Fundamentals of Conduction and Radiation
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology – Guwahati
Lecture - 20
Semi-infinite Solid

Hello everyone. So in last class we have studied the lumped system analysis and found the temperature distribution with time. So we have solved two problems and we learnt how to solve a problem. In this class now we will consider a simple case of distributed system. What is distributed system? We have already studied that in distributed system temperature will vary in space as well as time. One such kind of application is semi-infinite medium.

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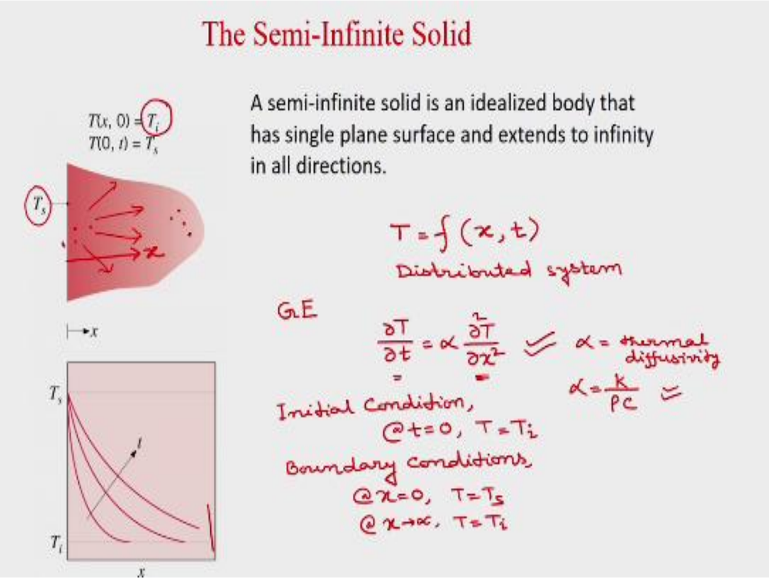

So we will consider semi-infinite solid so what is semi-infinite solid? A semi-infinite solid is an idealized body that has single plane surface and extend to infinity in all other directions. Consider this body. So this is having only single plane surface and in any direction you go okay; in this direction or in this direction or in this direction in any direction you go it is having infinite.

So in this case it is known as Semi-infinite Medium. So it is having a plane surface and in any direction it is infinity. Simple example is our earth surface. So earth surface is a plane surface in a particular place and in all other direction it continues to infinity. So this is one example of Semi-infinite Solid. So you can see that from the sunshine it gets heat okay, and it is transmitted to the soil. This phenomenon can be explained by using earth as a semi-infinite medium.

Now in Semi-infinite Solid what actually happens? Say the temperature at the surface if you change it, its effect will be just near to the surface only, it will not go deep inside okay. So if T_s is the surface temperature and if you are willing to calculate temperature near to this surface inside this body then you will find that it is a function of x and t or space and time.

But if you go away from this surface say at this location or at this location, so these locations are very far away from the surface. So its temperature will be the same as the initial temperature. The temperature which actually you have given at the surface that's effect has not reached up to this point okay. So that is known as a Semi-infinite medium. So if your surface temperature is T_s and initial temperature T_i in this case, you can see that your temperature will penetrate inside, but at x tends to infinity up to sometime t it has not reached the effect of the boundary temperature. So that is known as Semi-infinite medium. Today we will consider such Semi-infinite medium where your temperature is function of one space coordinate and time okay, so we will consider T as function of one space coordinate let us say x and t. So x is measured perpendicular from this surface like this okay.

So this is a distributed system okay because your temperature is function of space as well as time. So in this case now what is the governing equation? So governing equation already you have studied.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Where,

$$T = f(x, t)$$

So the assumptions are that there is no heat generation obviously, and it is a distributed system and only it is a one-dimensional transient heat transfer. So this is the governing partial differential governing equation okay; left side is your temporal term and right side is your diffusion term and α is the thermal diffusivity,

$$\alpha = \frac{k}{\rho C}$$

Where, k is the thermal conductivity ρ is the density and C is the heat capacity of the solid medium okay.

