frac{T_{m.\ n} - T_{m,\ n-1}}{\Delta y} + O(\Delta y^2)$$

So, central difference we are using, so these are second order accurate okay. So, now the first derivative we have found. Now, what is the governing equation?

**(Refer Slide Time: 22:44)**




So governing equation for 2D, steady state heat conduction without heat generation and constant thermal conductivity will be

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

So, now you have to discretize this using finite difference method. So, if you use finite difference method, so the second derivative what we can write?

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

So, now if we write at point (m, n) it will be

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \left.\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\right|_{m,n} = \frac{\left.\frac{\partial T}{\partial x}\right|_{m+\frac{1}{2},n} - \left.\frac{\partial T}{\partial x}\right|_{m-\frac{1}{2},n}}{\Delta x} + O(\Delta x^2)$$

So here also we are taking central differencing scheme for finding the derivative of the variable $\frac{\partial T}{\partial x}$. $\Delta x$ is the distance between points and (m+1/2, n) and (m-1/2, n) as they are the central points between (m, n) and (m+1, n) and the central point between (m, n) and (m-1, n). So, what is the order of accuracy? Central difference order of accuracy is second order accurate okay. So, now you substitute this first derivative with the expression in last slide we have derived. And what is the order of accuracy of that first derivative finite difference approximation? That is also second order accurate, so overall it will be second order accurate scheme okay. So, now we can write it

$$\left.\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\right|_{m,n} = \frac{\left\{\frac{T_{m+1.\ n} - T_{m,\ n}}{\Delta x}\right\} - \left\{\frac{T_{m.\ n} - T_{m-1,\ n}}{\Delta x}\right\}}{\Delta x}$$

So, you can see the order of accuracy is second order. So, now what you can write this, we can write it

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\bigg|_{m,n} = \frac{T_{m+1.\ n} - 2T_{m,\ n} + T_{m-1,\ n}}{(\Delta x)^2}$$

So this is the second derivative we have derived using central difference okay and it is a second order accurate. So, I think you understood how we have derived it okay. So, this is the central difference approximation we have used. Similarly, you write second derivative in y direction okay

$$\frac{\partial^2 T}{\partial y^2}\bigg|_{m,n} = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)\bigg|_{m,n} = \frac{\frac{\partial T}{\partial y}\Big|_{m,n+\frac{1}{2}} - \frac{\partial T}{\partial x}\Big|_{m,n-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

We can find the distance between points (m, n+1/2) and (m, n-1/2) to be $\Delta y$ following previous method. Hence

$$\frac{\partial^2 T}{\partial y^2}\bigg|_{m,n} = \frac{\left\{\frac{T_{m.\ n+1} - T_{m,\ n}}{\Delta y}\right\} - \left\{\frac{T_{m.\ n} - T_{m,\ n-1}}{\Delta y}\right\}}{\Delta y}$$

$$= \left\{\frac{T_{m.\ n+1} - 2T_{m,\ n} + T_{m,\ n-1}}{(\Delta y)^2}\right\}$$

Here also final order of accuracy will be of $\Delta y^2$. Now finally we can write

$$\frac{\partial^2 T}{\partial x^2}\bigg|_{m,n} + \frac{\partial^2 T}{\partial y^2}\bigg|_{m,n} = \frac{T_{m+1.\ n} - 2T_{m,\ n} + T_{m-1,\ n}}{(\Delta x)^2} + \frac{T_{m.\ n+1} - 2T_{m,\ n} + T_{m,\ n-1}}{(\Delta y)^2} = 0$$

So, now we have not assumed anything that whether $\Delta x = \Delta y$ okay but it is a general expression we have written where $\Delta x$ is constant okay. So, it is uniform in x direction. And $\Delta y$ is constant in y direction. So, $\Delta y$ is uniform okay. But now let us assume that $\Delta x = \Delta y$ then how will get the expression? $\Delta x = \Delta y$ means uniform grid okay. So, as $\Delta x = \Delta y$ we can take them common from both the terms and they can vanish if we take it to the right side. So final term we can write