So now you can see that here T is function of x and t. Now as it is a transient heat conduction you need some initial condition as well as boundary condition because you will actually march in time so at t = 0 you need to specify what is the temperature distribution inside the solid. So that is known as initial condition. So what is the initial condition in this case?

$$at\ t = 0; T = T_i$$

So now you how many boundary conditions do we need in this case? You see this equation; so it 2nd order partial differential equation okay, so we need two boundary conditions. So let us write boundary conditions,

$$at\ x = 0; T = T_s$$

$$at\ x \rightarrow \infty; T = T_i$$

We will assume in this case that your surface temperature remains constant and that is T_s okay so that is reflected in the first boundary condition. Now, when x tends to infinity where the effect of surface temperature has not reached. So that means it will be in the initial temperature so that is reflected in the second boundary condition.

Now we need to find the temperature distribution with respect to time and space as well as what is the heat transfer rate okay because that we are interested in okay. So let us find.

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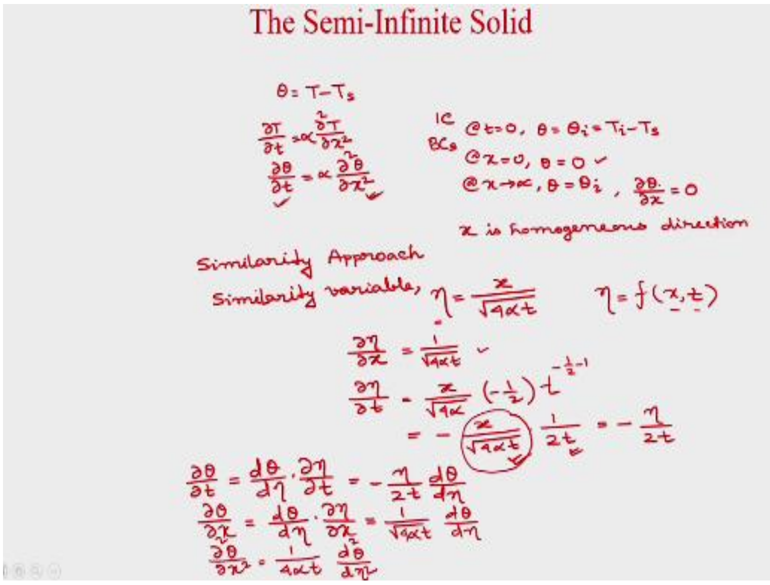

Now we will write

$$\theta = T - T_s$$

Transforming the governing equation like we have done in earlier classes

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$$

Now the initial and boundary conditions are

$$at\ t = 0; \theta_i = T_i - T_s$$

$$at\ x = 0; \theta = 0$$

$$at\ x \to \infty; \theta = \theta_i, or \frac{\partial \theta}{\partial x} = 0$$

So now how to solve this partial differential equation? Let's check whether we can apply separation of variable method here. Looking at the boundary conditions we can find that in x direction both the end boundary conditions are homogeneous. So we can say that x direction is homogeneous direction. But the second boundary condition is not finite. As there x tends to infinity. So we cannot apply separation of variable method as the homogeneous direction is not finite.

So the other method what we can use just to convert this partial differential equation to ordinary differential equation is similarity approach. What we do in similarity approach? We find a similarity variable okay such a way that this similarity variable is function of two dependent variables okay and we can convert this partial differential equation to ordinary differential equation.

So in this case we will use the similarity variable

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

Where, α is the thermal diffusivity and t is the time okay. How we got this similarity variable that we will not cover in this lecture. You can see that

$$\eta = f(x, t)$$

So we are actually converting this two dependent variables x and t to η. Hence, we can convert this partial differential equation to ordinary differential equation. Now,

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}$$

$$\frac{\partial \eta}{\partial t} = \frac{x}{\sqrt{4\alpha}}\left(-\frac{1}{2}\right) t^{-\frac{1}{2}-1}$$

Multiplying $t^{\frac{1}{2}}$ in both numerator and denomenator

$$= -\frac{x}{\sqrt{4\alpha t}} \frac{1}{2t} = -\frac{\eta}{2t}$$

Now,

$$\frac{\partial\theta}{\partial t} = \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial t} = -\frac{\eta}{2t}\frac{d\theta}{d\eta}$$

So you can see we have converted the partial derivative to ordinary differential as we have only one independent variable now. Similarly

$$\frac{\partial\theta}{\partial x} = \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{d\theta}{d\eta}$$

And,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{4\alpha t}\frac{\partial^2\theta}{\partial\eta^2}$$

Now you put these values in the governing equation. So if you put so what you will get.

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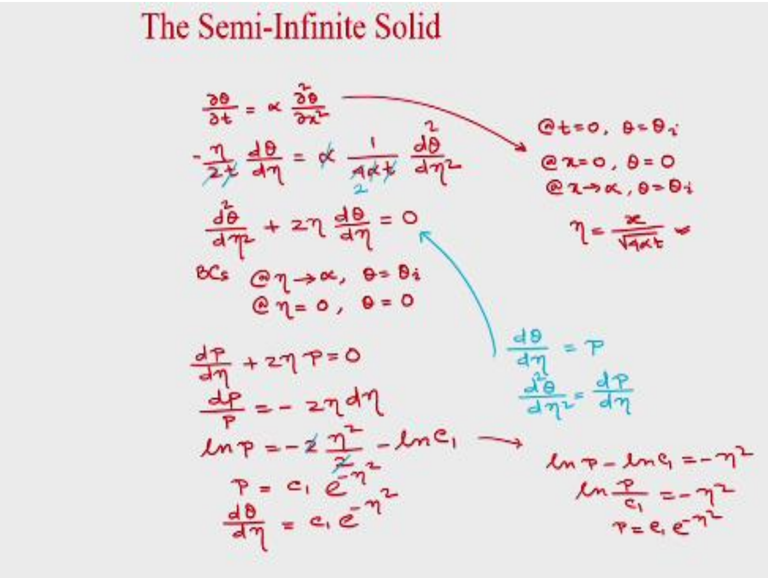

So our equation was

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}$$

Now we can write one ordinary differential equation by putting the values of the partial differentials which is

$$\Rightarrow -\frac{\eta}{2t}\frac{d\theta}{d\eta} = \alpha\frac{1}{4\alpha t}\frac{\partial^2\theta}{\partial\eta^2}$$

Rearranging the terms

$$\Rightarrow \frac{\partial^2\theta}{\partial\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0$$

So now you see that partial differential equation we have converted to ordinary differential equation using similarity approach. So this is your governing ordinary differential equation okay. So now including initial condition and boundary condition we have three conditions.

But we need to solve this equation with two boundary conditions. So now if we transform them one boundary condition will collapse.

So let us see. So we have seen that

$$at\ t = 0; \theta = \theta_i = T_i - T_s$$

$$at\ x = 0; \theta = 0$$

$$at\ x \to \infty; \theta = \theta_i$$

So now as $\eta = \frac{x}{\sqrt{4\alpha t}}$ they will be

$$at\ t = 0; \eta \to \infty;\ \theta = \theta_i$$

$$at\ x = 0; \eta = 0; \theta = 0$$

So now we'll be able to solve this problem right; so just you integrate it put the boundary condition solve it you will get the temperature distribution.

So let us do the steps. So now what you can do let us write that let us assume that

$$\frac{d\theta}{d\eta} = P \Rightarrow \frac{\partial^2\theta}{\partial\eta^2} = \frac{dP}{d\eta}$$