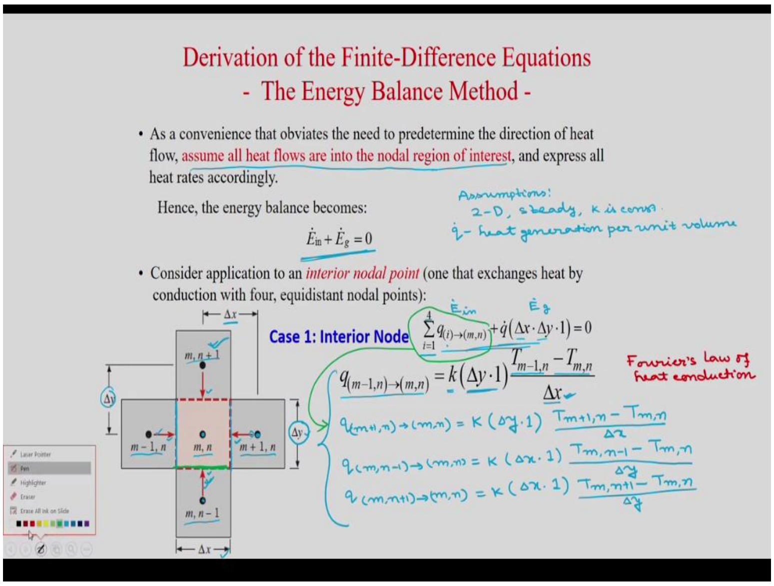
$$T_{m+1.\ n} + T_{m-1,\ n} - 4T_{m,\ n} + T_{m.\ n+1} + T_{m,\ n-1} = 0$$

So, this is the final discretized algebraic equation. So, we started with partial differential equation okay, we started with partial differential equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$, but using the finite difference scheme, we could write the discretized algebraic equation okay.

So, at any point (m, n) at the interior of the domain we have written this expression okay which is a discretized algebraic equation. So you can write this equation for all the interior nodal points for (m, n) which you can solve using some numerical techniques like Gauss Seidel or Jacobi method or some direct method like conjugate gradient method or BiCGSTAB okay.

So, depending on your problem, you can solve this equation okay. So, now for interior domain using $\Delta x = \Delta y$ you could write this. Now, we have to treat the boundary condition also right because there are boundary points where you will not have the 4 neighbor points depending on whether it is corner point or side point. Depending on that you will get one interior neighbor but other will be on the surface or outside there will be some boundary condition, so let us see that.

**(Refer Slide Time: 34:45)**




So, before going to that we will derive the same algebraic equation using energy balance method okay. We will also find the algebraic equation for both interior domain using energy balance method as well as at the boundary points using energy balance method. So, you can see that this is your nodal points of interior nodal points (m, n) okay. And 4 neighbor points are (m + 1, n), (m - 1, n), (m, n+ 1) and (m, n- 1) okay and this is $\Delta x$ and this is the $\Delta y$ . Now we will do the energy balance at this control volume okay. So let's consider the domain inside the red dotted line about the (m, n) node. Here the red arrows are pointing towards the domain. So we will assume whatever heat is flowing it is coming into the domain because we do not know whether heat is going out or in. If the heat is leaving the domain then the value of q will become negative.

Now, what is the energy balance method? So, assuming that it is a steady state and no heat out, so you will get

$$\dot{E}_{in} + \dot{E}_g = 0$$

So, let us assume that there is a heat generation per unit volume okay and that is $\dot{q}$. And

$$\dot{E}_g = \dot{q}V = \dot{q}(\Delta x \Delta y)$$

Where, V is the volume of a unit cell represented by the red dotted line. Here we are considering unit width so that $V = \Delta x \Delta y$. So, that is the assumption we are taking and obviously we have assumed that it is 2D and steady and another point will assume that K is constant. So, these are the assumptions. And, there is no heat out because all the heat we have taken inflow towards the nodal point.

Now as we are assuming heat flow is only towards inside net heat flow towards node (m, n) will be summation of heat flow from each of the neighboring nodes that is (m + 1, n), (m - 1, n), (m, n+ 1) and (m, n- 1). That we can write as

$$\dot{E}_{in} = \sum_{i=1}^{4} q_{(i)\to(m,n)}$$

Hence the energy balance will become

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q}(\Delta x \Delta y) = 0$$

So, now you can write the heat flow from this left neighbor point (m - 1, n) to (m, n) using Fourier's law of heat conduction okay. Here, what is the area? In this case, you can see, in this case it is $\Delta y$ okay and per unit width, perpendicular to the plane of paper, so width will be 1. Hence area will be $(\Delta y. 1)$. Now heat is flowing from (m - 1, n) to (m, n) so temperature difference we can write as $T_{m-1,n} - T_{m,n}$. Writing the temperature difference like this also allows us the neglect the –ve sign associated with Fourier law. And the distance between the two nodal points is $\Delta x$. So finally we can write

$$q_{(m-1,n)\to(m,n)} = k(\Delta y. 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

So, now similarly you can write for other neighbor points.

$$q_{(m+1,n)\to(m,n)} = k(\Delta y. 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

Here, the area will be same and temperature gradient will be $T_{m+1,n} - T_{m,n}$ as heat is going from (m +1, n) to (m, n). Similarly, from (m, n- 1) to (m, n) the heat transfer rate is