So if you put it there okay what you are going to get? So you can write it now

$$\frac{dP}{d\eta} + 2\eta P = 0$$

$$\Rightarrow \frac{dP}{P} = -2\eta\ d\eta$$

Integrating

$$\Rightarrow \ln P = -2\frac{\eta^2}{2} - \ln C_1$$

$$\Rightarrow P = C_1 e^{-\eta^2}$$

$$\Rightarrow \frac{d\theta}{d\eta} = C_1 e^{-\eta^2}$$

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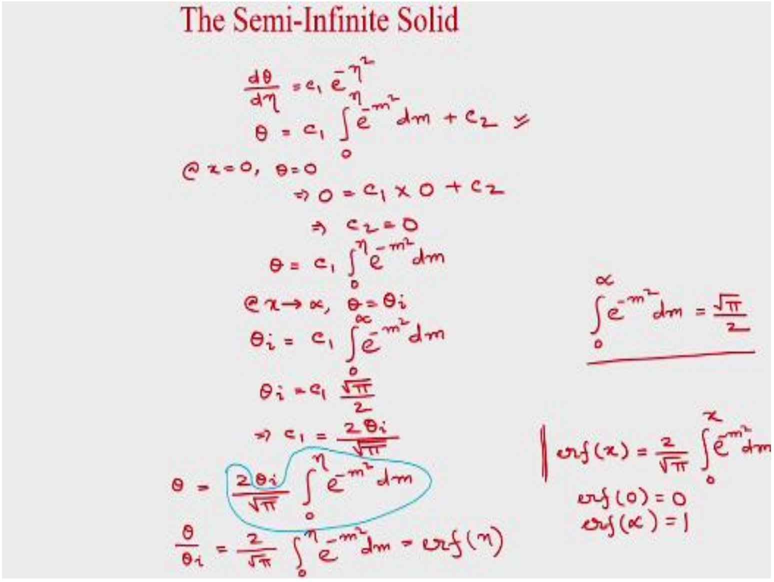

So integrating once more

$$\theta = C_1 \int_0^{\eta} e^{-m2}\, dm + C_2$$

Here, C_1 and C_2 are integration constants and m is a dummy variable used instead of η. Now you need to find these constants okay. So now what are boundary conditions?

$$at\ t = 0; \eta \to \infty;\ \theta = \theta_i$$

$$at\ x = 0; \eta = 0; \theta = 0$$

From the second boundary condition we can write

$$0 = C_1 \times 0 + C_2$$

$$\Rightarrow C_2 = 0$$

From the 1st boundary condition

$$\theta_i = C_1 \int_0^{\infty} e^{-m2}\, dm$$

The term inside integration if we integrate from 0 to ∞ we will get $\frac{\sqrt{\pi}}{2}$. So it becomes

$$\theta_i = \frac{C_1\sqrt{\pi}}{2}$$

$$\Rightarrow C_1 = \frac{2\theta_i}{\sqrt{\pi}}$$

So now you put it in the original equation

$$\theta = \frac{2\theta_i}{\sqrt{\pi}} \int_0^{\eta} e^{-m2}\, dm$$

So now we will define the Gaussian error function. What is error function? It is written as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-m^2}\,dm$$

So this is the definition of error function and $\mathrm{erf}(0)$ is 0 $\mathrm{erf}(\infty)$ is 1. So now we can write

$$\frac{\theta}{\theta_i} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-m^2}\,dm = \mathrm{erf}(\eta)$$

So we got the temperature distribution okay; so let's write it in terms of x and t.

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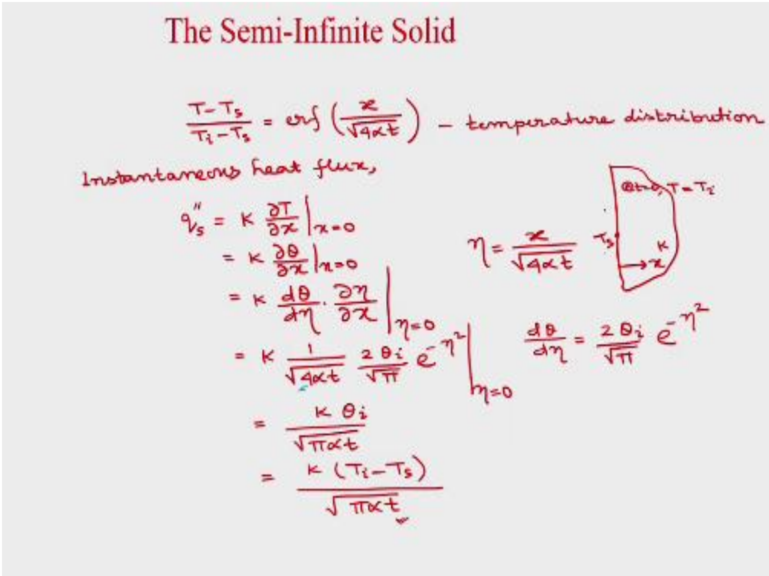

So it is

$$\frac{T - T_s}{T_i - T_s} = \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

So this is the temperature distribution inside the Semi-infinite Solid okay. This temperature variation I have already shown here you can see so this temperature variation will look like this okay. It will decay like this okay and at x tends to infinity it will remain at initial temperature. So now we are interested to find the instantaneous heat transfer rate. So what will be instantaneous heat transfer rate? So what is that at a particular time okay what will be your heat transfer rate?

So say this was your Semi-infinite Solid x is measured from here okay at t = 0 you have T= T_i and your surface temperature is T_s right and it is having thermal conductivity k. So now we will use Fourier law of heat conduction to find the heat transfer rate. So at x = 0 what is the heat transfer rate. Let's write that as q_s'' or heat flux as we do not know the area of heat transfer. Now you see here direction of q_s'' is opposite to the positive x direction. Hence, we can write

$$q"_s = k\frac{\partial T}{\partial x}\Big|_{x=0}$$

$$= k\ \frac{\partial \theta}{\partial x}\Big|_{x=0}$$

Now as $\eta = 0$ when x=0; we can write

$$q"_s = k\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial x}\Big|_{\eta=0}$$

Now

$$\frac{d\theta}{d\eta} = C_1 e^{-\eta^2} = \frac{2\theta_i}{\sqrt{\pi}} e^{-\eta^2}$$

And,

$$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}$$

Putting it in q''_s

$$q"_s = \frac{1}{\sqrt{4\alpha t}}\frac{2\theta_i}{\sqrt{\pi}} e^{-\eta^2}\Big|_{\eta=0}$$

$$= \frac{k\theta_i}{\sqrt{\pi\alpha t}} = \frac{k(T_i - T_s)}{\sqrt{\pi\alpha t}}$$

So that is the instantaneous heat flux from the surface. If you multiply it with area then you will get the instantaneous heat transfer rate. So you can see that time is involved in the equation so at particular time you will get this heat transfer rate. So we considered one semi-infinite medium and we considered a simplified case where surface temperature T_s is constant. So with that we have used similarity variable approach.

And similarity variable we have used $\eta = \frac{x}{\sqrt{4\alpha t}}$, and with that we converted the partial differential equation to ordinary differential equation and with those initial conditions and boundary conditions we solve these equation and found the constant C_1 and C_2 with that we got the temperature distribution. So this error function is Gaussian error function where its value at 0 is 0 and at ∞ it is 1.

Okay and after that we found the instantaneous heat flux. So this is a simplified case. Now let's consider other cases, we will not derive it, but I will show the expression.

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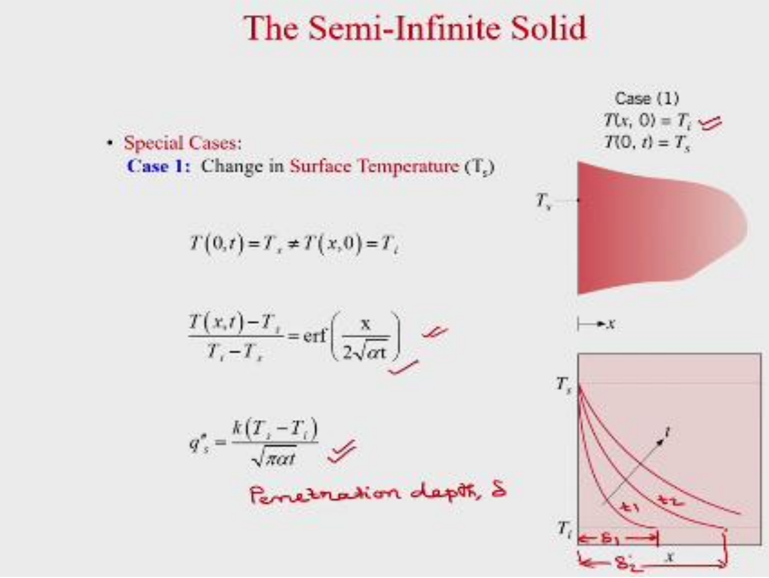

So you can see here, this is the case we consider just now

$$T(0,t) = T_s \neq T(x,0) = T_i$$

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$$q''_s = \frac{k(T_i - T_s)}{\sqrt{\pi\alpha t}}$$

Now if you increase the time obviously your heat will diffuse inside more. So you can see this is known as penetration depth (δ).

So if at some time t_1 this is the your penetration depth δ_1 okay if it is t_2 this is the temperature distribution and it is diffused up to this then after that there is it is at initial temperature so this is your penetration depth δ_2 okay. Similarly, if you it reaches T_i at in infinity so you will get that δ_3 so this is now penetration depth.