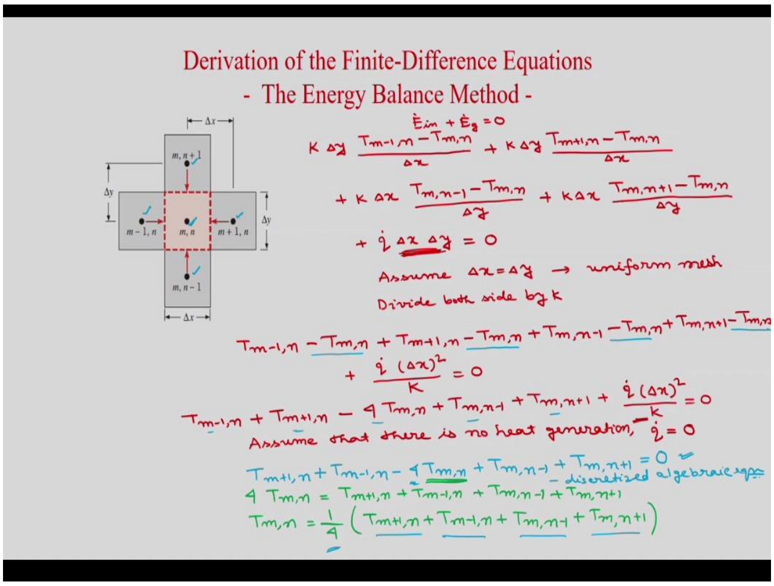
$$q_{(m,n-1)\to(m,n)} = k(\Delta x.1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

Here the area becomes $(\Delta x.1)$as it is the horizontal plane and the distance becomes$\Delta y$. Similarly we can write

$$q_{(m,n+1)\to(m,n)} = k(\Delta x.1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

Now, we can put all these values in energy balance equation okay.

**(Refer Slide Time: 43:40)**




So, we can write

$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q}(\Delta x\Delta y) = 0$$

$$\Rightarrow q_{(m-1,n)\to(m,n)} + q_{(m+1,n)\to(m,n)} + q_{(m,n-1)\to(m,n)} + q_{(m,n+1)\to(m,n)} + \dot{q}(\Delta x\Delta y) = 0$$

$$\Rightarrow k(\Delta y.1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k(\Delta y.1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k(\Delta x.1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$+ k(\Delta x.1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + \dot{q}(\Delta x\Delta y) = 0$$

So the first 4 terms in the left hand side is for the $\dot{E}_{in}$ and last term is heat generation term. Now if we divide the whole equation by k and assume uniform mesh or $\Delta x = \Delta y$ to get a simple form we can write

$$\Rightarrow T_{m-1,n} - T_{m,n} + T_{m+1,n} - T_{m,n} + T_{m,n-1} - T_{m,n} + T_{m,n+1} - T_{m,n} + \dot{q}\left(\frac{\Delta x^2}{k}\right) = 0$$

Here we are writing $\Delta x\Delta y = \Delta x^2$ as both are same

$$\Rightarrow T_{m-1,n} + T_{m+1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} + \dot{q}\left(\frac{\Delta x^2}{k}\right) = 0$$

Now assume that there is no heat generation hence $\dot{q} = 0$.

$$\Rightarrow T_{m+1,n} + T_{m-1,n} - 4T_{m,n} + T_{m,n-1} + T_{m,n+1} = 0$$

So, this is the final discretized algebraic equation okay. So, this you can see that we started with the energy balance and considering that all the heat flow is from the neighbor to the nodal points then we have derived these equation; this is the discretized algebraic equation and this is you can see that is same as whatever we have derived using finite difference method okay.

So, using finite difference method where we have used Taylor series expansion, so using the approximation $\Delta x = \Delta y$ and there is no heat generation whatever discretized equation we derived same discretized equation we got using energy balance method okay.

Now, say we want to find the temperature at (m, n) okay. So, we can write

$$4T_{m,n} = T_{m+1,n} + T_{m-1,n} + T_{m,n-1} + T_{m,n+1}$$

$$T_{m,n} = \frac{1}{4}\left(T_{m+1,n} + T_{m-1,n} + T_{m,n-1} + T_{m,n+1}\right)$$

So, all these temperatures at (m + 1, n), (m - 1, n), (m, n+ 1) and (m, n- 1) are all temperatures at the neighboring points and T (m, n) is the main nodal point, so what will be the main nodal point temperature? It will be just the arithmetic average.

So, you can see that if you have a uniform mesh that means $\Delta x = \Delta y$ and if there is no heat generation then in two-dimensional case, the temperature at main nodal point T (m, n) is equal to arithmetic average of all the neighbor temperatures.

So, today I will stop here. In the next class, we will use this energy balance method and treat the different boundary conditions. Thank you.