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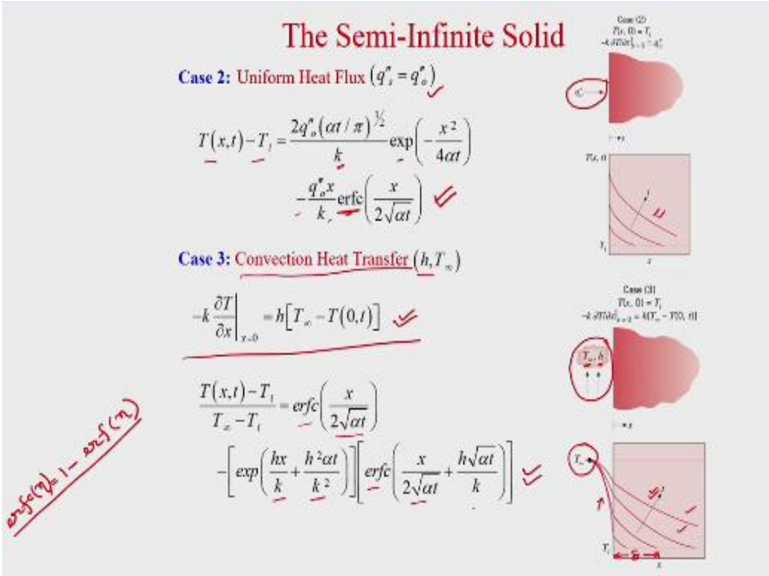

So now let's take the other condition of uniform heat flux at the surface. So you can see here you are giving a uniform heat flux $q"_0$. So you can see that if you solve it you are going to get this expression

$$T(x,t) - T_i = \frac{2q"_0\left(\frac{\alpha t}{\pi}\right)^{\frac{1}{2}}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q"_0 x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Here, it is written complementary error function. What is complimentary error function? Your error function complementary is nothing but

$$\operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta)$$

So this is the temperature distribution and you can find what is the heat flux just using the Fourier's Law. And you can see how the temperature distribution looks like. So earlier case at x = 0, T_s was constant, but here now T_s will vary as you are giving a constant heat flux.

Now the other case you see convection heat transfer. So you have one convective environment where heat is lost due to convection and your T_∞ is the ambient temperature, h is the heat transfer coefficient okay. And your boundary condition in this case will be

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0,t)]$$

So if you consider this case then temperature distribution you will get

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

So you can see so you can put these boundary conditions and find what is the temperature distribution as well as what is the heat transfer or instantaneous heat flux okay. You can see the temperature distribution for the 3rd case. Here, T_∞ is constant okay, but now you have a convective heat loss from here so convection boundary condition. And temperature will vary like this okay, so this is at a particular time this is the penetration depth okay. So for three cases I have shown the temperature distribution.

So now we got the temperature distribution in function of space as well as time which is a distributed system and we have carried out this analysis for a simplified case which is a semi-infinite medium. So let us solve one problem then we will understand that how you can apply this knowledge, so let us consider this problem.

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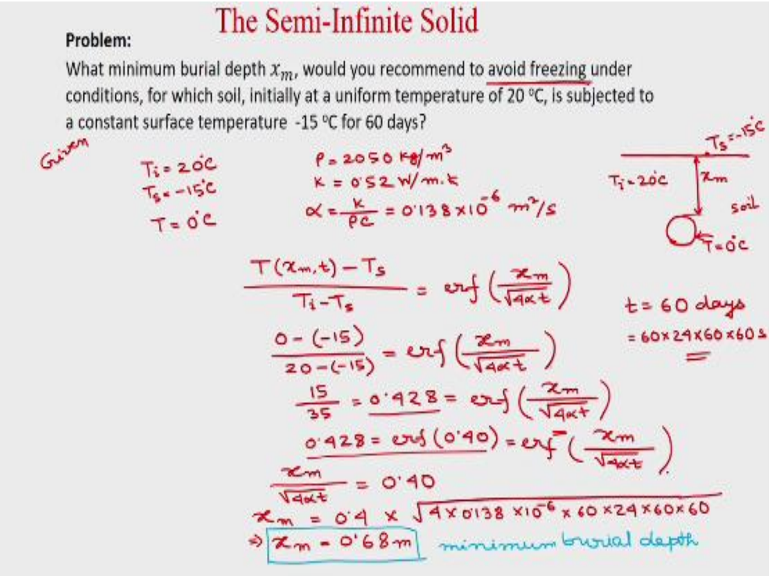

I am reading this problem; what minimum burial depth x_m would you recommend to avoid freezing under conditions for which soil initially at a uniform temperature of 20 ºC is subjected to a constant surface temperature - 15 ºC for 60 days okay. So one pipeline is there okay it is buried under the Earth's surface. So now your initial temperature is given, initial temperature T_i is 20 ºC okay, but on the surface of the earth it is having temperature -15 ºC okay.

So T_s is -15 ºC okay, and the other properties are given. So ρ is 2050 kg/m^3, thermal conductivity k is given as 0.52 W/mK and α is given as 0.138 X10^{-6} m^2/s okay. So now we have to find the distance from the earth surface where this pipe we will keep such that it will not go for freezing okay.

Avoid freezing means it should not go below 0 ºC. You see surface temperature is -15 ºC. So if it goes below 0 ºC at this pipe surface then your water will freeze. To avoid that you have to find the minimum depth from the earth surface to the pipe such that your temperature T will be 0 ºC. so this is the condition.

So if you draw this, so let us say this is your art surface where T_s is given okay and you have one pipe okay and this is the x_m you have to find. And initial T_i is 20 ºC okay T_s is -15 ºC okay, and here now you have to find T such that it will not go below 0 ºC okay. So now you know the temperature distribution

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

So now in this expression in left hand side all the terms you know in the right hand side α you know t also you know because t is given as 60 days. So all the terms you know here. You have to find now xm.

$$\frac{0 - (-15)}{20 - (-15)} = 0.428 = \text{erf}\left(\frac{x_m}{\sqrt{4\alpha t}}\right)$$

So now you have to know the chart of error function okay. So any textbook which we are following you can find the value of error function table. So for which value x this error function value will be 0.428 that we have to find. If you find that, it will be

$$0.428 = \text{erf}(0.40) = 0.428 = \text{erf}\left(\frac{x_m}{\sqrt{4\alpha t}}\right)$$

So now you can write it from

$$\frac{x_m}{\sqrt{4\alpha t}} = 0.40$$

$$\Rightarrow x_m = 0.4 \times \sqrt{4 \times 0.138 \times 10^{-6} \times 60 \times 24 \times 60 \times 60} = 0.68\, m$$

So this is the minimum burial depth such that your water will not freeze okay, because your surface temperature is -15 ºC okay that is why we have taken that condition that it will not go below 0 ºC so with that we found the distance x_m from the earth's surface okay. So here we will stop and I will just conclude this Transient Heat Conduction.

So we started with the lumped capacitance system where we first used the condition that Biot number is less than 0.1, so that we can use the lump system analysis where temperature is function of t only okay. So spatial variation you can neglect. So with that we found the temperature distribution in a generalized case considering the uniform heat generation

q'''after that putting the $q''' = 0$ we have written the temperature distribution for a simplified case where

$$\frac{(T - T_\infty)}{(T_i - T_\infty)} = e^{-\left(\frac{hA_s}{\rho cV}\right)t}$$

And then we define also the Fourier number which is dimensionless time and we have written in terms of Biot number and Fourier number then we have solved two problems then today we have studied this semi-infinite medium so where you have one plane surface and other directions are infinite. So one such example we have given that your earth surface is a best example because the earth surface if you can use at a plane surface and all other directions are infinite.

Another example also I can give as; for a short period of time any solid medium you can consider as a semi-infinite medium, let us say consider one metal okay and for a fraction of second you have given a laser light there. So the temperature on the surface it will change, but at x tends to infinity that means if you go depth of the solid you will find that its surface temperature change that effect has not come to the inside okay.

So that also you can assume as a semi-infinite medium and we used similarity approach and we use similarity variable $\eta = \frac{x}{\sqrt{4\alpha t}}$ and with that we converted the partial differential equation with one initial condition and two boundary conditions to ordinary differential equation with two boundary conditions and we got the temperature distribution in terms of error function okay.

So for a simplified case where surface temperature is constant we have found this temperature distribution and the instantaneous heat flux. After that we have shown the temperature distribution if you have the constant heat flux case and convective boundary condition case. So we will conclude the Transient Heat Conduction here. Thank you.